COMPLETE MODULI FOR FIBERED SURFACES

Preliminary version

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1. Introduction

All the schemes with which we work will be schemes over the field $\mathbf{Q}$ of rational numbers.[2]

1.1. The problem.

1.1. The problem. Fix a base field k of characteristic 0, and let C be a smooth, projective, geometrically integral curve of genus g. By a *fibered surface over* C we mean a morphism $X \to C$, with sections $\sigma_1, \ldots, \sigma_\nu : C \to X$ forming a family of stable ν-pointed curves of some genus γ. (This notion will be generalized below for singular C.)

A fibered surface naturally corresponds to a morphism $C \to \overline{\mathcal{M}}_{\gamma,\nu}$, into the moduli stack of stable ν-pointed curves of genus γ. In [Vistoli2], the second author showed that $Hom(C, \overline{\mathcal{M}}_{\gamma,\nu})$ has the structure of a Deligne - Mumford stack. This stack is clearly not complete in general. A natural question to ask is, can one complete it in a meaningful way? Moreover, can one do this as the source curve C itself moves in a family?

1.2. Our approach. One construction which may spring to mind is that of *stable maps.* Recall that if $Y \subset \mathbf{P}^r$ is a projective variety, there is a complete Deligne - Mumford stack $\overline{\mathcal{M}}_g(Y, d)$ parametrizing Kontsevich stable maps of degree d, from curves of genus g to Y (see [Kont], [B-M], [F-P]). In order to reduce the possibility of later confusion, we will use the notation $\mathcal{K}_g^d(Y)$ rater than $\overline{\mathcal{M}}_g(Y, d)$ for this stack. It admits a projective coarse moduli space $\mathbf{K}_g^d(Y)$.

For instance, if $\gamma = 0$, the moduli stack $\overline{\mathcal{M}}_{0,\nu}$ is actually a projective variety. Fixing a projective embedding, we have natural complete Deligne - Mumford stacks $\mathcal{K}_g^d(\overline{\mathcal{M}}_{0,\nu})$ parametrizing families of stable ν-pointed curves of genus 0 over nodal curves of genus g, with a suitable stability condition. We can think of d, the degree of the image of C via the fixed projective embedding, as an additional measure of complexity of the family of curves. One is led to ask, is there a complete Deligne - Mumford stack of stable maps $\mathcal{K}_g^d(\overline{\mathcal{M}}_{\gamma,\nu})$ in general?

We need to define our terms: given a nodal curve C, define a morphism $f : C \to \overline{\mathcal{M}}_{\gamma,\nu}$ to be a stable map of degree d, if once projected into the coarse moduli space, we obtain a stable map $f' : C \to \overline{\mathbf{M}}_{\gamma,\nu}$ of degree d:

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[2]The theory below could be extended to the case that the characteristic is large with respect to the degrees and genera of the curves involved, but we will not pursue this here.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[dr, "f'"'] & \overline{\mathcal{M}}_{\gamma,\nu} \arrow[d] \\ & \overline{\mathbf{M}}_{\gamma,\nu} \end{tikzcd}$$

As we will see later, these stable maps are parametrized by a Deligne - Mumford stack. However, this stack fails to perform its first goal: it is not always complete! This can be seen via the following example:

Consider the case $\gamma = 1, \nu = 1$ of elliptic curves. It is well known (and easily follows from the formula of Grothendieck - Riemann - Roch) that given a family of stable elliptic curves $E \to C$ over a curve C, the degree of the corresponding moduli map $j_E : C \to \overline{\mathbf{M}}_{1,1} \simeq \mathbf{P}^1$ is divisible by 12. However, it is very easy to construct a family of moduli maps $j_t : C_t \to \mathbf{P}^1, \quad t \neq 0$, where when $t \to 0$ the curve C_t breaks in two components, and the degrees of the limit map j_0 on these components are not divisible by 12. Thus j_0 is not a moduli map of a family of stable curves.

1.3. **Compactifying the space of stable maps.** Our main goal in this paper is to correct this deficiency. In order to do so, we will enlarge the category of stable maps into $\overline{\mathcal{M}}_{\gamma,\nu}$. The source curve $\mathcal{C}$ of a new stable map $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ will acquire an orbispace structure at its nodes. Specifically, we endow it with the structure of a Deligne-Mumford stack.

To see how these come about, consider again the example of a one-parameter family of elliptic curves sketched above. Let $C \to S$ be the one-parameter family of base curves, and let C_{sm} be the smooth locus of this morphism. A fundamental purity lemma (see 2.1 below) will show, that after a suitable base change we can extend the family E of stable elliptic curves over C_{sm}. On the other hand, if $p \in C$ is a node, then on an étale neighborhood U of p, the curve C looks like

$$uv = t^r,$$

where t is the parameter on the base. By taking r-th roots:

$$u = u_1^r;\ v = v_1^r$$

we have a nonsingular covering $V_0 \to U$ where $u_1 v_1 = t$. The fundamental purity lemma applies to V_0, so the pullback of E to V_0 extends over all of V_0. There is a minimal intermediate covering $V_0 \to V \to U$ such that the family extends already over V. This V gives the orbispace structure $\mathcal{C}$ over C.

The precise definitions will follow in section 4. We can define stable maps $f : \mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ where $\mathcal{C}$ is a Deligne-Mumford stack and f is a representable morphism satisfying certain properties; such objects naturally form a 2-category. Our main theorem states that this 2-category is equivalent to a 1-category $\mathcal{F}_g^d(\gamma,\nu)$, which we call *the category of fibered surfaces*; this category forms a complete Deligne-Mumford stack (over schemes over $\mathbf{Q}$), admitting a projective coarse moduli space. Thus the original moduli problem has a solution of the same nature as that of stable curves.

We provide an explicit description of the category $\mathcal{F}_g^d(\gamma,\nu)$ of fibered surfaces in terms of charts and atlases over schemes, in analogy to Mumford's treatment of $\mathbf{Q}$-varieties in [Mum]. Thus given a stable map $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ we have a stable pointed curve $\mathcal{X} \to \mathcal{C}$, which can be described via an atlas of charts over the associated coarse moduli scheme $X \to C$.

1.4. **Comparison with Alexeev's work.** The latter, coarse object $X \to C$, which we call a *coarse fibered surface*, has another interpretaion: the associated morphism $X \to \overline{\mathbf{M}}_{\gamma,\nu}$ turns out to be a stable map in the sense of Alexeev (see [Al3]). Alexeev has shown the existence of complete moduli of (smoothable) surface stable maps, using the theory of semi-log-canonical surfaces developed in [K-SB] and [Al2]. In the last section we show that in the case of fibered surfaces, Alexeev's approach coincides with the space of (smoothable) coarse fibered surfaces. We sketch a proof of the existence of the latter space (even as a stack) which is independent of Alexeev's boundedness result.

1.5. **Natural generalizations in forthcoming work.** It should be evident from our work that this approach should apply to stable maps $\mathcal{C} \to \overline{\mathcal{M}}$ into any Deligne-Mumford stack which admits a projective coarse moduli scheme; moreover, the source "curve" $\mathcal{C}$ should be allowed to be "pointed" as well. We are now in the process of writing up the necessary details, circumventing the various shortcuts we have taken in this paper. Various applications promise to be of interest; e.g. the case $\overline{\mathcal{M}} = B\mathfrak{S}_n$, the classifying space of the symmetric group on n-letters, provides us with a natural desingularization of the moduli stack of admissible covers (whose coarse moduli scheme was introduced by Harris and Mumford [H-M]; see [Mo1] for a rigorous construction using logarithmic structures). Also, the construction of stable maps is amenable to an inductive approach, allowing us to construct a moduli space of "plurifibered varieties", giving complete moduli in arbitrary dimension for varieties with "plurifibrations". This is closely related to work of Mochizuki [Mo2, Mo3], where the existence of level structure is assumed.

1.6. **Gromov - Witten invariants.** Originally, the Kontsevich spaces of stable maps were introduced for the purpose of defining Gromov - Witten invariants. It seems likely that, using our construction, one could extend the formalism of [BF] and [B] of virtual fundamental classes to the case of stable maps into an arbitrary Deligne Mumford stack $\overline{\mathcal{M}}$ admitting a projective coarse moduli space. This should allow one to define Gromov - Witten invariants in this generality, which may have interesting application for specific choices of stacks $\overline{\mathcal{M}}$.

