Orbifold cohomology and quantum cohomology of orbifolds

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Part 2: Algebraic Orbifold Quantum Products

- Work over $\mathbb{C}$
- This work with Tom Graber and Angelo Vistoli
- Quantum products based on W. Chen + Y. Ruan
- global quotient case by Fantechi + Göttche.

Suppose that $\mathcal{X}$ is an algebraic stack, with moduli space X. Let $\mathcal{C}$ be a twisted curve

Contraction Lemma: Given (possibly unstable)

$$\begin{array}{ccc} \mathcal{C} & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ C & \to & X \end{array}$$

Then if $C \to X$ can be stabilized,

so does $\mathcal{C} \to \mathcal{X}$.

Applications:

(a) functoriality in $\mathcal{X} \to \mathcal{Y}$

(b) forgetful maps for untwisted markings and universal curve.

proof aside

Recall: Gromov-Witten theory

Let X be a smooth projective variety. Consider the correspondence

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1}(X,\beta) \arrow[r, "e_{n+1}"] \arrow[d, "e_1\times\cdots\times e_n"'] & X \\ X^n & \end{tikzcd}$$

Define

$$\begin{aligned} H^*(X)^n &\longrightarrow H^*(X) \\ \gamma_1 \times \cdots \times \gamma_n &\mapsto \langle \gamma_1, \dots, \gamma_n, * \rangle_{g,\beta}, \end{aligned}$$

where

$$\langle \gamma_1, \dots, \gamma_n, * \rangle_{g,\beta} =$$

$$(e_{n+1})_* \left(e^*_{1,\dots,n}(\gamma_1 \times \cdots \times \gamma_n) \cap [\overline{\mathcal{M}}_{g,n+1}(X,\beta)]^v \right)$$

In genus zero these *GW invariants* satisfy *WDVV relations*, giving *associativity*.

“Small” quantum product is defined by

$$\gamma_1 * \gamma_2 = \sum_{\beta \in H_2(X)} \langle \gamma_1, \gamma_2, * \rangle_{0,\beta} \; q^{\beta}.$$

(grading of q^{β} is $2c_1(X) \cdot \beta$)

Associativity (assuming nothing odd):

$$\sum_{\beta_1+\beta_2=\beta} \Big\langle \langle \gamma_1, \gamma_2, * \rangle_{0,\beta_1}, \; \gamma_3, \; * \Big\rangle_{0,\beta_2} =$$

$$\sum_{\beta_1+\beta_2=\beta} \Big\langle \langle \gamma_1, \gamma_3, * \rangle_{0,\beta_1}, \; \gamma_2, \; * \Big\rangle_{0,\beta_2}.$$

A key step in proving associativity is a morphism from

$$\overline{\mathcal{M}}_{0,3}(X,\beta_1) \underset{X}{\times} \overline{\mathcal{M}}_{0,3}(X,\beta_2)$$

to a "boundary divisor" in

$$\overline{\mathcal{M}}_{0,4}(X,\beta_1+\beta_2)$$

$$\begin{tikzcd}
 & & & \mathcal{M}_2 \arrow[dl] \arrow[dr] \arrow[rr] & & X \\
 & \mathcal{M}_1 \arrow[dl] \arrow[dr] \arrow[r] & X & & X & \\
X & & X & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{M}_1 \times_X \mathcal{M}_2 \arrow[r, hook] & D_{(12|34)}(X) \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{0,4}(X,\beta) \arrow[d] \\
 & D_{(12|34)} \arrow[r, hook] & \overline{\mathcal{M}}_{0,4}
\end{tikzcd}$$

In order to generalize this picture, we will need

1. an analogue of $H^*(X)$ or $A^*(X)$ for a smooth Deligne–Mumford stack,
2. an analogue of the evaluation maps e_i and e_{n+1},
3. an analogue of the virtual fundamental class, and
4. an analogue of the fiber product description of the boundary divisors.

