SUBVARIETIES OF SEMIABELIAN VARIETIES

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0. Introduction

Let $X \subset A$ be a reduced, irreducible, closed subvariety of a semiabelian variety A over an algebraically closed field k. We would like to study the structure of such an X. Our point of departure is arithmetic: motivated by Lang's conjectures [15, 17, 16], as manifested by the theorems of Faltings [6, 7] and Vojta [27], we study the Mordell exceptional locus:

$$Z(X) = \{x \in X | \exists B, \dim B > 0, B \text{ a subgroup}, x + B \subset X\},$$

that is, the union of translated positive dimensional subgroups of A inside X. It is shown that $Z(X)$ is closed in X. Each component Z_0 of $Z(X)$ has a maximal stabilizing semiabelian variety B_0 such that $Z_0 + B_0 = Z_0$, and Z_0/B_0 is shown to be of logarithmic general type.

This structure was previously known in characteristic 0. Kawamata [14] (following Ochiai [23]) showed in the case of A an abelian variety that the Mordell exceptional locus is closed. A different proof was given by Bogomolov [4]. This proof is in fact valid in arbitrary characteristic. Ueno [29] gave the characterization of the Kodaira dimension for a subvariety of an abelian variety and Noguchi [22] extended the result to semiabelian varieties. The exposition of Mori ([20]) makes Ueno's construction algebraic. With the addition of several ingredients (mainly [25] and [18]) one can make his approach work in positive characteristic as well, although this has not been done in the past, to the author's knowledge.

Our goal here is to give alternative, algebraic proofs which are valid in arbitrary characteristic (as well as in the analytic case).

First, we show that the analogue of the Mordell exceptional locus is closed for subvarieties of arbitrary algebraic or complex groups. This is done by studying the infinitesimal properties of the Mordell exceptional locus.

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We then extend the characterization of the (logarithmic) Kodaira dimension for arbitrary characteristic. We do this by constructing certain maps to projective spaces. An important feature of these maps is that they are stable under field extension. The motivation comes from Voloch's work on Lang's conjectures in positive characteristic [28] and the effort to generalize it to higher dimension [3]. The maps we produce here allows us to obtain a weak notion of "field of moduli" for our variety X, which behaves well under certain birational modifications. This proves to be useful in [3].

Recall that a group variety A over a field K is called a *semiabelian variety* if it fits in an exact sequence

$$0 \to T \to A \to B \to 0,$$

where B is an abelian variety and T is a torus, that is, after extending to the algebraic closure T becomes a product of multiplicative groups: $T \otimes_K \overline{K} \simeq G_m^d$. Analogousely, we say that a commutative complex group A is a *semitorus* if it is an extension of a compact complex torus by $(\mathbf{C}^*)^n$, or equivalently, a quotient of $(C^*)^g$ by a discrete subgroup. A semiabelian variety A over an algebraically closed field (or a complex semitorus) is always obtained as a subgroup of the automorphism group of a completion $\overline{A}$ which is a projective bundle over B.

The geometric results in this paper can be summarized by the following theorem:

Theorem 1. *Let $X \subset G$ be a reduced, irreducible, closed subvariety of G, where G is either a complex group or an algebraic group over an algebraically closed field. Let*

$$Z(X) = \{x \in X | \exists B, \dim B > 0, B \text{ a subgroup, } xB \subset X\}$$

be the Mordell exceptional locus on X. Then $Z(X)$ is a closed subvariety of X.

Theorem 2. *Let $X \subset A$ be a reduced, irreducible, closed subvariety of A, where A is either a complex semitorus or a semiabelian variety over an algebraically closed field. Assume $Z(X) = X$. Then there is a positive dimensional subgroup B of A such that $B + X = X$, that is, $\dim(Stab(X)) > 0$.*

We denote by $\bar{\kappa}(X)$ the *logarithmic* Kodaira dimension of X. In the complex case we need to assume that X is meromorphic, that is, the closure of X in a compactification of A is a complex space.

Theorem 3. *Let $X \subset A$ be as in the previous theorem. In the analytic case assume that X is meromorphic, that is, it extends to a complex subspace of a compactification of A. Let $B = Stab(X)$, that is, B is the maximal closed subgroup B of A such that $B + X = X$. Then $\bar{\kappa}(X) = \dim(X/B)$.*

The above theorem is proved using a generalized Gauss map defined using jets. The usual Gauss map for $X \subset A$ is defined by sending a smooth point x of X to the point on the Grassmann variety representing the tangent space of X at x, inside the tangent space to A. Here we have:

Theorem 4. *Let $X \subset A$ be a subvariety of an abelian variety. Assume that X is non-singular and has a finite stabilizer in A. Then the Gauss map $X \to \mathbf{Gr}(\dim X, \dim A)$ is finite.*

This theorem was proved in characteristic 0 by Griffiths and Harris [8] and Ziv Ran [24] gave an alternative proof.