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2. The purity lemma

There are several results in the litterature, which give conditions under which a family of curves can fail to be stable only in pure codimension 1 (see [MB], [dJ-O]). For our purposes, the following case will be most useful:

2.1. **Purity Lemma.** *Let $\overline{\mathcal{M}}$ be a separated Deligne - Mumford stack, $\overline{\mathcal{M}} \to \overline{\mathbf{M}}$ the coarse moduli space. Let X be a separated scheme of dimension 2 satisfying Serre's condition S_2. Let $P \subset X$ be a finite subset consisting of closed points, $U = X \smallsetminus P$. Assume that the local fundamental groups of U around the points of P are trivial.*

Let $f : X \to \overline{\mathbf{M}}$ be a morphism. Suppose there is a lifting $\tilde{f}_U : U \to \overline{\mathcal{M}}$:

$$\begin{tikzcd} & & \overline{\mathcal{M}} \arrow[d] \\ U \arrow[r] \arrow[urr, "\tilde{f}_U"] & X \arrow[r, "f"] & \overline{\mathbf{M}} \end{tikzcd} \tag{1}$$

Then the lifting extends uniquely to X:

$$\begin{tikzcd} & & \overline{\mathcal{M}} \arrow[d] \\ U \arrow[r] \arrow[urr, "\tilde{f}_U"] & X \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & \overline{\mathbf{M}} \end{tikzcd} \tag{2}$$

Proof. By descent theory the problem is local in the étale topology, so we may replace X and $\overline{\mathbf{M}}$ with the spectra of their strict henselizations at a geometric point; then we can also assume that we have a universal deformation space $V \to \overline{\mathcal{M}}$ which is *finite*. Now U is the complement of the closed point, U maps to $\overline{\mathcal{M}}$, and the pullback of V to U is finite and étale, so it has a section, because U is simply connected; consider the corresponding map $U \to V$. Let Y be the scheme-theoretic closure of the graph of this map in $X \times_{\overline{\mathbf{M}}} V$. Then $Y \to X$ is finite and is an isomorphism on U. Since X satisfies S_2, the morphism $Y \to X$ is an isomorphism. ♣

2.2. **Corollary.** *Let X be a smooth surface, $p \in X$ a closed point with complement U. Let $X \to \overline{\mathbf{M}}$ and $U \to \overline{\mathcal{M}}$ be as in the purity lemma. Then there is a unique lifting $X \to \overline{\mathcal{M}}$.*

2.3. **Corollary.** *Let X be a normal crossings surface, namely a surface which is étale locally isomorphic to* Spec $k[u, v, t]/(uv)$. *Let $p \in X$ a closed point with complement U. Let $X \to \overline{\mathbf{M}}$ and $U \to \overline{\mathcal{M}}$ be as in the purity lemma. Then there is a unique lifting $X \to \overline{\mathcal{M}}$.*

Proof. In both cases X satisfies condition S_2 and the local fundamental group around p is trivial, hence the purity lemma applies. ♣

3. Group actions on nodal curves

Fix two nonnegative integers γ and ν. An important ingredient in the theory will be formed of data as follows:

1. a diagram

$$\begin{array}{c} Y \\ \downarrow \uparrow \\ V \\ \downarrow \\ S \end{array}$$

(we think of Y and V as S-schemes);

2. the morphism $\rho\colon Y \to V$ comes with sections $\tau_1\colon V \to Y, \dots, \tau_\nu\colon V \to Y$, forming a family of stable ν-pointed curves of genus γ on V.

Given a finite group Γ, by *an action of* Γ *on* ρ we mean a pair of actions of Γ on V and Y as S-schemes such that the morphisms $\rho, \tau_1, \dots, \tau_\nu$ are Γ-equivariant. Such an action induces a morphism $Y/\Gamma \to V/\Gamma$ together with sections $V/\Gamma \to Y/\Gamma$.

We adopt the convention that all families of curves are assumed to be *of finite presentation* over the base.

3.1. **Definition.** Let $\rho\colon Y \to V$ be a family of ν-pointed curves of genus γ, Γ a finite group. An action of Γ on ρ is called *essential* if no nontrivial element of Γ leaves a geometric fiber of ρ fixed.

Another way to state this condition is to require that if γ is an element of Γ leaving a geometric point v_0 of V fixed, then γ acts nontrivially on the fiber of ρ on v_0.

The following lemma will help us replace an action of Γ on ρ by an essential action on a related family of curves.

3.2. **Lemma.** *Let V be the spectrum of a local ring R, $Y \to V$ a flat family of nodal curves, Γ_0 a group acting compatibly on Y and V. Suppose that the action of Γ_0 on the residue field k of R and on the fiber Y_0 of Y over* Spec k *is trivial. Then $Y/\Gamma_0 \to V/\Gamma_0$ is again a flat family of nodal curves.*

3.3. **Remark.** It is easy to give an example showing that this fails in positive characteristic.

Proof. We need to show that the map $\rho\colon Y/\Gamma_0 \to V/\Gamma_0$ is flat, and the natural map $Y_0 \to \rho^{-1}(\operatorname{Spec} k)$ is an isomorphism. Replacing R with the completion of its strict henselization we may assume that R is complete and k is algebraically closed. Choose a rational point $p \in X(k)$, and let M be the completion of the local ring of Y at p. Write $M \otimes_R k$ as a quotient of a power series algebra $k[[t_1, \dots, t_r]] = k[[\mathbf{t}]]$, and lift the images of the t_i in $M \otimes_R k$ to invariant elements in M. We get a surjective homomorphism $R[[\mathbf{t}]] \to M$ which is equivarinant, if we let Γ_0 act on $R[[\mathbf{t}]]$ leaving the t_i fixed. Then the result follows from the lemma below.

3.4. **Lemma.** *Let M be a finitely generated $R[[\mathbf{t}]]$-module which is flat over R, such that Γ_0 acts trivially on $M \otimes_R k$. Then M^{Γ_0} is flat over R^{Γ_0}, and the natural homomorphism $M^{\Gamma_0} \otimes_{R^{\Gamma_0}} R \to M$ is an isomorphism.*

Proof. We have $R[[\mathbf{t}]]^{\Gamma_0} = R^{\Gamma_0}[[\mathbf{t}]]$, and from this we see that the statement holds when M is free over $R[[\mathbf{t}]]$. In general, take a finite set of generators of the finite $k[[\mathbf{t}]]$-module $M \otimes_R k$ and lift them to a set of invariant generators of M. We obtain an equivariant surjective homomorphism $F \to M$, where F is a free $R[[t]]$-module. Let K be the kernel:

$$0 \to K \to F \to M \to 0.$$

Note that by the finite presentation assumption on $Y \to V$, K is finitely generated. Since M is flat we get $Tor_i(K, \cdot) = Tor_{i+1}(M, \cdot) = 0$, thus K is also flat. Moreover, from the exact sequence

$$0 \to K \otimes_R k \to F \otimes_R k \to M \otimes_R k \to 0$$

we get that Γ_0 acts trivially on $K \otimes_R k$. We obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & K^{\Gamma_0} \otimes_{R^{\Gamma_0}} R & \longrightarrow & F^{\Gamma_0} \otimes_{R^{\Gamma_0}} R & \longrightarrow & M^{\Gamma_0} \otimes_{R^{\Gamma_0}} R & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

where the middle column is an isomorphism. It follows that the right column is surjective; since K is also flat and Γ_0 acts trivially on $K \otimes_R k$ we can apply the same argument to K, and it follows that the left column is also surjective. Therefore the right column is also injective, as desired.

Applying the same argument to the module K, we have that the left column is also an isomorphism, and the first arrow in the top row is injective; the flatness of M^{Γ_0} over R^{Γ_0} then follows from Grothendieck's local criterion of flatness. ♣

The following lemma will help us identify a situation where an essential action is free:

3.5. **Lemma.** *Let G be a group of automorphisms of a stable ν-pointed curve X of genus γ over an algebraically closed field, and assume that the quotient X/G is also a stable ν-pointed curve of genus γ. Then G is trivial.*

Proof. If C is a complete curve we denote by $\mathrm{CH}^i(C)$ the Chow group of classes of cycles of codimension i on C; so $\mathrm{CH}^0(C)$ is a direct sum of n copies of $\mathbf{Z}$, where n is the number of irreducible components of C, while $\mathrm{CH}^1(C)$ is the divisor class group of C. If $\mathcal{F}$ is a coherent sheaf on C we denote by $\tau^C(\mathcal{F}) \in \mathrm{CH}(C)$ its Riemann–Roch class, and by $\tau_i^C(\mathcal{F}) \in \mathrm{CH}^i(C)$ its component of codimension i; $\tau_0^C(\mathcal{F})$ is the cycle associated with $\mathcal{F}$.

Set $Y = X/G$, and call $\pi\colon X \to Y$ the projection. Call Δ_X and Δ_Y the divisors corresponding to the distinguished points in X and Y respectively, ω_X and ω_Y the dualizing sheaves.

A local calculation reveals that $\chi\bigl(\omega_X(\Delta_X)\bigr) \geq \chi\bigl(\pi^*\omega_Y(\Delta_Y)\bigr)$. Indeed, for this calculation we may replace X and Y by their normalization, where we add markings on the normalized curves at the points above the nodes. Then the natural pullback $\pi^*\omega_{Y^{nor}} \to \omega_{X^{nor}}$ extends to a homomorphism of invertible sheaves $\pi^*\omega_{Y^{nor}}(\Delta_{Y^{nor}}) \to \omega_{X^{nor}}(\Delta_{X^{nor}})$ which is injective, and whose cokernel is supported on the points of ramification of π not in $\Delta_{X^{nor}}$.

