Orbifold cohomology and quantum cohomology of orbifolds

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lectures by Dan Abramovich

Part 1: Moduli of twisted stable maps

- with Angelo Vistoli
- other construction by W. Chen and Y. Ruan

Work over $\mathbb{C}$

recall that

nice moduli problems are typically Deligne-Mumford stacks admitting projective coarse moduli spaces.

Classic Examples:

- $\overline{\mathcal{M}}_{g,n}$; $\quad \overline{\mathcal{M}}_{g,n}(X,\beta)$

Notation: $\quad \overline{\mathcal{M}}_{g,n}(X,\beta) =: \mathcal{K}_{g,n}(X,\beta)$

- $BG = \{pt\}/G$
$=$ moduli of principal homogeneous G-spaces

- the quotient stack $[V/G]$

Stacks have been introduced as “moduli objects” for families of schemes, sheaves etc.

People have now come to accept them as basic algebraic geometry objects.

Is $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ likely to be interesting for a DM stack $\mathcal{X}$?

examples:

1. $\mathcal{X} = \overline{\mathcal{M}}_{\gamma,\nu} \rightsquigarrow$

$\mathcal{K}_{g,n}(\mathcal{X})$ = moduli of fibered surfaces: families of stable curves over stable curves

2. $\mathcal{X} = \mathbf{B}G \rightsquigarrow$ moduli of principal G bundles on stable curves

recall:

Definition. A *stable n-pointed map*

$$(C, \ldots x_i \ldots, f : C \to X)$$

of genus g and image class β is a morphism from a *prestable n-pointed curve* $(C, x_1, \ldots, x_n)$ to a variety X, such that

- the group $\mathrm{Aut}_X(f : C \to X, x_i)$ of automorphisms of f fixing x_i is finite

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Now let $\mathcal{X}$ be a Deligne–Mumford stack.

Preliminary Definition. A *stable n-pointed map*

$$(C,\, x_i,\, f : C \to \mathcal{X})$$

is a morphism from a *prestable n-pointed curve* (C, x_i) to the stack $\mathcal{X}$ such that

- $\mathrm{Aut}_{\mathcal{X}}(f : C \to \mathcal{X})$ is finite

-

PROBLEM: NOT COMPLETE!!

e.g. consider degeneration of a smooth curve of genus 2,

and map to BG, with $G = (\mathbb{Z}/r)^4$.

smooth fiber C_η has connected principal G bundles, one for each isomorphism

$$H_1(C_\eta, \mathbb{Z})/r \ \stackrel{\sim}{\to}\ G$$

C_s has $H_1(C_\eta, \mathbb{Z}) = \mathbb{Z}^3$, so no connected G bundle!!

Claim: this problem is resolved if we allow the degenerate curve to be a *stack.*

Consider the example above:

$$\begin{tikzcd} P_\eta \arrow[r, hook] \arrow[d] & ? \arrow[d] \\ C_\eta \arrow[r, hook] \arrow[d] & C \arrow[d] \\ \{\eta\} \arrow[r, hook] & S \end{tikzcd}$$

If we try to extend directly (e.g. normalization of C in P_η) we typically get ramification index r over C_s.

(i) after base change of order r may assume P_η extends over C_{sm}:

$$\begin{array}{ccccc} P_\eta & \subset & P_{\mathrm{sm}} & & \\ \downarrow & & \downarrow & & \\ C_\eta & \subset & C_{\mathrm{sm}} & \subset & C \end{array}$$

(Purity of Branch Locus)

(ii) at node p we have in local analytic coordinates $C_p : xy = t^r$

We can pick a "uniformization" $V_p \to C_p$

$V_p : \{\xi\eta = t\} \quad$ where $\quad u = \xi^r,\ v = \eta^r$

V_p nonsingular. Purity $\Longrightarrow$ the bundle extends:

$$P_p \to V_p$$

There is an action of μ_r on V_p which lifts to the bundle P_p.

(iii) $V_p \to C_p$ with μ_r action is a **chart** for an **orbifold structure** $\mathcal{C} \to C$,
namely a DM stack $\mathcal{C}$ with moduli space C

moreover we have a morphism $\mathcal{C} \to BG$

Twisted curves ($n = 0$)

Definition. A twisted (pre-stable) curve is a Deligne–Mumford stack $\mathcal{C}$ with coarse moduli space C such that

1. C is a pre-stable curve,

2. the map $\mathcal{C} \to C$ to the coarse moduli curve is an isomorphism away from singularities of C, and

3. $\mathcal{C}$ has at most nodes as singularities

Local description in a family

$$C :\ xy = f(t)^r$$

$$\mathcal{C} = [V/\mu_r],\ \ V :\ \xi\eta = f(t),\ \ x = \xi^r, y = \eta^r$$

Action of μ_r :

$$(\xi, \eta) \ \ \mapsto \ \ (\,\zeta_r \cdot \xi\,,\, \zeta_r^a \cdot \eta\,)$$

with $\gcd(a, r) = 1$.

Terminology:

$r = 1$: **untwisted**; $r > 1$: **twisted**.

$a \equiv -1 \mod r$: **balanced**; otherwise **unbalanced**.

Note: if $f(t) \neq 0$ then balanced.