Example: Let C be a smooth, projective curve over an algebraically closed field k. Let $W_d(C) \subset Pic^d(C)$ be the variety parametrizing effective line bundles of degree d on C, inside the Picard group. As long as $d < g$ we have $\kappa(W_d(C)) = d$. This is because $W_d(C)$ cannot be stabilized by a positive dimensional abelian subvariety: if it were, then $W_{g-1}(C)$ would also be stabilized, but then it could not give a polarization of $Pic(C)$. (More directly, one can identify the projectivized tangent space of $W_d(C)$ as a linear subspace in the canonical space, and show that the Gauss map has a finite degree.) On the other hand, the behavior of $Z(W_d(C))$ is rather subtle. More detailed discussions may be found in [12, 2, 1, 5].

In the appendix we will study a general question of intersection of families, which is used in the proof of theorem 1.

In section 1 we give a proof of theorem 1 as well as an alternative proof for the case when the group is a semiabelian variety, which gives us also a proof of theorem 2.

Section 2 gives a proof of theorem 3. Section 3 proves theorem 4. We also prove that if $X \subset A$ is a nonsingular hypersurface in an abelian variety admitting a vector field, then X is stabilized by a positive dimensional subgroup. We conjecture that this is true for arbitrary nonsingular subvariety $X \subset A$.

This paper is an extension of part 1 of the author's Ph.D. thesis [1], where the case of an abelian variety was considered.

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1. The union of translated subgroups inside X.

We will give two proofs of theorem 1. One proof follows ideas of Joe Harris and uses the local structure of A. The other, following Faltings, uses torsion points, and is valid only in the semi - abelian case.

All schemes are assumed to be of finite type over a fixed algebraically closed field.

Remark: In the proofs we will use the language of schemes, and the reader may make the suitable changes for the non-algebraic complex case.

1.1. The first proof of theorem 1.

Notation: We use the map $m_1 : G \times G \to G$ defined by $m_1(a,b) = a^{-1}b$. Also, whenever we have subschemes X, Y, Z of G, we write $X^{-1}Y \subset Z$ to mean $m_1(X \times Y) \subset Z$, that is, $X \times Y \subset m_1^{-1}Z$.

An important tool in the proof of the the theorem is the following notion, which is discussed in the appendix: Let $\mathcal{X} \to B; \mathcal{Y} \to B; \mathcal{Z} \to B$ be three schemes (or complex spaces) of finite type over the base B, and let $i_1 : \mathcal{Y} \to \mathcal{X}; i_2 : \mathcal{Z} \to \mathcal{X}$ be closed embeddings over B. Assume that $\mathcal{Z}$ is either *finite and flat* over B or that it is a *constant* family over B. Then we can find a closed subscheme S of B which is maximal with respect to the property $\mathcal{Z}|_S \subset \mathcal{Y}|_S$.

The main idea in the proof of the theorem is to single out a subset of X of points at which X contains a positive dimensional formal group, and to show that this is the required set $Z(X)$. The key point is the following lemma, which proves that the Zariski closure of a formal subgroup is a group subscheme:

Lemma 1. *Let I_n be a nested sequence of subschemes of G supported at the origin, with $l(I_n) \to \infty$, such that $(I_n)^{-1}I_n \subset I_{n+1}$ for all n. Let B be the intersection of all closed subschemes of G containing all the I_n. Then B is a positive dimensional group-subscheme of G.*

Proof: Clearly $l(B) = \infty$, so $\dim(B) > 0$. To show that $B^{-1}B = B$ we will need the following:

Claim: Let $Z \subset G$ be a scheme, $\mathcal{Y} \to G$ a closed subscheme of $\pi_2 : G \times G \to G$. If $Z \times I_n \subset \mathcal{Y}$ then $Z \times B \subset \mathcal{Y}$.

Proof of claim: Let S be the maximal closed subscheme where $Z \times S \subset \mathcal{Y}|_S$ as in part 2 of theorem 6 in the appendix. Since $I_n \subset S$ for all n, we must have $B \subset S$ by the definition of B $\square$

Back to our lemma. In order to show $B^{-1}B = B$ we need $B \times B \subset m_1^{-1}B$. Fixing $\mathcal{Y} = m_1^{-1}B$ and $Z = B$ in our claim, we see that it is enough to show that $B \times I_n \subset m_1^{-1}B$ for all n. Similarly, taking $Z = I_n$ and keeping $\mathcal{Y} = m_1^{-1}B$ we see (after switching the factors) that it is enough to show that $I_{n_1} \times I_{n_2} \subset m_1^{-1}B$ for all n_1 and n_2. But this is equivalent to $I_{n_1}^{-1}I_{n_2} \subset B$, and if $n = \max(n_1, n_2)$ then $I_{n_1}^{-1}I_{n_2} \subset I_n^{-1}I_n \subset I_{n+1} \subset B$ $\square$

We proceed to prove our theorem. We will construct the locus on X of points where X contains a positive dimensional formal group by approximating it by finitely many closed conditions on X.

Denote by $\mathcal{H}_{n,0}$ the Hilbert scheme parametrizing subschemes of length n supported at the origin. It is important that $\mathcal{H}_{n,0}$ is a proper scheme.