Thus we get an inequality of Euler characteristics of sheaves:

$$\begin{aligned} \chi\bigl(\omega_X(\Delta_X)\bigr) &\geq \chi\bigl(\pi^*\omega_Y(\Delta_Y)\bigr) \\ &= \chi\bigl(\pi_*\pi^*\omega_Y(\Delta_Y)\bigr) \\ &= \chi\bigl(\omega_Y(\Delta_Y) \otimes \pi_*\mathcal{O}_X\bigr) \\ &= \chi\bigl(\omega_Y(\Delta_Y)\bigr) + \chi\bigl(\omega_Y(\Delta_Y) \otimes (\pi_*\mathcal{O}_X/\mathcal{O}_Y)\bigr). \end{aligned}$$

By hypothesis

$$\chi\bigl(\omega_X(\Delta_X)\bigr) = \gamma + \nu - 1 = \chi\bigl(\omega_Y(\Delta_Y)\bigr).$$

From this, together with the theorem of Grothendieck–Riemann–Roch, we obtain

$$\begin{aligned} 0 &\geq \chi\bigl(\omega_Y(\Delta_Y) \otimes (\pi_*\mathcal{O}_X/\mathcal{O}_Y)\bigr) \\ &= \int_Y \mathrm{ch}\bigl(\omega_Y(\Delta_Y)\bigr)\tau^Y(\pi_*\mathcal{O}_X/\mathcal{O}_Y) \\ &= \int_Y \mathrm{c}_1\bigl(\omega_Y(\Delta_Y)\bigr) \cap \tau_0^Y(\pi_*\mathcal{O}_X/\mathcal{O}_Y) + \int_Y \tau_1^Y(\pi_*\mathcal{O}_X/\mathcal{O}_Y) \\ &= \int_Y \mathrm{c}_1\bigl(\omega_Y(\Delta_Y)\bigr) \cap (\pi_*[X] - [Y]) + \int_X \tau_1^X(\mathcal{O}_X) - \int_Y \tau^Y(\mathcal{O}_Y) \\ &= \int_Y \mathrm{c}_1\bigl(\omega_Y(\Delta_Y)\bigr) \cap (\pi_*[X] - [Y]) + \chi(\mathcal{O}_X) - \chi(\mathcal{O}_Y) \\ &= \int_Y \mathrm{c}_1\bigl(\omega_Y(\Delta_Y)\bigr) \cap (\pi_*[X] - [Y]). \end{aligned}$$

But $\omega_Y(\Delta_Y)$ is an ample line bundle on Y and $\pi_*[X] - [Y]$ is an effective cycle; the only possibility is that $\pi_*[X] - [Y] = 0$. This implies that π is birational, so G is trivial. ♣

4. Fibered surfaces: definitions

If $C \to S$ is a flat family of nodal curves, we denote by C_{sm} the open subscheme of C consisting of points where the morphism $C \to S$ is smooth.

4.1. **Definition.** Let $C \to S$ be a flat (not necessarily proper) family of nodal curves, $X \to C$ a proper morphism with one dimensional fibers, and $\sigma_1, \ldots, \sigma_\nu\colon C \to X$ sections of ρ. We will say that

$$\begin{array}{c} X \\ \downarrow \\ C \\ \downarrow \\ S \end{array}$$

is a *family of generically fibered surfaces* if X is flat over S, and the restriction of ρ to C_{sm} is a flat family of stable pointed curves. If S is the spectrum of a field we will refer to $X \to C$ as a generically fibered surface.

4.2. **Remark.** Notice that we do not require the morphism $X \to C$ to be flat.

4.3. **Definition.** A *chart* $(U, Y \to V, \Gamma)$ for a family of generically fibered surfaces $X \to C \to S$ consists of a diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \times_C U \arrow[r] \arrow[d] & X \arrow[d] \\ V \arrow[r] \arrow[drr] & U \arrow[r] \arrow[dr] & C \arrow[d] \\ & & S \end{tikzcd}$$

together with a group action $\Gamma \subset \mathrm{Aut}_S(Y \to V)$ satisfying:

1. The morphism $U \to C$ is étale;
2. $V \to S$ is a flat (but not necessarily proper) family of nodal curves;
3. $\rho \colon Y \to V$ is a flat family of stable ν-pointed curves of genus γ,
4. the action of Γ on ρ is essential;
5. we have isomorphisms of S-schemes $V/\Gamma \simeq U$ and $Y/\Gamma \simeq U \times_C X$ compatible with the projections $Y/\Gamma \to V/\Gamma$ and $U \times_C X \to U$, such that the sections $U \to U \times_C X$ induced by the σ_i correspond to the sections $V/\Gamma \to Y/\Gamma$.

4.4. **Lemma.** *Let $(U, Y \to V, \Gamma)$ be a chart for a family of generically fibered surfaces $X \to C \to S$. If $S' \to S$ is an arbitrary morphism, then $(S' \times_S U, S' \times_S Y \to S' \times_S V, \Gamma)$ (with the obvious definitions of the various maps and of the action of Γ on $S' \times_S Y$ and $S' \times_S V$) is a chart for the family of generically fibered surfaces $S' \times_S X \to S' \times_S C \to S'$.*

Proof. Conditions (1) through (4) in the definition are immediately verified for the pullback diagram. The only point that requires a little care is to check that $(S' \times_S V)/\Gamma \simeq S' \times_S U$ and $(S' \times_S Y)/\Gamma \simeq (S' \times_S U) \times_C X$, which requires the hypothesis that S be a scheme over $\mathbf{Q}$. ♣

4.5. **Proposition.**

1. *Let $(U, Y \to V, \Gamma)$ be a chart for a family of generically fibered surfaces $X \to C \to S$. Let $V' \subset V$ be the inverse image of C_{sm}. Then*
   (a) *the action of Γ on V' is free; and*
   (b) *the natural morphism $Y|_{V'} \to V' \times_S X$ is an isomorphism.*
2. *Furthermore, if t_0 is a geometric point of S and v_0 a nodal point of the fiber V_{t_0} of V over t_0, then*

(a) *the stabilizer* Γ' *of* v_0 *is a cyclic group which sends each of the branches of* V_{t_0} *to itself.*
(b) *If* n *is the order of* Γ'*, then a generator of* Γ' *acts on the tangent space of each branch by multiplication with a primitive* n*-th root of 1.*

In particular, $V' = V_{\mathrm{sm}}$.

Proof. The claim (1b), that the natural morphism $Y|_{V'} \to V' \times_S X$ is an isomorphism, is a consequence of the first statement (1a).

Let v_0 be a geometric point of the inverse image of U_{sm} in V, Γ' its stabilizer. By definition Γ' acts faithfully on the fiber Y_0 of Y on v_0. The fiber X_0 of X over the image of v_0 in C is the quotient of Y_0 by Γ'; from Lemma 3.5 above we get that Γ' is trivial, as claimed.

For part (2) of the Proposition we observe that if the stabilizer Γ' of v_0 did not preserve the branches of V_{t_0} then the quotient V_{t_0}/Γ', which is étale at the point v_0 over the fiber U_{t_0}, would be smooth over S at v_0, so v_0 would be in the inverse image of U_{sm}. From part (1) of the Proposition it would follow that Γ' is trivial, a contradiction.

So Γ' acts on each of the two branches individually. The action on each branch must be faithful because it is free on the complement of the set of nodes; this means that the representation of Γ' in each of the tangent spaces to the branches is faithful, and this implies the final statement. ♣

4.6. **Definition.** A chart is called *balanced* if for any nodal point of any geometric fiber of V, the two roots of 1 describing the action of a generator of the stabilizer on the tangent spaces to each branch of V are inverse to each other.

