Orbifold cohomology and quantum cohomology of orbifolds

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Part 3: Stringy Orbifold Cohomology: Theory and Examples

• Work over $\mathbb{C}$

• This work with Tom Graber and Angelo Vistoli

• Quantum products based on W. Chen + Y. Ruan

• global quotient case by Fantechi + Göttsche.

Equivariant Riemann-Roch

Let $\mathcal{C}$ be a smooth twisted curve, $\mathcal{E}$ a vector bundle on $\mathcal{C}$.

Let x be a twisted marking, locally $\mathcal{C}$ is $[V/\mu_r]$,

V has coordinate z,

Say μ_r acts on $T_{V,x}$ via $v \mapsto \zeta_r \cdot v$.

(Then μ_r ants on z via $z \mapsto \zeta_r^{-1} \cdot z$.)

say μ_r acts on $\mathcal{E}_x$ with eigenvalues $e^{(2\pi i)\cdot r_i}$, with $0 \leq r_i < 1$.

the **age** is

$$a(\mathcal{E}, x) = \sum r_i$$

Theorem.

$$\chi(\mathcal{C}, \mathcal{E}) = d(\mathcal{E}) - rk(\mathcal{E})(g-1) - \sum_x a(\mathcal{E}, x)$$

Exercise: quantum product is graded.

Stringy Orbifold cohomology:

We defined

$$\gamma_1 * \gamma_2 = \sum_{\beta \in H_2(X)} \langle \gamma_1, \gamma_2, * \rangle_{0,\beta} \ q^\beta.$$

Now we can plug in $q^\beta \mapsto 0$ and obtain an associative ring structure on $H^*_{orb}(\mathcal{X}, \mathbb{Q})$

Exercise: This ring has unit $1 \in H^*(\mathcal{X}, \mathbb{Q})$.

Defined $\langle \gamma_1, \gamma_2, * \rangle_{0,\beta}$ using $\overline{\mathcal{M}}_{0,3}(\mathcal{X}, \beta)$

Claim: when $\beta = 0$ the three gerbes are trivial, and there are evaluation maps

$$\mathcal{K}_{0,3}(\mathcal{X}, \beta) \to \mathcal{X}_1.$$

These maps are **representable**

So the above lifts to a product

$$\gamma_1 \smile \gamma_2$$

on $H^*_{orb}(\mathcal{X}, \mathbb{Z})$

Theorem [$\aleph$GV]: this is well defined for arbitrary smooth separated Deligne–Mumford stack, independent of choices and associative.

Proof of associativity is much more delicate.

The Fantechi-Göttsche ring

In case $\mathcal{X} = [Y/G]$ write

$$\widetilde{Y} = \bigsqcup_{g \in G} Y^g$$

The group G acts by sending $y \in Y^g$ to $h(y) \in Y^{hgh^{-1}}$. So we have $\mathcal{I}(\mathcal{X}) \simeq [\widetilde{Y}/G]$

Notation:

$$H^*(Y,G) = H^*(\widetilde{Y}, \mathbb{Q}).$$

Theorem [Fantechi-Göttsche]:

1. There is a (natural) associative (noncommutative) ring structure on $H^*(Y,G)$, with an action of G.

2. There is a ring isomorphism $H^*(Y,G)^G \simeq H^*_{orb}(Y)$

The construction and proof is very much analogous to the one for $H^*_{orb}(Y)$

Example 0

$$\mathcal{X} = BG$$

$$\mathcal{I}(\mathcal{X}) \;=\; [G \overset{adj}{/} G] \;=$$

$$\coprod_{(g)} B(C(g)) \quad =: \quad \coprod_{(g)} \mathcal{X}_{(g)}$$

$$H^*_{orb}(BG, \mathbb{Z}) \quad = \quad \bigoplus_{(g)} H^*(B(C(g)), \mathbb{Z})$$

$$T_{BG} = 0 \quad \Longrightarrow a(Z) = 0$$

Component $\mathcal{K}_{g,h}$ of $\mathcal{K}_{0,3}(BG,0)$ with $g, h, (gh)^{-1}$:

$$\mathcal{K}_{g,h} \;\simeq\; B(C(g) \cap C(h))$$

product:

$$x_{(g)} \smile x_{(h)} = \sum_{(g',h')} \left| \frac{C(g'h')}{C(g') \cap C(h')} \right| x_{(gh)}$$

$$H^*_{orb}(BG, \mathbb{Q}) \;=\; C(\mathbb{Q}[G]) = (\mathbb{Q}[G])^G$$
$$(H^*(\{pt\}, G) = \mathbb{Q}[G])$$

$$H^*_{orb}(B(\mu_r), \mathbb{Z}) \;=\; \mathbb{Z}[s,t]/(s^r - 1, rt)$$

Example 1

$$\mathcal{X} \ = \ [\,\mathbb{A}^1 \,/\, \mu_r\,]$$

$$\mathcal{I}(\mathcal{X}) \ = \ \mathcal{X} \ \sqcup \ \bigsqcup_{k=1}^{r-1} B(\mu_r)$$

$$H^*(\mathcal{X}_i, \mathbb{Z}) \ = \ \mathbb{Z}[t_i]/(rt_i)$$

Age of X_i is i/r

Components:

$$\mathcal{K}_{i,j} = \begin{cases} \mathcal{X} & i = j = 0 \\ B(\mu_r) & \text{otherwise} \end{cases}$$

$$[\mathcal{K}_{i,j}]^v \;\; = \;\; [\mathcal{K}_{i,j}] \text{ whenever } i + j \le r \text{ by grading.}$$