Let $\mathcal{G}_n = \mathcal{H}_{2^n,0}$.

Claim: In $\mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_r$ there is a closed subscheme $\mathcal{K}_r$ parametrizing sequences of schemes $G_1, \ldots, G_r$ such that $(G_i)^{-1}G_i \subset G_{i+1}$.

Proof: Use induction on part 1 of theorem 6: Let $\mathcal{Y}_0$ be the pullback of the universal family of subschemes over $\mathcal{G}_r$ to $\mathcal{K}_{r-1} \times \mathcal{G}_r$, and let $\mathcal{Z}_0$ be the pullback of the universal family over $\mathcal{G}_{r-1}$ to $\mathcal{K}_{r-1} \times \mathcal{G}_r$. Now (by abuse of notation) let $\mathcal{Y} = m^{-1}\mathcal{Y}_0$ and $\mathcal{Z} = \mathcal{Z}_0 \times_{(\mathcal{K}_{r-1} \times \mathcal{G}_r)} \mathcal{Z}_0$. The scheme $\mathcal{K}_r$ is the maximal subscheme over which $\mathcal{Z} \subset \mathcal{Y}$ $\square$

We denote by $m_0 : G \times G \to G$ the usual multiplication map $(a, b) \mapsto ab$.

Let $\mathcal{Z}_r$ be the pullback of the universal family over $\mathcal{G}_r$ to $\tilde{S}_r = \mathcal{K}_r \times G$. Let $\mathcal{Y}_r = \mathcal{K}_r \times m_0^{-1}X$. We denote by $\tilde{S}_r^0$ the maximal closed subscheme of $\tilde{S}_r$ over which $\mathcal{Z}_r \subset \mathcal{Y}_r$. The scheme $\tilde{S}_r^0$ parametrizes sequences $(G_1, \ldots, G_r, q)$ such that $G_i^{-1}G_i \subset G_r \subset q^{-1}X$ for all $i < r$.

We need some notation. The projections $f_{s,r} : \tilde{S}_s \to \tilde{S}_r$ and $\pi_r : \tilde{S}_r \to G$ induce *proper* maps $f_{s,r}^0 : \tilde{S}_s^0 \to \tilde{S}_r^0$ and $\pi_r^0 : \tilde{S}_r^0 \to X$. We write $S_r^0 = \pi_r^0 \tilde{S}_r^0$. Clearly S_r^0 is a closed subscheme of X. Let $T = \cap_r S_r^0$ and $Z = T_{\mathrm{red}}$. Similarly, let $\tilde{T}_r = \cap_{s \geq r} f_{s,r}^0(\tilde{S}_s^0)$. Again, $\tilde{T}_r$ is the intersection of closed subschemes, therefore a closed subscheme. Moreover, $f_{s,r}(\tilde{T}_s) = \tilde{T}_r$. From this follows:

Lemma 2. *For a closed point $q \in X$, we have: $q \in Z$ if and only if there is a sequence of closed points $\tilde{t}_r \in \tilde{T}_r$ such that $f_{s,r}(\tilde{t}_r) = \tilde{t}_r$ and $\pi_r(\tilde{t}_r) = q$. In other words, $q \in Z$ if and only if there is a nested sequence of subgroups $G_r \subset q^{-1}X$ supported at the origin such that the length: $l(G_r) = 2^r$.*

Let us complete the proof of our theorem. If $B \subset q^{-1}X$ is a positive dimensional subgroup then X contains a sequence of subschemes I_r of length 2^r for all r, supported at q, such that $I_r^{-1}I_r \subset I_{r+1}$. Therefore q is in S_r^0 for all r, and therefore in Z. Conversely, if $q \in Z$ then by the lemma there is a nested sequence of subschemes I_n supported at the origin such that $I_n^{-1}I_n \subset I_{n+1}$, all contained in $q^{-1}X$. By lemma 1 there is a positive dimensional subgroup inside $q^{-1}X$.

We need to deal with non-closed points. In general we can deal with points on X defined over an arbitrary field by interpreting the proof appropriately. The Hilbert schemes we used have the universal property for families of schemes of a certain type, and are proper over X. Therefore, if $x \in X$ is a point in the image of the Hilbert scheme, after a suitable field extension there will be a subscheme of the type parametrized by the Hilbert scheme, supported at x. Therefore, if x is non closed points in Z, it is in the image of a family of translated positive dimensional group-subschemes lying in X. Similarly, if the base field is not algebraically closed, then a closed point $x \in Z$ if and only if after a field extension there is a positive dimensional translate of a subgroup in X through x.

1.2. The second proof. We give now a different proof of theorem 1 in the case where $G = A$ is a semiabelian variety.

We will use the following lemma:

Lemma 3 (Hindry [13]**).** *(See also Faltings*[6] *or Neeman*[21].*) Let $V \subset A$ be a geometrically integral subvariety of a semiabelian variety. Let l be an integer prime to the characteristic. Assume that $l^k V = V$ for all k. Then V is a translate of a semiabelian subvariety of A.*

Remark. In charactersitic 0 this lemma follows from the theorems of Faltings and Vojta on the Mordell-Lang conjecture, since the graphs of multiplication by l^k give infinitely many points on V with values in a finitely generated field, namely the function field of V. On the other hand, this lemma (and theorem 1) play a role in the proof of the theorems [6, 27].