4.7. **The transition scheme of two charts.** Let $X \to C \to S$ be a family of generically fibered surfaces, $(U_1, Y_1 \to V_1, \Gamma_1)$, $(U_2, Y_2 \to V_2, \Gamma_2)$ two charts; call $\mathrm{pr}_i \colon V_1 \times_C V_2 \to V_i$ the i^{th} projection. Consider the scheme

$$I = \operatorname*{Isom}_{V_1 \times_C V_2} (\mathrm{pr}_1^* Y_1, \mathrm{pr}_2^* Y_2)$$

over $V_1 \times_C V_2$ representing the functor of isomorphisms of the two families $\mathrm{pr}_1^* Y_1$ and $\mathrm{pr}_2^* Y_2$. There is a section of I over the inverse image $\widetilde{V}$ of C_{sm} in $V_1 \times_C V_2$ which corresponds to the isomorphism $\mathrm{pr}_1^* Y_1|_{\widetilde{V}} \simeq \mathrm{pr}_2^* Y_2|_{\widetilde{V}}$ coming from the fact that both $\mathrm{pr}_1^* Y_1$ and $\mathrm{pr}_2^* Y_2$ are pullbacks to $\widetilde{V}$ of the restriction of X to C_{sm}. We will call the scheme-theoretic closure R of this section in I the *transition scheme* from $(U_1, Y_1 \to V_1, \Gamma_1)$ to $(U_2, Y_2 \to V_2, \Gamma_2)$; it comes equipped with two projections $R \to V_1$ and $R \to V_2$. There is also an action of $\Gamma_1 \times \Gamma_2$ on I, defined as follows. Let $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$, and $\phi \colon \mathrm{pr}_1^* Y_1 \simeq \mathrm{pr}_2^* Y_2$ an isomorphism over $V_1 \times_C V_2$; then define $(\gamma_1, \gamma_2) \cdot \phi = \gamma_2 \circ \phi \circ \gamma_1^{-1}$.[3] This action of $\Gamma_1 \times \Gamma_2$ on I is compatible with the action of $\Gamma_1 \times \Gamma_2$ on $V_1 \times_C V_2$, and leaves R invariant. It follows from the definition of an essential action that the action of $\Gamma_1 = \Gamma_1 \times \{1\}$ and $\Gamma_2 = \{1\} \times \Gamma_2$ on I is free.

4.8. **Definition.** Two charts $(U_1, Y_1 \to V_1, \Gamma_1)$ and $(U_2, Y_2 \to V_2, \Gamma_2)$ are compatible if their transition scheme R is étale over V_1 and V_2.

Let us analyze this definition. Start from two charts $(U_1, Y_1 \to V_1, \Gamma_1)$ and $(U_2, Y_2 \to V_2, \Gamma_2)$. Fix two geometric points $v_1 \colon \operatorname{Spec} \Omega \to V_1$ and $v_2 \colon \operatorname{Spec} \Omega \to V_2$ mapping to the same geometric point $v_0 \colon \operatorname{Spec} \Omega \to C$, and call $\Gamma_i' \subseteq \Gamma_i$ the stabilizer of v_i. Also call V_1^{sh}, V_2^{sh} and C^{sh} the spectra of the strict henselizations of V_1, V_2 and C at the points v_1, v_2 and v_0 respectively. The action of Γ_i on V_i induces an action of Γ_i' on V_i^{sh}. Also call Y_i^{sh} the pullback of Y_i to V_i^{sh}; there is an action of Γ_i' on Y_i^{sh} compatible with the action of Γ_i' on V_i.

[3] Angelo - I changed the order of the gammas here, I hope correctly

4.9. **Proposition.** *The two charts are compatible if and only if for any pair of geometric points v_1 and v_2 as above there exist an isomorphism of groups $\eta\colon \Gamma_1' \simeq \Gamma_2'$ and two compatible η-equivariant isomorphisms $\phi\colon V_1^{\mathrm{sh}} \simeq V_2^{\mathrm{sh}}$ and $\psi\colon Y_1^{\mathrm{sh}} \to Y_2^{\mathrm{sh}}$ of schemes over C^{sh}.*

Proof. Consider the spectrum $(V_1 \times_C V_2)^{\mathrm{sh}}$ of the strict henselization of $V_1 \times_C V_2$ at the point $(v_1, v_2)\colon \operatorname{Spec} \Omega \to V_1 \times_C V_2$, and call R^{sh} the pullback of R to $(V_1 \times_C V_2)^{\mathrm{sh}}$. Assume that the two charts are compatible. The action of $\Gamma_1 \times \Gamma_2$ on I described above induces an action of $\Gamma_1' \times \Gamma_2'$ on R^{sh}, compatible with the action of $\Gamma_1' \times \Gamma_2'$ on $(V_1 \times_C V_2)^{\mathrm{sh}}$. The action of $\Gamma_1' = \Gamma_1' \times \{1\}$ on the inverse image of C_{sm} in R^{sh} is free, and its quotient is the inverse image of C_{sm} in V_2^{sh}; but R^{sh} is finite and étale over V_2^{sh}, so the action of Γ_1' on all of R^{sh} is free, and $R^{\mathrm{sh}}/\Gamma_1' = V_2$. Analogously the action of Γ_2' on R^{sh} is free, and $R^{\mathrm{sh}}/\Gamma_2' = V_1$.

Now, each of the connected components of R^{sh} maps isomorphically onto both V_1 and V_2, because V_i is the spectrum of a strictly henselian ring and the projection $R^{\mathrm{sh}} \to V_i$ is étale; this implies in particular that the order of Γ_1 is the same as the number n of connected components, and likewise for Γ_2. Fix one of these components, call it R_0^{sh}; then we get isomorphisms $R_0^{\mathrm{sh}} \simeq V_i$, which yield an isomorphism $\phi\colon V_1 \simeq V_2$.

Call Γ' the stabilizer of the component R_0^{sh} inside $\Gamma_1' \times \Gamma_2'$; the order of Γ' is at least $|\Gamma_1' \times \Gamma_2'|/n = n^2/n = n$. But the action of Γ_2' on R^{sh} is free, and so $\Gamma' \cap \Gamma_2 = \{1\}$; this implies that the order of Γ' is n, and the projection $\Gamma' \to \Gamma_1$ is an isomorphism. Likewise the projection $\Gamma' \to \Gamma_2$ is an isomorphism, so from these we get an isomorphism $\eta\colon \Gamma_1 \to \Gamma_2$, and it is easy to check that the isomorphism of schemes $\phi\colon V_1 \simeq V_2$ is η-equivariant.

There is also an isomorphism of the pullbacks of Y_1^{sh} and Y_2^{sh} to R_0^{sh}, coming from the natural morphism $R_0^{\mathrm{sh}} \to I$, which induces an isomorphism $\psi\colon Y_1^{\mathrm{sh}} \to Y_2^{\mathrm{sh}}$. This isomorphism is compatible with ϕ, and is it also η-equivariant.

Let us prove the converse. Suppose that there exist η, ϕ and ψ as above. Then there is a morphism $\sigma\colon V_1^{\mathrm{sh}} \times \Gamma_1' \to I$ which sends a point (v_1, γ_1) of $V_1^{\mathrm{sh}} \times \Gamma_1'$ into the point of I lying over the point $(v_1, \phi\gamma_1 v_1) = (v_1, \eta(\gamma_1)\phi v_1)$ corresponding to the isomorphism $\gamma_1\psi$ of the fiber of Y_1 over v_1 with the fiber of Y_2 over $\phi\gamma_1 v_1$. The morphism σ is an isomorphism of $V_1^{\mathrm{sh}} \times \Gamma_1'$ with R^{sh} in the inverse image of C_{sm}; it also follows from the fact that the action of Γ' on $Y_1 \to V_1$ is essential that σ is injective. Since the inverse image of C_{sm} is scheme-theoretically dense in R^{sh} and $V_1^{\mathrm{sh}} \times \Gamma_1$ is unramified over V_1 we see that σ is an isomorphism; it follows that R^{sh} is étale over V_1^{sh}; analogously it is étale over V_2^{sh}. So R is étale over V_1 and V_2 at the points v_1 and v_2; since this holds for all v_1 and v_2 mapping to the same point of C the conclusion follows. ♣

In 8.14 and 8.15 below, we give two examples of incompatible charts on the coarse fibered surface.

4.10. **The product chart.** Given two compatible charts $(U_1, Y_1 \to V_1, \Gamma_1)$ and $(U_2, Y_2 \to V_2, \Gamma_2)$, the graph $Y \subseteq (Y_1 \times_{V_1} R) \times_R (R \times_{V_2} Y_2)$ over R of the canonical isomorphism of the two families $Y_1 \times_{V_1} R$ and $R \times_{V_2} Y_2$ is invariant under the action of $\Gamma_1 \times \Gamma_2$, and it is a family of pointed curves on R. Then

$$(U_1 \times_C U_2, Y \to R, \Gamma_1 \times \Gamma_2)$$

is a chart, called *the product chart.* It is compatible with both the original charts.

Compatibility of charts is stable under base change:

4.11. **Proposition.** *Let $(U_1, Y_1 \to V_1, \Gamma_1)$ and $(U_2, Y_2 \to V_2, \Gamma_2)$ be two compatible charts for a family of generically fibered surfaces $X \to C \to S$. If $S' \to S$ is an arbitrary morphism, then*

$$(S' \times_S U_1, S' \times_S Y_1 \to S' \times_S V_1, \Gamma_1)$$

and

$$(S' \times_S U_2, S' \times_S Y_2 \to S' \times_S V_1, \Gamma_2)$$

are compatible charts for the the family $S' \times_S X \to S' \times_S C \to S'$.

This is easy.

We now come to the definition of our basic object:

4.12. **Definition.** A *family of fibered surfaces*

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathcal{C} \\ \downarrow \\ S \end{array}$$

is a family of generically fibered surfaces $X \to C \to S$ such that $C \to S$ is proper, together with a collection $\{(U_\alpha, Y_\alpha \to V_\alpha, \Gamma_\alpha)\}$ of mutually compatible charts, such that the images of the U_α cover C.