We'll start with the **last**

Interpretation of boundary: suppose $C = C_1 \overset{p}{\cup} C_2$

Then

$$Hom(C, X) =$$

$$Hom(C_1, X) \underset{Hom(\{p\},X)}{\times} Hom(C_2, X)$$

Now say $\mathcal{C}$ is a nodal **twisted** curve.

$$\mathcal{C} = \mathcal{C}_1 \overset{\wp}{\cup} \mathcal{C}_2,$$

where

$$\wp \simeq B(\mu_r)$$

$$Hom(\mathcal{C}, \mathcal{X}) =$$

$$Hom(\mathcal{C}_1, \mathcal{X}) \underset{Hom(\wp, \mathcal{X})}{\times} Hom(\mathcal{C}_2, \mathcal{X})$$

So evaluation lands in

$$Hom\,(\,B(\mu_r)\,,\;\mathcal{X}\,)$$

(and not just in $\mathcal{X}$)

Inertia stack

Here

$$\begin{aligned}
\mathcal{X}_1 &= \bigcup_r HomRep(\mathcal{B}\boldsymbol{\mu}_r, \mathcal{X}) \\
&= \left\{ (x, H, \chi) \;\middle|\; \begin{array}{c} x \in Ob(\mathcal{X}), \\ H \subset \mathrm{Aut}\ x, \\ \chi : H \tilde{\to} \boldsymbol{\mu}_r \end{array} \right\} \\
&= \left\{ (x, g) \;\middle|\; \begin{array}{c} x \in \mathcal{O}b(\mathcal{X}), \\ g \in \mathrm{Aut}\ x \end{array} \right\} \\
&= \mathcal{I}(\mathcal{X})
\end{aligned}$$

The points of the coarse moduli space $I(X)$ are

$$(\,[x]\,,\ (g)\,)$$

where $[x] \in X$ is the isom. class of an object x and

(g) is the *conjugacy* class of an element

$$g \in \mathrm{Aut}(x)$$

Let $\mathcal{X}$ be a *smooth* Deligne–Mumford stack of dimension d with projective coarse moduli space X

WE WORK WITH

$$H^*_{orb}(\mathcal{X}) = H^*(\mathcal{X}_1)$$

Age

Let $([x], (g)) \in I(X)$

Can write locally $\mathcal{X} = [V / \Gamma]$,
where $\Gamma = \mathrm{Aut}(x)$, and V smooth.

Diagonalize the action of g on the tangent $T_x(V)$, with eigenvalues

$$e^{(2\pi i)\cdot r_j}, \quad j = 1, \ldots, d$$

with $0 \leq r_j < 1$.

Definition: the age

$$a(x, g) = \sum_{i=1}^{d} r_j.$$

Claim: $a(x,g)$ is constant on every connected component $Z \subset I(X)$ (notation $a(Z)$)

Claim: $a(x,g) + a(x,g^{-1}) \; = \; \dim X - \dim Z$

GRADING

Definition: The degree of an element $\alpha \in H^i(Z)$ in $H^*_{orb}(\mathcal{X})$ is defined to be

$$i + 2a(Z)$$

Note: $\dim H^i_{orb}(\mathcal{X}) = \dim H^{d-i}_{orb}(\mathcal{X})$.

Evaluation

Unless $g = 0, n = 3, \beta = 0$, the markings of $\mathcal{K}_{g,n}(X, \cdot)$ are nontrivial grbes.