We now prove the theorem.

Pick a prime l different from the characteristic. We define the Faltings maps $F_m : A^m \to A^{m-1}$ by $(a_1, \dots, a_m) \mapsto (la_1 - a_2, la_2 - a_3, \dots, la_{m-1} - a_m)$. Notice that the fibers of F_m are isomorphic, via a_1, to A, and in fact the endomorphism (a_1, F_m) of A^m is an isomorphism. We look at F_m^X, defined to be F_m restricted to X^m. Notice that the projections onto the first factors induces an injections of the fibers of the restricted map: if $F_m(a_1, \dots a_m) = F_m(b_1, \dots b_m)$ then $F_{m-1}(a_1, \dots a_{m-1}) = F_{m-1}(b_1, \dots b_{m-1})$.

Define a map $D : X \to A^{m-1}$ via $D(x) = (l-1) \cdot (x, x, \dots, x)$. Define Y_m to be the pullback of F_m^X to X: $Y_m = \{(x_1, \dots, x_m, x) \in A^m \times X | F_m(x_1, \dots, x_m) = D(x)\}$. We have $Y_m \hookrightarrow A \times X$ via (x_1, x). Now Y_m give a descending sequence of subschemes of $A \times X$, which has to stabilize, say $Y_n = Y_{n+1} = \cdots$.

Take any component $B' \subset Y_m \cap A \times \{x\}$ of a fiber over some point $x \in X$ which passes through (x, x), and set $B = (B' - x)_{\mathrm{red}}$. Then we have $l^k B = B$. This is because every point of B' represents a sequence $(b_1, b_2, \dots)$ satisfying $lb_i - b_{i+1} = (l-1)x$, or $l(b_i - x) = b_{i+1} - x$. From the lemma we see that B is a semiabelian subvariety of A. Conversely, assume there is a translate of a semiabelian variety B, contained in X through x, and assume that it is maximal. Then it will appear as a component of the fiber at x.

To define $Z(X)$, take the locus V in Y_m where the fiber over X is positive dimensional, take the union U of those components in V which contain the diagonal $\{(x,x)\} \subset A \times X$ and take the image of U in X. This image is $Z(X)$. $\square$

1.3. A generalization and Theorem 2.

1.3.1. *T.* he first proof can be easily generalized to give the following:

Theorem 5. *Let $X \subset G$ as in theorem* 1, *and let $d \geq 0$ be an integer. Then the set*

$$Z_d = \cup\{xB | B \textit{ group variety, } \dim B \geq d, xB \subset X\}$$

is a closed subvariety of X.

To adjust the proof, one simply needs to add closed conditions requiring the schemes I_n to have large enough intersection with the infinitesimal neighborhoods of the origin, which can be done using theorem 6. A similar result can be achieved using the line of the second proof.

1.3.2. ***Proving theorem 2.*** The second proof gives theorem 2 readily. For the proof simply notice that in case $Z = X$, the component of Y_m dominates X, and has as reduced fibers translated semi abelian varieties, which have to be translates of a constant variety, since there are no families of semi abelian subvarieties. One can use the first proof as well: since the generic point of X is contained in Z, we have that X is covered by a family of positive dimensional translated subgroups, whose reduced fibers are again translates of a constant subgroup.

2. The Kodaira dimension.

The results are stated and proved in the algebraic language, the analytic case being an easy modification.

2.1. Some definitions. Again, let A be a semi abelian variety over an algebraically closed field. Let $X \subset A$ be a closed, reduced, irreducible subvariety.

2.1.1. *S.* uppose $B = Stab(X)$ is the maximal subgroup such that $B + X = X$. Let

$$\pi : X \to X_0 = X/B$$

be the quotient map.

Provisional definition 1. *We define the **"abelian Kodaira dimension"***

$$\kappa_a(X) = \dim X_0 \ (\ = \dim X - \dim B).$$

The abelian Kodaira dimension will be a natural upper bound for the logarithmic Kodaira dimension $\bar\kappa(X)$.

2.1.2. *I.* n order to give a lower bound on $\bar\kappa(X)$, we produce rational maps of X defined using jets.

Let $\Delta_m = \Delta_m^A \subset A \times A$ be the subscheme defined by the ideal I_Δ^{m+1} where Δ is the diagonal in $A \times A$. Let $g : A \times A \to A \times A$ be the map defined by $(a, b) \mapsto (a, a + b)$. Let A_m be defined by $A_m = g^{-1}\Delta_m$. A_m is simply the m-th order neighborhood of the zero section $A \times \{0\}$ in $A \times A$, that is, the product $A \times \operatorname{Spec} \mathcal{O}_{A,0}/\mathfrak{m}^{m+1}$, where $\mathfrak{m}$ is the maximal ideal of 0. Let A_m^X be the restriction of this product to X.