Such a collection of charts is called an *atlas*.

A family of fibered surfaces is called *balanced* if each chart in its atlas is balanced.

The family of generically fibered surfaces $X \to C \to S$ supporting the family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ will be called a family of *coarse fibered surfaces*.

4.13. **Lemma.** *If two charts for a family of fibered surfaces are compatible with all the charts in an atlas, they are mutually compatible.*

Furthermore, if the family is balanced, then any chart which is compatible with every chart of the atlas is balanced.

The proof is straightforward.

4.14. **Remark.** The lemma above allows us to define a family of fibered surfaces using a maximal atlas, if we want.

4.15. **Definition.** A *morphism of fibered surfaces* $\mathcal{X} \to \mathcal{C} \to S$ to $\mathcal{X}' \to \mathcal{C}' \to S'$ is a *cartesian* diagram of coarse fibered surfaces

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow & & \downarrow \\ C & \longrightarrow & C' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S', \end{array}$$

such that the pullback of the charts of an atlas of $\mathcal{X}' \to \mathcal{C}' \to S$ are all compatible with the atlas of $\mathcal{X} \to \mathcal{C} \to S$.

The composition of morphisms is the obvious one.

We will soon reinterpret this definition of a morphism.

4.16. **Fibered surfaces as stacks.** Consider a family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ with an atlas $\{(U_\alpha, Y_\alpha \to V_\alpha, \Gamma_\alpha)\}$. For each pair of indices (α, β) let $R_{\alpha\beta}$ be the transition scheme from $(U_\alpha, Y_\alpha \to V_\alpha, \Gamma_\alpha)$ to $(U_\beta, Y_\beta \to V_\beta, \Gamma_\beta)$. Let V be the disjoint union of the V_α, and let R be the disjoint union of the $R_{\alpha,\beta}$. These have the following structure:

- there are two projection $R \to V$, which are étale;
- there is a natural diagonal morphism $V \to R$ which sends each V_α to $R_{\alpha\alpha}$; and

- there is a product $R \times_V R \to R$, sending each $R_{\alpha\beta} \times_{V_\beta} R_{\beta\gamma}$ to $R_{\alpha\gamma}$ via composition of isomorphisms

These various maps give $R \rightrightarrows V$ the structure of a groupoid. Since the diagonal map $R \to V \times_C V$ is unramified the groupoid defines a quotient Deligne-Mumford stack. In a slight abuse of notation, we call this stack $\mathcal{C}$ as well.

A similar groupoid structure can be formed using Y_α and their pullback to $R_{\alpha\beta}$, endowing $\mathcal{X}$ with the structure of a Deligne-Mumford stack as well. Note that $\mathcal{X} \to \mathcal{C}$ is representable, and the stack $\mathcal{X}$ is a family of stable ν-pointed curves of genus γ over the stack $\mathcal{C}$.

Let us list some properties of $\mathcal{X}$ and $\mathcal{C}$:

- There is a moduli morphism $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ associated to the family $\mathcal{X} \to \mathcal{C}$.
- The stack $\mathcal{C}$ is a proper nodal stack over S, namely, it has an étale cover, given by the schemes V underlying the charts, which are nodal over S.
- Over the inverse image of C_{sm} the scheme R is isomorphic to the fibered product $V \times_C V$, so the quotient stack $\mathcal{C}_{\mathrm{sm}}$ coincides with C_{sm}. A similar statement holds for $\mathcal{X}$.
- Since U_α is the schematic quotient of V_α by the action of Γ_α, and $U \to C$ is étale, it is immediate that C is the schematic quotient of the groupoid $R \rightrightarrows V$, in other words, the stack morphism $\mathcal{C} \to C$ exhibits C as the coarse moduli scheme of $\mathcal{C}$. Similarly, X is the coarse moduli scheme of $\mathcal{X}$.

As mentioned above, a fibered surface being *balanced* is tantamount to the existence of local smoothing of the fibered surface. It is interesting to interpret the requirement that the action of Γ is *essential* within the language of stacks.

We claim that the action being essential is equivalent to the condition that the moduli morphism $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ be *representable*. This follows from the definition of essential action, using the following well known lemma:

4.17. **Lemma.** *Let $g : \mathcal{G} \to \mathcal{F}$ be a morphism of Deligne Mumford stacks. The following two conditions are equivalent:*

1. *The morphism $g : \mathcal{G} \to \mathcal{F}$ is representable.*
2. *For any algebraically closed field k and any $\xi \in \mathcal{G}(k)$, the natural group homomorphism $\mathrm{Aut}(\xi) \to \mathrm{Aut}(g(\xi))$ is a monomorphism.*

Proof. By definition, the morphism $g : \mathcal{G} \to \mathcal{F}$ is representable if and only if the following condition holds:

- For any algebraic space X and any morphism $f : X \to F$, the stack $\mathcal{Y} := \mathcal{G} \times_{\mathcal{F}} X$ is equivalent to an algebraic space.

For fixed $f : X \to F$ latter condition is equivalent to the following:

- The diagonal morphism $\Delta : \mathcal{Y} \to \mathcal{Y} \times \mathcal{Y}$ is a monomorphism.

By definition, this means:

- Given an algebraically closed field k, and an element $q \in \mathcal{Y} \times \mathcal{Y}(k)$ and two elements $p_i \in \mathcal{Y}(k)$, with isomorphisms $\beta_i : \Delta(p_i) \to q$, there exists a unique isomorphism $\phi : p_1 \to p_2$ such that $\beta_1 = \beta_2 \circ \Delta(\phi)$.

We can write

$$q = ((\mathfrak{g}_1, x_1, \alpha_1), (\mathfrak{g}_1, x_1, \alpha_1))$$

where $\mathfrak{g}_i \in \mathcal{G}(k), x_i \in X(k)$, and $\alpha_i : g(\mathfrak{g}_i) \xrightarrow{\sim} f(x_i)$. Similarly $p_i = (\mathfrak{g}'_i, x'_i, \alpha'_i)$ as above. The existence of β_i implies $x_1 = x_2 = x'_1 = x'_2$. Also, using the given isomorphisms we may assume that, in fact, the same is true for the $\mathfrak{g}_i, \mathfrak{g}'_i, \alpha_i$ and α'_i. Thus the condition above is equivalent to saying that for any $\mathfrak{g} \in G(k)$ there is a unique $\beta \in \mathrm{Aut}(\mathfrak{g})$ such that $g(\beta) = id_{g(\mathfrak{g})}$, which is what we wanted. ♣

4.18. **A stack-theoretic formulation of the category of fibered surfaces.** We will now formulate stack-theoretic data similar to that obtained above, and then compare them to the data of a fibered surface.

Let S be a scheme over $\mathbf{Q}$. Consider a proper Deligne–Mumford stack $\mathcal{C} \to S$, such that its fibers are purely one-dimensional and geometrically connected, with nodal singularities. Call C the moduli space of $\mathcal{C}$; this automatically exists as an algebraic space.

4.19. **Proposition.** *The morphism $C \to S$ is a flat family of nodal curves.*

Proof. First of all let us show that C is flat over S. We may assume that S is affine; call R its ring. Fix a geometric point $c_0 \to C$, and call C^{sh} the strict henselization of C at c_0. Let U be an étale cover of $\mathcal{C}$, u_0 a geometric point of U lying over c_0, and call U^{sh} the strict henselization of U at u_0. If Γ is the automorphism group of the object of $\mathcal{C}$ corresponding to u_0, then Γ acts on U^{sh}, and C^{sh} is the quotient U^{sh}/Γ. Because the schemes are defined over $\mathbf{Q}$, the ring of C^{sh} is a direct summand of the ring of Y^{sh}, as an R-module, so it is flat over R.

The fact that the fibers are nodal follows from the fact that, over an algebraically closed field, the quotient of a nodal curve by a group action is again a nodal curve. ♣

4.20. **Definition.** A stack-like family of curves of genus g over a scheme S consists of a proper stack $\mathcal{C} \to S$, whose fibers are purely one-dimensional and geometrically connected, with nodal singularities, such that:

1. the fibers of the morphism $C \to S$, where C is the moduli space of $\mathcal{C}$, have genus g;
2. over the inverse image of the smooth locus C_{sm} of the map $C \to X$, the projection $\mathcal{C} \to C$ is an isomorphism.

If $\mathcal{C} \to S$ and $\mathcal{C}' \to S'$ are stack-like families of curves of genus g, a morphism $F\colon \mathcal{C} \to \mathcal{C}'$ consists of a cartesian diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{C}' \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & S'. \end{array}$$

The composition of morphisms of stack-like families of curves is defined in the obvious way. In this way stack-like families of curves of genus g form a 2-category. The 2-arrows being defined in the usual way.