Define

$\overline{\mathcal{M}}_{g,n}(X, \cdot) =$ twisted stable maps with sections

$=$ fibered product of universal gerbes over $\mathcal{K}_{g,n}(X, \cdot)$

Have evaluation maps

$$\overline{\mathcal{M}}_{g,n}(X, \cdot) \xrightarrow{e_i} \mathcal{X}_1$$

recall that in

$$\mathcal{C} = \mathcal{C}_1 \overset{\wp}{\cup} \mathcal{C}_2,$$

we can naturally identify $B(\mu_r) \simeq \wp \subset \mathcal{C}_2$,

via the action of μ_r on the normal bundle to the marking in $\mathcal{C}_2$.

but then the action on the $\mathcal{C}_1$ side is opposite!

$$(\xi, \eta) \ \mapsto \ (\, \zeta_r \cdot \xi \,,\ \zeta_r^{-1} \cdot \eta \,)$$

Define involution

$$\iota : \mathcal{X}_1 \to \mathcal{X}_1$$

via

$$g \mapsto g^{-1}$$

Define:

$$\check{e}_i \ = \ \iota \circ e_i$$

Deformations / Obstructions: general case

(i) Twisted curves are unobstructed:

$$Ext^2(\Omega^1_{\mathcal{C}}, \mathcal{O}_{\mathcal{C}}) = 0$$

(ii) Obstructions of $f : \mathcal{C} \to \mathcal{X}$ lie in

$$H^1(\mathcal{C}\,,\ f^*T_{\mathcal{X}}\,)$$

(iii) Relative infinitesimal deformations are

$$H^0(\mathcal{C}\,,\ f^*T_{\mathcal{X}}\,)$$

Behrend - Fantechi, Kresch $\Longrightarrow$

virtual fundamental class $[\mathcal{K}_{g,n}(X, \cdot)]^v$

Because of sections need to use

$[\overline{\mathcal{M}}_{g,n}(X, \cdot)]^w \ = \ r_1 \cdots r_n \, [\overline{\mathcal{M}}_{g,n}(X, \cdot)]^v$

Special case: every deformation is **un**obstructed.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ \pi \downarrow & & \\ \overline{\mathcal{M}} & & \end{array}$$

then

$$[\overline{\mathcal{M}}]^v \;=\; c_{top}(E) \;=\; c_{top}\left(\mathbb{R}^1\pi_* f^* T_{\mathcal{X}}\right)$$

If, moreover, $\mathcal{C} \;=\; [D/H]$ with $\tilde{f} : D \to \mathcal{X}$, then

$$E \;=\; (\mathbb{R}^1\tilde{\pi}_* \tilde{f}^* T_{\mathcal{X}})^H$$

3-point orbifold G–W invariants

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_{0,3}(\mathcal{X},\beta) & \stackrel{\check{e}_3}{\to} & \mathcal{X}_1 \\
e_1 \times e_2 \downarrow & & \\
\mathcal{X}_1 \times \mathcal{X}_1 & &
\end{array}
$$

for $\gamma_i \in H^*_{orb}(\mathcal{X})$ Define

$\langle \gamma_1, \gamma_2, * \rangle_\beta =$

$$(\check{e}_3)_* \left(e_1^*(\gamma_1) e_2^*(\gamma_2) \cap [\overline{\mathcal{M}}_{0,3}(X,\beta)]^w \right)$$

"Small" quantum product is again defined by

$$\gamma_1 * \gamma_2 = \sum_{\beta \in H_2(\boldsymbol{X})} \langle \gamma_1, \gamma_2, * \rangle_{0,\beta} \; q^{\beta}.$$

Theorem: this is a graded skew-commutative associative ring.

grading of q^{β} is $2(c_1(\boldsymbol{\mathcal{X}}) \cdot \beta)$

Will only prove when every deformation is unobstructed, and curves smooth freely

Associativity (assuming nothing odd):

$$\sum_{\beta_1+\beta_2=\beta} \Big\langle \langle \gamma_1, \gamma_2, * \rangle_{0,\beta_1},\ \gamma_3,\ * \Big\rangle_{0,\beta_2} =$$

$$\sum_{\beta_1+\beta_2=\beta} \Big\langle \langle \gamma_1, \gamma_3, * \rangle_{0,\beta_1},\ \gamma_2,\ * \Big\rangle_{0,\beta_2} .$$

$$\begin{tikzcd}
 & & & \mathcal{M}_2 \arrow[r] \arrow[dl] \arrow[dr] & \mathcal{X}_1 \\
 & \mathcal{M}_1 \arrow[r] \arrow[dl] \arrow[dr] & \mathcal{X}_1 & & \mathcal{X}_1 \\
\mathcal{X}_1 & & \mathcal{X}_1 & &
\end{tikzcd}$$