Let $\Delta_m^X \subset \Delta_m$ be the subvariety defined by $I_\Delta^{m+1} + I_{X\times X}$. Let $X_m = g^{-1}\Delta_m^X$. Clearly $X_m \subset A_m^X$. Let $\pi_1 : A_m^X \to X$ be the projection.

Let $\mathcal{F}_m = \pi_{1*}\mathcal{O}_{A_m^X}$ and let $\mathcal{G}_m = \pi_{1*}\mathcal{O}_{X_m}$. The sheaf $\mathcal{F}_m$, which is sometimes called the sheaf of m-jets of A along X, or the sheaf of m-th order principal parts of A along X, is free of a certain rank N_m. Moreover, on the smooth locus X_{sm} the sheaf $\mathcal{G}_m$ (the sheaf of m-jets of X) is locally free of some rank L_m.

We denote by $G = \mathbf{Gr}(L_m, N_m)$ the grassmannian of L_m-dimensional planes in an N_m space. The surjection of sheaves $\mathcal{F}_m \to \mathcal{G}_m$ defines, by the universal property of grassmannians, maps $f_m : X_{sm} \to \mathbf{Gr}(L_m, N_m) = G$. In effect, the map f_m sends a point x in X to the point on the punctual Hilbert scheme corresponding to the m-th infinitesimal neighborhood of x in X, translated back to the origin. This Hilbert scheme sits naturally in the Grassmannian G.

Notice that via the surjections $\mathcal{G}_m \to \mathcal{G}_{m-1}$ we have $\mathcal{G}_{m-1} = \mathcal{G}_m \otimes_{\mathcal{F}_m} \mathcal{F}_{m-1}$ (since $X_m \cap A_{m-1} = X_{m-1}$). This induces surjections $G_m \to G_{m-1}$ compatible with the maps f_m. Therefore the fibers of f_m inject in the fibers of f_{m-1}. To be precise, the sequence of closed subvarieties

$$Y_m = X_{sm} \times_{f_m} X_{sm} \subset X_{sm} \times X_{sm}$$

is decreasing. Since we are dealing with noetherian schemes, this sequence stabilizes. From now on, let m be chosen such that $Y_m = Y_{m+k}$ for $k \geq 0$.

Provisional definition 2. *We define the **"crude Kodaira dimension"***

$$\kappa_c(X) = \dim f_m(X_{sm}).$$

The crude Kodaira dimension will be a natural lower bound for $\bar{\kappa}(X)$.

2.1.3. *W.* e now compare our two provisional definitions.

Lemma 4. *The map f_m factors as $f_m^0 \circ \pi$, where f_m^0 is birational.*

Proof. The set theoretic idea is as follows: let $x_1 \in X_{sm}$ and $x_2 \in X_{sm}$ be two smooth points on X such that $f_m(x_1) = f_m(x_2)$. Denote by t_x the translation by x. We denote by $I_{Y,Z}$ the ideal of a subscheme Y in a scheme Z. By definition of the maps and the choice of m, $t^*_{(x_1-x_2)}(I_{x_1,X} + I^{k+1}_{x_1,A}) = I_{x_2,X} + I^{k+1}_{x_2,A}$, for all k. This means that the translation by $x_1 - x_2$ gives an isomorphism of formal subschemes $\widehat{X}_{x_1} \subset \widehat{A}_{x_1}$ with $\widehat{X}_{x_2} \subset \widehat{A}_{x_2}$. Since the completion is faithfully flat over the localization, this means that $t_{(x_1-x_2)}$ gives an isomorphism of the local schemes X_{x_1} and X_{x_2}. Since X is irreducible, $t_{(x_1-x_2)}$ gives an automorphism of X, and therefore $x_1 - x_2$ lies in B.

In order to complete the proof scheme theoretically, we replace x_1 and x_2 by a subscheme Y in a fiber of f_m, and $y \in Y$ a closed point. We have $Y \times X \subset Y \times A$, we have the diagonal $Y \simeq \Gamma \subset Y \times X$, and we have the morphism $g : Y \times A \to Y \times A$ defined by $(y, a) \mapsto (y, a - y)$. By the definition of f_m and the choice of m, we have that

$$g^*(I_{Y\times y, Y\times X} + I^{k+1}_{Y\times y, Y\times A}) = (I_{\Gamma, Y\times X} + I^{k+1}_{\Gamma, Y\times A})$$

for all k. Therefore g identifies the formal completion of $Y \times X$ along Γ with the completion along $Y \times y$. Again, by faithful flatness and irreducibility Y lies in the stabilizer of X. □

Corollary 1. *We have the equality*

$$\kappa_a(X) = \kappa_c(X).$$

2.1.4. *W.* e now look at the logarithmic Kodaira dimension. There is a slight difficulty with the notion of logarithmic Kodaira dimension of nonsmooth varieties, when the existence of a resolution of singularities is not known. As we will see later, in our situation any reasonable definition will give the same answer. However, we propose the following (ad hoc) definition (which coincides in our case with a more general definition of Luo [18, 19]):