4.21. **Proposition.** *The 2-category of stack-like families of curves of genus g is equivalent to a 1-category.*

Proof. This is the same as saying that a 1-arrow in the category can not have nontrivial automorphisms. The point here is that the stack $\mathcal{C}$ has an open subscheme which is stack-theoritically dense in $\mathcal{C}$, which is sufficient by the following lemma[4].

4.22. **Lemma.** *Let $F\colon \mathcal{X} \to \mathcal{Y}$ be a representable morphism of Deligne–Mumford stacks over a scheme S. Assume that there exists an open subscheme $U \subseteq \mathcal{X}$ which is stack-theoretically dense, and that the diagonal $\mathcal{Y} \to \mathcal{Y} \times_S \mathcal{Y}$ is separated. Then any automorphism of F is trivial.*

[4] which must be corrected...

Proof. Let α be an automorphism of F; for each object ξ of $\mathcal{X}$ over a scheme X we are given an automorphism α_ξ of $F(\xi)$ over X, satisfying the usual condition for being a natural transformation. We are going to need the following two facts. First of all, if $\xi \to \eta$ is a morphism in $\mathcal{X}$, then it follows from the fact that $\mathcal{X}$ is a category fibered in groupoids that if α_η is trivial then also α_ξ is trivial. Also, if $X' \to X$ is an étale surjective map of schemes and ξ' is the pullback of ξ to X', then it follows from the definition of a stack that if $\alpha_{\xi'}$ is trivial then also α_ξ is trivial.

Now, the lemma is clear if $\mathcal{X}$ is isomorphic to a scheme. Let $X \to \mathcal{X}$ be an étale cover, and call ξ the corresponding object of $\mathcal{X}$ over X. The restriction of α_ξ to the open subscheme $V = X \times_{\mathcal{X}} X \subseteq U$ is trivial, and V is scheme-theoretically dense in U; from the fact that the diagonal of $\mathcal{Y}$ is separated it follows that α_ξ is trivial.

Take an arbitrary object τ of $\mathcal{X}$ over a scheme T; then there is an étale cover $T' \to T$ such that the pullback τ' of τ to T' admits a morphism $\tau \to \xi$. By applying the two facts mentioned above we get the result. ♣

4.23. **Definition.** A stack-like fibered surface $f\colon \mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ over a scheme S is a stack-like family $\mathcal{C} \to S$ of curves of genus g over S, with projective coarse moduli space $C \to S$, together with a representable morphism $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$.

As pointed out above, stack-like fibered surfaces form a 2-category. A 1-arrow from $f\colon \mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ to $f'\colon \mathcal{C}' \to \overline{\mathcal{M}}_{\gamma,\nu}$ consists of a morphism of stack-like families of curves $\phi\colon \mathcal{C} \to \mathcal{C}'$, together with an isomorphism of the composition α_ϕ of $f' \circ \phi$ with f; the compostion of 1-arrows is defined in the obvious way. It is to be noticed that, because of the Proposition above, the isomorphism α_ϕ is uniquely determined by ϕ. A 2-arrow from a 1-arrow ϕ from $f\colon \mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ to $f'\colon \mathcal{C}' \to \overline{\mathcal{M}}_{\gamma,\nu}$ to another arrow ψ with the same domain and codomain consists of an isomorphism β of the two functors ϕ and ψ; there should be a compatibility condition of the isomorphism β with α_ϕ and α_ψ, but this follows from the Proposition.

Again because of the Proposition, this category is equivalent to a nice unproblematic 1-category, which we can think of as the category of stack-like fibered surfaces, the arrows being given by isomorphism classes of morphisms.

We can now compare the two categories we have defined.

4.24. **Theorem.** *The category of stack-like fibered surfaces over schemes over* $\mathbf{Q}$ *is equivalent to the category of fibered surfaces.*

Proof. The construction in 4.16 shows that a fibered surface gives rise to a stack-like fibered surface in a functorial way.

To go from stack-like fibered surfaces to honest fibered surfaces, let us take a stack-like fibered surface $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$, and then take as a family of pointed nodal curves its coarse moduli space $C \to S$. The charts are given by étale morphism from schemes to $\mathcal{C}$. ♣

From now on we will use fibered surfaces and stack-like fibered surfaces interchangeably.

4.25. **Stable fibered surfaces.** We first define a moduli morphism on a coarse fibered surface:

4.26. **Lemma.** *Let* $\mathcal{X} \to \mathcal{C} \to S$ *be a family of fibered surfaces. Then the morphism* $C_{\mathrm{sm}} \to \overline{\mathbf{M}}_{\gamma,\nu}$ *induced by the restriction of* $X \to C$ *to* C_{sm} *extends uniquely to a morphism* $C \to \overline{\mathbf{M}}_{\gamma,\nu}$.

Proof. The unicity is clear from the fact that $\overline{\mathbf{M}}_{\gamma,\nu}$ is separated and C_{sm} is scheme-theoretically dense in C. To prove the existence of an extension is a local question in the étale topology; but if $\{(U_\alpha, Y_\alpha \to V_\alpha, \Gamma_\alpha)\}$ is an atlas then the families $Y_\alpha \to V_\alpha$ induce morphisms $V_\alpha \to \overline{\mathbf{M}}_{\gamma,\nu}$, which are Γ_α-equivariant, yielding morphisms $U_\alpha \to \overline{\mathbf{M}}_{\gamma,\nu}$. These morphisms are extensions of the restriction to the U_α of the morphism $C_{\mathrm{sm}} \to \overline{\mathbf{M}}_{\gamma,\nu}$. Therefore they descend to C. ♣

We can use this lemma to finally define stable fibered surfaces:

4.27. **Definition.** A family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ is *stable* if the associated morphism $C \to \overline{\mathbf{M}}_{\gamma,\nu}$ is Kontsevich stable.

4.28. **Natural line bundles on a coarse fibered surface.**

4.29. **Proposition.** *To each family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ it is possible to assign a canonically defined line bundle $\mathcal{L}_X$ on the coarse fibered surface X, which is relatively ample along the map $X \to C$. This line bundle satisfies:*

1. *For any morphism of fibered surfaces*

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\phi} & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

   there is an isomorphism of line bundles $\sigma_\phi \colon \mathcal{L}_{X'} \simeq \phi^*\mathcal{L}_X$.
2. *These isomorphisms satisfy the cocycle condition, in the sense that $\sigma_{\mathrm{id}_X} = \mathrm{id}_{\mathcal{L}_X}$ for all families of fibered surfaces $X \to S$, and if we have two morphisms*

$$\begin{array}{ccccc} \mathcal{X}'' & \xrightarrow{\psi} & \mathcal{X}' & \xrightarrow{\phi} & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}'' & \longrightarrow & \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow & & \downarrow \\ S'' & \longrightarrow & S' & \longrightarrow & S \end{array}$$

   then

$$\sigma_{\phi\psi} = \psi^*\sigma_\phi \circ \sigma_\psi \colon \mathcal{L}_{X''} \longrightarrow (\phi\psi)^*\mathcal{L}_X = \psi^*\phi^*\mathcal{L}_X$$

Proof. Fix an integer N which is divisible by the order of the automorphism group of any ν-pointed stable curve of genus γ. The construction below will depend on the choice of the integer N. For each index α consider the line bundle $\omega_\alpha = \omega_{Y_\alpha/V_\alpha}(\Delta_\alpha)$, the relative dualizing sheaf of Y_α over V_α twisted by the divisor Δ_α of marked points. This line bundle has a natural action of Γ_α, and the stabilizers of all the geometric points act trivially on the fiber of $\omega_\alpha^{\otimes N}$; therefore $\omega_\alpha^{\otimes N}$ descends to a line bundle $\mathcal{L}_\alpha$ on $X \times_C U_\alpha$.

Given two indices α and β, let $R_{\alpha\beta}$ be the transition scheme of two corresponding charts and $Y_{\alpha\beta} \to R_{\alpha\beta}$ the family of pointed stable curves constructed above; there is a canonical isomorphism between the pullback of ω_α to $Y_{\alpha\beta}$ and the analogous bundle $\omega_{\alpha\beta}$ on the family $Y_{\alpha\beta}$, so we can compose these to get isomorphisms between the pullbacks of ω_α and ω_β. These isomorphisms satisfy the cocycle condition and give the collection of the ω_α the structure of a line bundle on the groupoid $R \rightrightarrows V$.