The product diagram

$$\begin{tikzcd}
\mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \arrow[r] \arrow[d] & \mathcal{M}_2 \arrow[d] \\
\mathcal{M}_1 \arrow[r] & \mathcal{X}_1
\end{tikzcd}$$

can be expanded:

$$\begin{tikzcd}
\mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \arrow[r] \arrow[d] & \mathcal{M}_2 \arrow[r] \arrow[d] & \mathcal{X}_1 \arrow[d] \\
\mathcal{M}_1 \times \mathcal{M}_2 \arrow[r] \arrow[d] & \mathcal{X}_1 \times \mathcal{M}_2 \arrow[r] \arrow[d] & \mathcal{X}_1 \times \mathcal{X}_1 \\
\mathcal{M}_1 \arrow[r] & \mathcal{X}_1 &
\end{tikzcd}$$

It follows that the coefficient in $(\gamma_1 * \gamma_2) * \gamma_3$ is

$$r_\times^2 \;\; e^\times_{4\,*}\left(\left(\prod_{i=1}^{3} e_i^{\times\,*}(\gamma_i) \right) \cdot \mathfrak{C} \right)$$

where

$$\mathfrak{C} \;\; = \;\; c_{top}(\, p_1^* E_1 \;\oplus\; p_2^* E_2 \,)\; c_{top}(\, N_\delta / N_{p_1 \times p_2} \,)$$

Here

$E_i =$ obstruction bundle of $f_i : \mathcal{C}_i \to \mathcal{X}$,

$N_\delta / N_{p_1 \times p_2} =$ excess normal bundle

Can we make this "symmetric"?

The divisor diagram

$$\begin{tikzcd}
 & & & \mathcal{M}_1 \times_{\mathcal{X}_1} \mathcal{M}_2 \arrow[d] \\
\mathcal{M}_{0,4}(X) \arrow[d] & D \times_{\mathbf{M}_{0,4}} \mathcal{M}_{0,4}(X) \arrow[l] \arrow[d] & \supset & D(X) \\
\mathbf{M}_{0,4} & \supset \quad D & &
\end{tikzcd}$$

So enough to show

$$\mathfrak{C} = gl^* c_{top}(E)$$

$E =$ obstruction bundle of $f_{0,4} : \mathcal{C}_{0,4} \to \mathcal{X}$.

We consider the normalization sequence

$$0 \to f^*T\mathcal{X} \to \nu_* f'^*T\mathcal{X} \to (f^*T\mathcal{X})|_\Sigma \to 0.$$

Pushing forward to $\mathcal{Y}$ gives

$$0 \to \pi_* f^*T\mathcal{X} \to \pi'_* f'^*T\mathcal{X} \to (\pi_*(f^*T\mathcal{X})|_\Sigma)$$

$$\to gl^*E \to p_1^*E_1 \oplus p_2^*E_2 \to 0.$$

Let Q be the bottom kernel.

Need: $Q =$ excess bundle.

Have on top

$$0 \to \; T\mathcal{Y} \; \to \; p_1^* T\mathcal{M}_1 \oplus p_2^* T\mathcal{M}_2$$

$$\to \; \pi_*((f^* T\mathcal{X})|_\Sigma) \; \to \; Q \; \to 0$$

So we got

$$0 \to N_{p_1 \times p_2} \to \pi_*((f^* T\mathcal{X})|_\Sigma) \to Q \to 0$$

So the following suffices:

Lemma.

$$\pi_* \left((f^* T\mathcal{X})|_\Sigma\right) \; \simeq \; e_\times^* \, T\mathcal{X}_1$$