Definition 1. *Let Y be an integral scheme, and $K = K(Y)$. An element ω of $\Omega(Y_{sm})$ (similarly, any tensor) is said to be **absolutely logarithmic** if the following holds:*

(1) *If $v : K^* \to \mathbf{Z}$ is a divisorial valuation of K whose center lies over Y, then ω is regular at v, that is, $\omega = \sum f_i dz_i$ where $v(f_i) \geq 0$ and $v(z_i) \geq 0$.*

(2) *If $v : K^* \to \mathbf{Z}$ is a divisorial valuation of K whose center does not lie over Y, then $\omega = f_0 du/u + \sum f_i dz_i$ where $v(f_i) \geq 0$, $v(z_i) \geq 0$ and $v(u) > 0$.*

In case Y is complete, the element ω satisfying the first condition is said to be absolutely regular.

Remark. On a scheme which has a smooth completion this notion coincides with the usual logarithmic forms.

Definition 2. *We define $\bar{\kappa}(X)$ as in the smoothable case, using absolutely logarithmic elements $K^m_{\rm reg} \subset \det \Omega(X_{sm})^{\otimes m}$: take the transcendence degree of the ring $\oplus K^m_{\rm reg}$ minus one, and set it to $-\infty$ if it comes out as -1.*

Remark. For a space whose completion admits a resolution of singularities, this coincides with the usual definition.

Lemma 5. *Given a morphism $f : X \to A$ such that A has a smooth completion, any logarithmic form (or tensor) on A pulls back to an absolutely logarithmic element of $\Omega(K)$ (or the corresponding tensors).*

Proof: The pull back of the form $f^*\omega$ to the spectrum a valuation is the same as the pullback all the way from A. Let v be a valuation on X and let w be the image of the center of v in $\overline{A}$. If $w \in A$ then $\omega = \sum f_i dz_i$ where f_i, z_i are regular, and if w lies in the boundary of $\overline{A}$ then $\omega = f_0 du/u + \sum f_i dz_i$, and u vanishes at w. Pulling back f_i, z_i and u we get the result. $\square$

2.2. Proof of theorem 3. We will show the equality

$$\kappa_a(X) = \kappa_c(X) = \bar{\kappa}(X).$$

Let again $B = Stab(X)$ and let B_0 be the reduced connected component of the identity in B.

It is important that that every invariant tensor on A is logarithmic (since the invariant forms on G_m are of the type $c\frac{du}{u}$).

Notice that B acts on the space V of absolutely logarithmic forms on X. If B is complete, the action of B_0 is trivial, since the image of a complete variety in an affine group is finite. Therefore any pluri-log-canonical map factors through X/B_0. So we only need to worry about the multiplicative part T_0 of B. This we do (alas) using an explicit computation. Decompose V into weight spaces under T_0, and let ω be an element of a weight space. We need to show that the character of the action of T_0 on the space spanned by ω is 1. Let $G \subset B$ be a one parameter subgroup. Let $x \in X$ be a smooth point where ω does not vanish. Let $\eta \in V$ be an invariant form which does not vanish at x. We can write $\omega = f\eta$ for some rational function f on X. We write

Gx for the orbit of x under G and by abuse of notation $0x$ and ∞x are the endpoints of the orbit in $\overline{X}$. Now $X \to X/G$ makes X into an open set in a $\mathbf{P}^1$ bundle, so we can write $f = \sum g_i u^i$, where g_i are functions on X/G regular and nonvanishing at the inage of x, and u is a rational function which restricts to a parameter of G along the orbit Gx. We now check the valuation of ω at $0x$ and ∞x. The valuation of η at both points is already -1, and that of g_i is 0. So in the sum we must have only $i = 0$, therefore the weight of ω is 1, and it is invariant. This gives us the inequality $\kappa_a(X) \geq \bar{\kappa}(X)$

Now to the other direction. The map to $f_m : X \to G$ followed by the Plücker embedding is given by the determinant of the pullback of the space of invariant jets on A, namely by the pullback of a space of invariant tensors on A, which are absolutely logarithmic on X. Moreover, this determinant is certainly pluricanonical on X_{sm}. We get an inequality $\kappa_c(X) \leq \bar{\kappa}(X)$. $\square$

Remark. A similar construction of jet Gauss maps can be defined for arbitrary groups and homogeneous spaces. One can obtain results of the following flavor:

Let $X \subset G/H$ be a reduced irreducible complex subspace of a homogeneous space of a complex group G, where H is a complex subgroup of G. Assume that X is not covered by homogeneous spaces of complex subgroups of G. Then the global meromorphic functions on X separate smooth points on X. In particular, when X is compact, it is Moisheson. This is related to some unpublished results of Bogomolov.