Passing to the N-th power, the isomorphisms above also descend to isomorphisms between the pullbacks of $\mathcal{L}_\alpha$ and $\mathcal{L}_\beta$ on the fiber product $X \times_C (U_\alpha \times_C U_\beta)$ satisfying the cocycle condition, so the $\mathcal{L}_\alpha$ are the pullback of a well defined line bundle $\mathcal{L}_X$, which is the one we want. ♣

4.30. **Proposition.** *To each family of stable fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ it is possible to assign a canonically defined line bundle $\mathcal{A}_X$ on the coarse fibered surface X, which is relatively ample along the map $X \to S$. This line bundle satisfies the properties listed in Proposition 4.29.*

Proof. Fix N as in the proof of Proposition 4.29. Fix an ample line bundle H_0 on $\overline{\mathbf{M}}_{\gamma,\nu}$, and let $H = H_0^{\otimes 3}$. Let $\mathcal{X} \to \mathcal{C} \to S$ be a stable fibered surface, and let $m_C : C \to \overline{\mathbf{M}}_{\gamma,\nu}$ be the associated moduli morphism. Consider the line bundle $A_C := (\omega_{C/S} \otimes m_C^* H)^{\otimes N}$ on C. By the theory of stable maps, the line bundle A_C is ample relatively to the morphism $C \to S$. It clearly satisfies the invariance condition in the proposition. Also, by Kollár's semipositivity lemma (see [Kollar1]) the line bundle $\mathcal{L}_X$ constructed above is nef. Set $\mathcal{A}_X = \mathcal{L}_X \otimes f^* A_C$. By Kleiman's criterion for ampleness we clearly have that $\mathcal{A}_X$ is ample. The invariance conditions follow by construction.

4.31. **Remark.** It is important to note that both $\mathcal{L}_X$ and $\mathcal{A}_X$ are invariants of the *coarse* fibered surfaces, independent of the given atlas. This is because, on the open subscheme U where $X \to S$ is Gorenstein, these coincide with $\omega_{U/C}^N$ and $\omega_{U/S}^N$. Since U has complement of codimension ≥ 2, the extensions to X are unique.

5. The stack of fibered surfaces

Consider the stack $\overline{\mathcal{M}}_{\gamma,\nu}$ of stable ν-pointed curves of genus γ over $\mathbf{Q}$, and the associated moduli space $\overline{\mathbf{M}}_{\gamma,\nu}$; choose an ample line bundle $\mathcal{H}$ on $\overline{\mathbf{M}}_{\gamma,\nu}$. Fix two nonnegative integers g and d.

We define a category $\mathcal{F}_g^d(\gamma,\nu)$, fibered over the category $\mathrm{Sch}/\mathbf{Q}$ of schemes over $\mathbf{Q}$, as follows. The objects are balanced stable families $\mathcal{X} \to \mathcal{C} \to S$ such that for the associated morphism $f\colon C \to \overline{\mathbf{M}}_{\gamma,\nu}$, the degree of the line bundle $f^*\mathcal{H}$ on each fiber of C over S is d. The arrows are morphism of fibered surfaces.

There is an obvious morphism from $\mathcal{F}_g^d(\gamma,\nu)$ to the stack $\mathcal{K}_g^d\big(\overline{\mathbf{M}}_{\gamma,\nu}\big)$ of Kontsevich - stable maps of genus g and degree d into $\overline{\mathbf{M}}_{\gamma,\nu}$ which sends each stable family of fibered surfaces $\mathcal{X} \to \mathcal{C} \to S$ to the associated morphism $C \to \overline{\mathbf{M}}_{\gamma,\nu}$.

Our main result is:

5.1. **Theorem.** *The category $\mathcal{F}_g^d(\gamma,\nu)$ is a complete Deligne-Mumford stack, admitting a projective coarse moduli space $\mathbf{F}_g^d(\gamma,\nu)$.*

Proof.

By definition, $\mathcal{F}_g^d(\gamma,\nu)$ is fibered by groupoids over $\mathrm{Sch}/\mathbf{Q}$, since all the morphisms of objects over a scheme S are isomorphisms.

The next step is to check the sheaf axioms, to prove that $\mathcal{F}_g^d(\gamma,\nu)$ is indeed a stack. First, we need to show that we have effective descent for objects (see [Artin], 1.1 (ii)). Given a scheme S, an étale cover $\{S_\alpha \to S\}$, and fibered surfaces $\mathcal{X}_\alpha$ over S_α together with isomorphisms between the pullbacks of $\mathcal{X}_\alpha$ and $\mathcal{X}_\beta$ to $S_\alpha \times_S S_\beta$ satisfying the cocycle condition, we claim that these descend to a fibered surface $\mathcal{X}$ on S. The existence of the coarse fibered surface X as an algebraic space is immediate; its projectivity follows from the existence of the ample line bundles $\mathcal{A}_{\mathcal{X}_\alpha}$ (Proposition 4.30). Now, any chart

$$\begin{tikzcd}
Y_\alpha \arrow[r] \arrow[d] & X_\alpha \times_{C_\alpha} U_\alpha \arrow[r] \arrow[d] & X_\alpha \arrow[d] \\
V_\alpha \arrow[r] \arrow[drr] & U_\alpha \arrow[r] \arrow[dr] & C_\alpha \arrow[d] \\
& & S_\alpha
\end{tikzcd}$$

immediately gives rise to a chart

$$\begin{tikzcd}
Y_\alpha \arrow[r] \arrow[d] & X \times_{C} U_\alpha \arrow[r] \arrow[d] & X \arrow[d] \\
V_\alpha \arrow[r] \arrow[drr] & U_\alpha \arrow[r] \arrow[dr] & C \arrow[d] \\
& & S
\end{tikzcd}$$

The charts coming from one X_α are obviousely compatible. Compatibility of the charts coming from X_α and X_β follows since they are compatible over $X_{\alpha\beta}$.

The other sheaf axiom (see [Artin], 1.1 (i)) requires the Isom functor to be a sheaf in the étale topology. Since we would like to show that $\mathcal{F}^d_g(\gamma,\nu)$ is an *algebraic* stack, we might as well address the representability of the Isom functor at the same time.

Take two stable families of fibered surfaces $\mathcal{X}_1$ and $\mathcal{X}_2$ over the same scheme S; we want to show that the functor $\mathrm{Isom}_S(\mathcal{X}_1,\mathcal{X}_2)$ of isomorphisms of fibered surfaces is representable by a scheme of finite type over S. By Grothendieck's theorem, $\mathrm{Isom}_S(X_1,X_2)$ and $\mathrm{Isom}_S(C_1,C_2)$ are represented by schemes locally of finite type over S. There is an obvious functorial embedding

$$\underset{S}{\mathrm{Isom}}(\mathcal{X}_1,\mathcal{X}_2) \hookrightarrow \underset{S}{\mathrm{Isom}}(X_1,X_2)\underset{S}{\times}\underset{S}{\mathrm{Isom}}(C_1,C_2);$$

we will prove that $\mathrm{Isom}_S(\mathcal{X}_1,\mathcal{X}_2)$ is a closed subscheme of $\mathrm{Isom}_S(X_1,X_2)\times_S \mathrm{Isom}_S(C_1,C_2)$. This shows that $\mathrm{Isom}_S(\mathcal{X}_1,\mathcal{X}_2)$ is a scheme locally of finite type over S; the fact that a stable fibered surface has only finitely many automorphisms means that $\mathrm{Isom}_S(\mathcal{X}_1,\mathcal{X}_2)$ is quasifinite over S, so it is of finite type.

First of all, there is a closed subscheme $I \subseteq \mathrm{Isom}_S(X_1,X_2)\times_S \mathrm{Isom}_S(C_1,C_2)$ representing the functor of compatible pairs of isomorphisms $X_1 \simeq X_2$ and $C_1 \simeq C_2$; and $\mathrm{Isom}_S(\mathcal{X}_1,\mathcal{X}_2)$ is obviously contained in I. So we only need to prove that given a scheme S, two fibered surfaces $\mathcal{X}_1$ and $\mathcal{X}_2$ on S, and compatible isomorphisms $f\colon X_1 \simeq X_2$ and $g\colon C_1 \simeq C_2$, there exists a closed subscheme $S' \subseteq S$ such that given a morphism $T \to S$, the pullbacks of f and g to T give an isomorphism of fibered surfaces if and only if $T \to S$ factors through S'. By considering the given atlas on X_1 and the pullback of the atlas of X_2 we see that we need the following.

5.2. **Lemma.** *Let $X \to C \to S$ be a coarse fibered surface with two atlases $\mathcal{A}_1$ and $\mathcal{A}_2$. Then there exists a closed subscheme $S' \subseteq S$ such that given a morphism $T \to S$, the pullbacks of $\mathcal{A}_1$ and $\mathcal{A}_2$ to $T \times_S X$ are compatible if and only if $T \to S$ factors through S'.*

Proof. We will prove a local version of the fact above.

5.3. **Lemma.** *Let there be given a coarse fibered surface $X \to C \to S$ and two charts $(U, Y \to V, \Gamma)$ and $(U', Y' \to V', \Gamma')$. Let there be given a geometric point s_0 of S; assume that the fiber of C over s_0 has a unique nodal point c_0, and that there are two unique geometric points $v_0 \in V$ and $v_0' \in V'$ over c_0. Then, after passing to an étale neighborhood of s_0, there exists a closed subscheme $S' \subseteq S$ such that given a morphism $T \to S$, the pullbacks of the two charts to $T \times_S X$ are compatible if and only if $T \to S$ factors through S'.*

The local lemma clearly implies the global lemma.