$n > 0$: Twisted curves at a marking

One way to arrive at marked curves is by normalizing a nodal curves and "separating" a node in two markings.

Local description via parameters

$$C: \; x$$

$$\mathcal{C} = [V/\mu_r], \;\; V : \xi, \;\; x = \xi^r$$

Action of μ_r : $\quad \xi \;\; \mapsto \;\; \zeta_r \cdot \xi$

Global description: a marking is a closed substack which is a **gerbe** banded by μ_r.

NOT NECESSARILY A SECTION

Definition

Let $\mathcal{X}$ be a D-M stack, $\mathcal{X} \to X$ projective coarse moduli scheme.

An n-pointed **twisted stable map** $f : \mathcal{C} \to \mathcal{X}$ to $\mathcal{X}$ of genus g, class β, is a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ C & \xrightarrow{\overline{f}} & X \end{array}$$

where

(i) $\mathcal{C}$ twisted curve with coarse moduli space C

(ii) $\mathcal{C} \to \mathcal{X}$ representable

(no unnecessary branchings)

(iii) $C \to X$ stable n-pointed map of genus g and class β

($\Leftrightarrow$ automorphism group is finite)

Theorem: the moduli of n-pointed twisted stable maps of genus g and class β to $\mathcal{X}$ is a proper D-M stack $\mathcal{K}_{g,n}(\mathcal{X},\beta)$ admitting a projective coarse moduli scheme $\mathbf{K}_{g,n}(\mathcal{X},\beta)$.

In fact, there is a commutative diagram of finite maps (not cartesian)

$$\begin{array}{ccc} \mathcal{K}_{g,n}(\mathcal{X},\beta) & \to & \mathbf{K}_{g,n}(\mathcal{X},\beta) \\ \downarrow & & \downarrow \\ \mathcal{K}_{g,n}(X,\beta) & \to & \mathbf{K}_{g,n}(X,\beta) \end{array}$$

Issues coming in proof:

(i) existence of bottom line in diagram

(ii) valuative criterion for properness like for $\mathbf{B}G$

(iii) annoying technicalities

Problem: a stack is a category, but this is a-priori a 2-category.

Claim: the 2-category of twisted curves is *equivalent* to the associated category.

Lemma Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne–Mumford stacks over a scheme S. Assume that there exists a dense open representable substack (i.e. an algebraic space) $U \subseteq \mathcal{X}$ and an open representable substack $V \subseteq \mathcal{Y}$ such that F maps U into V. Then any automorphism of F is trivial.

proof aside

in fact we have:

Proposition: the category of twisted curves is an algebraic stack.

Purity Lemma

S a nonsingular surface, $U \subset S$, $\quad U = S \smallsetminus \{pt\}$.

Given a diagram

$$
\begin{tikzcd}
 & & \mathcal{X} \arrow[d] \\
U \arrow[r, hook] \arrow[urr, "f_U"] & S \arrow[r, "\overline{f}"'] \arrow[ur, dotted, "\exists ?\, f"'] & X
\end{tikzcd}
$$

Then the f_U extends uniquely to $f : S \to \mathcal{X}$

(up to unique isomorphism...)

proof aside

Valuative criterion for properness (one case):

Suppose $S = \mathrm{Spec} R$, the spectrum of a discrete valuation ring;

Generic point $\eta = \mathrm{Spec} K$, special point s.

Suppose C_η / K is an untwisted curve, $C_\eta \to \mathcal{X}$ a (twisted) stable map.

Then, after a base change $S' \to S$, there is an extension

$$\begin{tikzcd} C_{\eta'} \arrow[r, hook] \arrow[d] & \mathcal{C} \arrow[r] \arrow[d] & \mathcal{X} \\ \{\eta\} \arrow[r, hook] & S & \end{tikzcd}$$

Steps:

(i) Kontsevich: extend the map to the coarse moduli space X.

(ii) Properness of $\mathcal{X}$: extend the map $C_\eta \to \mathcal{X}$ over an open set containing the generic points of C_s

(iii) Purity Lemma: the map extends over the smooth locus of C.

(iv) local argument as for BG: lift the map to $[V_p/\mu_r] \to \mathcal{X}$

(v) descent: replace $[V_p/\mu_r]$ by something representable.

Global description of a twisted marking

Recall that a marking $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ is a gerbe. In fact, there is a nice canonical way to recover $\Sigma^{\mathcal{C}} \subset \mathcal{C}$ from $\Sigma^{C} \subset C$, away from nodes.

The point is that the pullback of Σ^{C} is $r \cdot \Sigma^{\mathcal{C}}$.

Definition. Let V be a scheme, L an invertible sheaf, s a global section of L. Define a stack $\sqrt[r]{(L,s)}$ whose objects over a scheme T are

$$(f, M, t, \phi),$$

where

- $f : T \to V$ is a morphism
- M is an invertible sheaf on V
- t is a section of M
- $\phi : M^{\otimes r} \xrightarrow{\sim} f^*L$ is an isomorphism, such that
- $\phi(t^{\otimes r}) = s$,

and arrows are fiber diagrams.

Proposition Near Σ^C, the twisted curve $\mathcal{C}$ is isomorphic to $\sqrt[r]{(L,s)}$ where L is $O_C(\Sigma^C)$ and s is the defining section.