2.2.1. ***Some arithmetic.*** Let $X \subset A$ be a geometrically integral subvariety of a semiabelian variety over an arbitrary field K. A regrettable feature of positive characteristic is, that pluricanonical maps do not commute with field extensions. Examples already exist for curves, where a curve which is not absolutely normal might have less absolutely regular differentials than the curve after field extensions. On the other hand our maps f_m may be defined over an arbitrary field of definition, and clearly commutes with field extension, once we choose a K basis for $O_A/I_{0,A}^m$. In case $Stab(X)$ is trivial, if we fix m so that f_m is birational, we can associate to X the point in the Hilbert scheme corresponding to its birational image by f_m, and this point is defined over any field of definition of $X \subset A$. Moreover, f_m is invariant under translations by definition, so the Hilbert point is a translation invariant of subvarieties of A. This invariant proves to be useful in [3].

Another useful feature of the maps f_m is their behavior when A is replaced by a dominating A_1. Suppose $\pi : A_1 \to A$ is a homomorphism of semiabelian varieties, and $X_1 \subset A_1$ such that $\pi(X_1) = X$ and π is a birational equivalence of X and X_1. Then the linear series associated with $f_m^{X_1}$ contains that of f_m^X. The reason is as follows: Let U be an open set of the smooth locus on X_1 where π is an isomorphism.

Then we have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}_m^X & \to & \mathcal{F}_m^{X_1} \\ \downarrow & & \downarrow \\ \mathcal{G}_m^X & \to & \mathcal{G}_m^{X_1} \end{array}$$

where the vertical arrows are surjections and over U the bottom map is an isomorphism. Since the diagram is commutative, over U the kernel of the top map is in the kernel of the left vertical map, and it does not contribute to the linear series (f_m^X maps X into a linear subspace, which is dominated by the space of $f_m^{X_1}$).

3. The Gauss map

In this section we assume that A is an abelian variety. We also assume that $X \subset A$ is smooth, and not swept by translates of abelian varieties, that is, the stabilizer B is finite. We start with the following:

Lemma 6. *The canonical sheaf of X is ample.*

Proof: We have shown in lemma 4 that the map f_m is finite for large m. This map was defined by a base point free pluricanonical linear system. $\square$

Proof of theorem 4: Again, the map f_1 is defined by a base point free canonical linear system, which by the lemma cannot collapse any curve. $\square$

Remark. In the theorem, one can relax the conditions and show that when the singular locus of X is of dimension d, the generic fiber of the Gauss map has dimension at most $d+1$. Assume the contrary, take a fiber of the Gauss map in the smooth locus of X such that its closure does not contain singular points of X in codimension 1, and pick a complete curve C in this fiber missing the singularities of X. On this curve f_m is ample, therefore $K_X|_C$ is ample. Since the base points of the Gauss map are contained in the singular locus of X, we have a contradiction.

Corollary 2. *Suppose now that $X \subset A$ is of codimension 1. Then X does not admit a global vector field.*

Proof: Since the Gauss map $F_1 : X \to \mathbf{P}T_0(A)$ is finite and since

$$\dim \mathbf{P}T_0(A) = \dim A - 1 = \dim X,$$

the map is surjective. But if X had a vector field, this vector field would definitely be invariant, and the Gauss map would land in the hyperplane parametrizing quotients of $T_0(A)$ annihilating this vector, which is a contradiction.

Example. This theorem is not true if X has bad singularities. Let $A = E \times E$ be a product of elliptic curves defined over a field of characteristic p. Let $Y \subset A$ be a curve which is generically étale over the first factor, of genus greater than 1. Let

$g : A \to A_1$ be given by $(x, y) \mapsto (x, Fr(y))$. Here Fr stands for the Frobenius, and $A_1 = E \times Fr(E)$. Let $X = g(Y)$. The curve X is now "horizontal": it admits an infinitesimal automorphism (vector field) which is induced by the action of a group - scheme H as follows: let $h : A_1 \to A_2$ be defined by $(x, y) \mapsto (Fr(x), y)$, where $A_2 = Fr(E) \times Fr(E)$ and let $H = ker(h)$.

Question. Let $X \subset A$ be regular (or regular in codimnsion 1), and assume that X admits a vector field. Does it follow that X is stabilized by an abelian variety?

Appendix A. Intersection of families

Let $\mathcal{X} \to B; \mathcal{Y} \to B; \mathcal{Z} \to B$ be three schemes (or complex spaces), and let $i_1 : \mathcal{Y} \to \mathcal{X}; i_2 : \mathcal{Z} \to \mathcal{X}$ be closed embeddings over B. We would like to know whether we can find a subscheme S of B which "parametrizes" points where $\mathcal{Z}|_S \subset \mathcal{Y}|_S$, in the sense of having a universal property:

Definition 3. *Let S be a closed subscheme of B. We say that S is the* ***maximal closed subscheme over which*** $\mathcal{Z} \subset \mathcal{Y}$ *if:*

(1) *$\mathcal{Z} \times_B S \subset \mathcal{Y} \times_B S$, and*
(2) *Whenever $f : S_1 \to B$ is a morphism such that $\mathcal{Z} \times_B S_1 \subset \mathcal{Y} \times_B S_1$, then f factors through the inclusion $S \subset B$.*

For the definition of S, note that we can first intersect: $\mathcal{T} = \mathcal{Y} \cap \mathcal{Z}$ and look for the locus where $\mathcal{T}$ equals $\mathcal{Z}$ itself. Since the object on which we need to make assumptions is $\mathcal{Z}$, it is convenient to make the following definition:

Definition 4. *We say that a map $\mathcal{Z} \to B$ is* ***strictly faithful*** *if for any closed subscheme $\mathcal{T} \subset \mathcal{Z}$, the maximal closed subscheme over which $\mathcal{Z} = \mathcal{T}$ exists.*

One condition which may help, is that $\mathcal{Z}$ be flat. However, this is not enough, as one sees from the following example:

Let K be a field, and let $B = \mathrm{Spec}(K[x])$. Let $\mathcal{Z} = \mathrm{Spec}(K[x, y]/(xy - 1))$ and $\mathcal{T} = \emptyset$. One does not have a maximal closed subscheme supported at 0 over which $\mathcal{Z} \subset \mathcal{T}$: both $\mathcal{T}$ and $\mathcal{Z}$ are empty over any scheme supported at 0.

Flatness is not even necessary, as can be seen by examples over zero dimensional base. E.g.: Let $B = \mathrm{Spec}(K(X)/(x^2))$ and $\mathcal{Z} = \mathrm{Spec}(K[x, y]/(x^2, xy))$. the image of any scheme S_1 in B is closed, and the union of subschemes stabilized after finitely many steps. However, if we take $\mathcal{T} = \mathrm{Spec}(K(x, y)/(x^2, y))$, there is no subscheme of B over which $\mathcal{T}$ takes the whole fiber, but thinking of B as an "arrow", we would like to say that "over the tip of the arrow, $\mathcal{T}$ takes the whole fiber". Basically, we would like to avoid unnecessary torsion, and we will consider only flat families from now on. Assuming flatness, it is easy to see that every primary component of $\mathcal{Z}$ surjects onto a primary component in B. This is why we chose the name "strictly faithful".

We now look for conditions for $\mathcal{Z} \to B$ to be strictly faithful.

Theorem 6. *Let $\mathcal{Z} \to B$ be a flat morphism of finite type of noetherian schemes (or complex spaces). Then in the following cases $\mathcal{Z} \to B$ is strictly faithful:*

(1) *$\mathcal{Z} \to B$ is finite;*
(2) *B is of finite type over a field k and $\mathcal{Z} \cong Z \times_k B$ for some k scheme Z;*
(3) *$\mathcal{Z} \to B$ is projective.*

Proof: In the algebraic case, this follows from [9, 10], since the morphisms in question are "essentially free"[1]. In cases (2) and (3) this is not clear in the complex case, but we can avoid it by reducing the product case and the projective case to the finite case as follows:

For case (2): every function on Z is determined by its values on all infinitesimal neighborhoods of all the points, that is, by the collections of its values in all the completed stalks. We now take the intersection

$$\cap\{S_{p,m} | p \in Z, m \geq 1\}$$

where $S_{p,m}$ in the maximal subscheme of B over which $\mathrm{Spec}(\mathcal{O}_Z/\mathfrak{m}_p^m) \times B$ is contained in T.

For case (3): we would like to use the affine cone $C = C(\mathcal{Z})$ over $\mathcal{Z}$, but a-priori this is not even flat. However, by the theory of Hilbert schemes, there is an n_0 such that the $n-th$ graded piece of the homogeneous ideal, I_n, generates I_{n+1} and I_n is locally free over the base for all $n \geq n_0$. This means that $(C \cup V_{n_0}) \cap V_n$ is finite and flat, where V_n is the n-th infinitesimal neighborhood of the vertex of the cone C. Replacing $\mathcal{T}$ by $(C(\mathcal{T}) \cup V_{n_0}) \cap V_n$ for all n we reduce to the finite case $\square$

We conclude by showing that strict faithfulness is equivalent to the commuting of Zariski closure and pullback:

Theorem 7. *Let $f : Z \to B$ be a morphism of finite type. Then f is strictly faithful if and only if for every collection of morphisms $g_\alpha : S_\alpha \subset B$ we have*

$$f^{-1}\overline{\cup Im(g_\alpha)} = \overline{\cup f^{-1} Im(g_\alpha)},$$

where the bar stands for Zarisky closure and Im stands for the scheme theoretic closure of the image.

[1] In [1] I gave the rather non-geometric condition "coherently locally free". Thanks to J.-F. Burnol for noting that Grothendieck preceded me by 30 years with an equally non-geometric condition. The two conditions are essentially coherent with each other.

Proof. If f is strictly faithful, let $\mathcal{T} = \overline{\cup f^{-1} Im(g_\alpha)}$. Then the maximal closed subscheme over which $Z = \mathcal{T}$ contains all the image of S_α, and therefore equals $\overline{\cup Im(g_\alpha)}$. Conversely: given $\mathcal{T}$, let S_α be the collection of all scheme over which $Z = \mathcal{T}$. Since T is closed in Z we have

$$Im(g_\alpha) \times_B Z = Im(g_\alpha) \times_B \mathcal{T},$$

and taking Zariski closure of the unions we obtain the result.

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