Product:

$$x_i \smile x_j = \begin{cases} x_{i+j} & i + j < r \\ t_0 & i + j = r \end{cases}$$

$$t_0 \smile x_i = t_i$$

$$H^*_{orb}(\mathcal{X}, \mathbb{Z}) = \mathbb{Z}[x_1]/(rx_1^r)$$

Example 2 The wighted projective stack

$$\mathbb{P}^1[a,b] = [\,(\mathbb{A}^2 \smallsetminus \{0\})\ /\mathbb{G}_m\,]$$

with the action $(x,y) \mapsto (t^a x, t^b y), \quad a,b > 0.$

Let $g = \gcd(a,b),\ l = ab/g.$

$$\mathcal{I}(\mathcal{X}) = \coprod_{i=0}^{g-1} \mathcal{X} \sqcup \coprod_{i=0}^{a-g} B(\mu_a) \sqcup \coprod_{i=0}^{b-g} B(\mu_b)$$

$$= \coprod_{i=0}^{g-1} X_i \sqcup \coprod_{\substack{i=1 \\ \gcd(i,a/g)=1}}^{a-1} A_i \sqcup \coprod_{\substack{i=1 \\ \gcd(i,b/g)=1}}^{b-1} B_i$$

$$H^*(X,\mathbb{Z}) = \mathbb{Z}[t]/(abt^2)$$

Age of A_i is ig/a, of B_i is ig/b

Ring:

$H^*_{orb}(\mathcal{X}, \mathbb{Z}) =$

$$\frac{\mathbb{Z}[X, A, B, T]}{(X^g - 1, aAT, bBT, A^{\frac{a}{g}} - bXT, B^{\frac{b}{g}} - aXT, AB)}$$

Exercise: Find $H^*_{orb}(\mathcal{X}, \mathbb{Q})$ and $H^*_{orb}(\mathcal{X}, \mathbb{Z})$ when $\mathcal{X} = \mathbb{P}^2[1, 1, a]$

Example:

$$\mathcal{X} \;\; = \;\; [\, V^n \,/\, \mathcal{S}_n \,]$$

Notation:

$\sigma \in \mathcal{S}_n$ has $l(\sigma) =$ length,

$o(\sigma) =$ number of cycles $= n \,-\, l(\sigma)$.

Also $\dim V = d$.

$$\mathcal{I}(\mathcal{X}) \; = \; \coprod_{(\sigma)} [\,(V^n)^\sigma \,/\, C(\sigma)]$$

$$(V^n)^\sigma \simeq V^{o(\sigma)} \;\; \Longrightarrow \;\; \dim \mathcal{X}_{(\sigma)} = d \cdot o(\sigma)$$

Age: $(\sigma) = (\sigma^{-1})$

$$\implies a(\mathcal{X}_{(\sigma)}) \;=\; \frac{d \cdot l(\sigma)}{2}$$

Components:

$$\mathcal{K}_{(\sigma,\tau)} = \Big[(V^n)^{\langle \sigma,\tau \rangle} \,/\, C(\sigma) \cap C(\tau) \Big]$$

Notation: $\mathcal{O}(\sigma, \tau)$ = set of orbits of $\langle \sigma, \tau \rangle$.

$W = \mathbb{C}^n$ standard representation of $\mathcal{S}_n$.

W_I = representation of $\langle \sigma, \tau \rangle$ on the subspace with coordinates in $I \subset \{1, \ldots, n\}, I \in \mathcal{O}(\sigma, \tau)$.

Have $\mathcal{C} \xrightarrow{\pi} \mathcal{K}_{(\sigma,\tau)} \to \mathcal{X}$

$f^* T_{\mathcal{X}} =$ "$\bigoplus_{I \in \mathcal{O}(\sigma,\tau)} (W_I \otimes \pi^* T_{V_I})$"

$$H^0(\mathcal{C},\, W_I \overset{G}{\otimes} \mathcal{O}_{\mathcal{C}}) = 1$$

$$\chi(\mathcal{C},\, W_I \overset{G}{\otimes} \mathcal{O}_{\mathcal{C}}) =$$
$$|I| - a(g, W_I) - a(h, W_I) - a((gh)^{-1}, W_I)$$

so $r_I = H^1_I = \frac{|I|+1-o_I(g)-o_I(h)-o_I(gh)}{2}$.

So W_I contributes a factor

$$\begin{cases} 1 & \text{if } r_i = 0 \\ c_{top}(T_V) & \text{if } r_i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Finally there is a constant factor of $\left|\frac{C(gh)}{C(g) \cap C(h)}\right|$.