Proof of the local lemma. By passing to the fiber product $U \times_C U'$ we may assume that $C = U = U'$. By refining V and U we may assume that they are affine, and that there exists an invariant effective Cartier divisor D on V containing the locus where the projection $V \to S$ is not smooth, but none of the fibers of $V \to S$.

We claim that such a divisor is flat over S. Indeed, we may assume that D is defined by an equation f:

$$0 \to \mathcal{O}_V \xrightarrow{\cdot f} \mathcal{O}_V \to \mathcal{O}_D \to 0.$$

Since f does not vanish along any component of any fiber we have that the natural map

$$\mathcal{O}_V \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_s \xrightarrow{\cdot f} \mathcal{O}_V \underset{\mathcal{O}_S}{\otimes} \mathcal{O}_s$$

is injective for any point $s \in S$.

By passing to an étale neighborhood of s_0 we can split D into a number of connected components, so that the components containing v_0 is finite over S; then by deleting the other connected components we see that we may assume that D is finite over S.

Consider the transition scheme R of the two charts. There is a free action of the group Γ' over R; set $E = R/\Gamma'$. The projection $E \to V$ is an isomorphism over the smooth locus of C, and is an isomorphism if and only if the projection $R \to V$ is étale. Take a morphism $T \to S$; then the transition schemes R_T of the pullbacks of the two charts to T is the scheme-theoretic closure of the inverse image of the smooth locus of C in $R \times_S T$, so R_T is étale over $V \times_S T$ if and only if the projection $(R \times_S T)/\Gamma' = E \times_S T \to V \times_S T$ has a section.

Set $S = \operatorname{Spec} \Lambda$, $V = \operatorname{Spec} A$, $E = \operatorname{Spec} B$, and call I the ideal of D in A. The coordinate ring of the complement of D inside V is $A' = \cup_{i=0}^{\infty} I^{-i}$, and there is a natural homomorphism $B \to A'$; given a Λ-algebra L the coordinate ring of the transition scheme of the pullbacks of the two charts to Spec L is the image of $B \otimes_\Lambda L$ in $A' \otimes_A L$, so $(R \times_S T)/\Gamma' = E \times_S T \to V \times_S T$ has a section if and only if the image of $B \otimes_\Lambda L$ in $A' \otimes_\Lambda L$ is equal to $A \otimes_\Lambda L \subseteq A' \otimes_\Lambda L$. Take a set of generators of B as an A-algebra, and call $e_1, \dots, e_n$ their images inside A'/A; the condition that the image of $B \otimes_\Lambda L$ be equal to $A \otimes_\Lambda L$ is equivalent to the condition that the image of the e_i in $(A'/A) \otimes_\Lambda L$ be zero. Fix an integer n such that the e_i are all contained in I^{-n}/A; then the sequence

$$0 \longrightarrow I^{-n}/A \longrightarrow A'/A \longrightarrow \cup_{i=n}^{\infty} I^{-i}/I^{-n} \longrightarrow 0$$

is exact and $\cup_{i=n}^{\infty} I^{-i}/I^{-n} = I^{-n} \otimes_A (A'/A)$ is flat over Λ, so the sequence stays exact after tensoring with L. The conclusion is that R_T is étale over $V \times_S T$ if and only if the images of the e_i in $(I^{-n}/A) \otimes_\Lambda L$ is zero. But I^{-n}/A is finite and flat as a Λ-module, so it is projective, and can be embedded as a direct summand of a free Λ-modules F. If $J \subseteq \Lambda$ is the ideal generated by the coefficients of the e_i with respect to a basis of F, it is clear that the closed subscheme $S' = \text{Spec } (\Lambda/J)$ has the desired property. ♣

Now let us conclude the proof that $\mathcal{F}_g^d(\gamma, \nu)$ is an algebraic stack, by verifying Artin's axioms ([Artin], Corollary 5.2). First note that Schlessinger's first condition, in the strong form ([Artin] (2.3) condition (S 1')), follows from standard principles. The finiteness condition ([Artin] (2.5) condition (S 2)) will follow from the properness of $\mathcal{C}$ once we construct the deformation space below. The representability of the diagonal ([Artin], Corollary 5.2, condition 1) has just been checked.

Let us verify condition 3 of [Artin], Corollary 5.2 about formal completions.

Let A be a complete local ring with residue field k and maximal ideal $\mathfrak{m}$; set $A_n = A/\mathfrak{m}^{n+1}$, $S_n = \text{Spec } A_n$. Let there be given a fibered surface $\mathcal{X}_n$ on S_n for each n, together with isomorphisms of the pullback of $\mathcal{X}_n$ to S_{n-1} with $\mathcal{X}_{n-1}$. From Grothendieck's existence theorem and the existence of a compatible system of very ample line bundles on the C_n we get that the formal scheme $\{C_n\}$ is algebraizable, that is, there exists a flat family of nodal curves C over Spec A together with isomorphisms $C_n \simeq C \times_S S_n$ compatible with the embeddings $C_{n-1} \hookrightarrow C_n$; this C is unique up a unique isomorphism. Likewise, we get a unique scheme X on C, inducing the formal scheme $\{X_n\}$ over C.

For each n consider the associated stack $\mathcal{C}_n$.

5.4. **Lemma.** *There is a nodal curve C_0', not necessarily geometrically connected, and a flat morphism of k-schemes $C_0' \to \mathcal{C}_0$, which is étale over an open subset of C_0 containing of the nodes.*

Proof. Assume that k is algebraically closed; assume also that we have chosen an isomorphism of the group of roots of 1 in k with $\mathbf{Q}/\mathbf{Z}$. For each nodal point p of C choose a chart $(U_p, Y_p \to C_p, \Gamma_p)$ such that V_p has a single point over p; then Γ_p is a finite cyclic group. Choose a cyclic group Γ whose order is divisible by the order of all the Γ_p, and an embedding $\Gamma_p \subseteq \Gamma$ for each p. Let Δ be a finite set of smooth points of C_0 such that the pointed curve (C_0, Δ) is stable, and call N the set of nodal points of C_0. Call Π the abelianization of the algebraic fundamental group of $C_0 \smallsetminus (N \cup \Delta)$, and for each nodal point p call A_p and B_p the spectra of the completions of the local rings of the normalization of C at the two branches of C at p. The fundamental groups of A_p and B_p are canonically isomorphic to the free rank one profinite group $\widehat{\mathbf{Z}}$; call α_p and β_p the images in Π of the canonical generators under the canonical homomorphisms $\pi_1(A_p) \to \Pi$ and $\pi_1(B_p) \to \Pi$. The fact that $C_0 \smallsetminus (N \cup \Delta)$ is stable insures that the α_p and β_p are part of a basis of the free abelian profinite group Π, so there exists a continuous homomorphism $\Pi \to \Gamma$ sending α_p to a generator of $\Gamma_p \subseteq \Gamma$ and β_p to the opposite generator; call C_0' the associated ramified covering of C_0.

Over the inverse image of N the morphism $\mathcal{C}_0 \to C_0$ is an isomorphism, so the morphism $\pi\colon C_0' \to C_0$ yields a morphism $C_0' \smallsetminus \pi^{-1}(N) \to \mathcal{C}_0$. There is also an isomorphism of some étale neighborhood of any point of C_0' mapping to some $p \in N$ with an étale neighborhood of the inverse image of p in V_p, and each V_p comes with an étale morphism $V_p \to \mathcal{C}_0$; this gives an extension of the morphism $C_0' \smallsetminus \pi^{-1}(N) \to \mathcal{C}_0$ over the inverse image of N to all of C_0', which is étale everywhere except possibly at the points of Δ. This morphism is automatically flat.

If k is not algebraically closed, then consider the pullback of the fibered surface $\mathcal{X}_0$ to the algebraic closure $\overline{k}$. Given a morphism $\overline{C}_0' \to \mathcal{C}_0 \times_{\text{Spec } k} \text{Spec } \overline{k}$ satisfying the desired conditions, this will be obtained by base change from some morphism $C_0' \to \mathcal{C}_0 \times_{\text{Spec } k} \text{Spec } k'$, where k' is a finite extension of k. The desired morphism is the composition of this $C_0' \to \mathcal{C}_0 \times_{\text{Spec } k} \text{Spec } k'$ with the projection $\mathcal{C}_0 \times_{\text{Spec } k} \text{Spec } k' \to \mathcal{C}_0$. ♣

Now starting from C'_0 we can construct, by induction on n, a sequence of flat nodal curves C'_n on $S_n = \text{Spec } A_n$, together with embeddings $C'_{n-1} \hookrightarrow C'_n$, inducing an isomorphism of C'_{n-1} with the restriction of C'_n to S_{n-1}, and proper flat morphisms $\phi_n\colon C'_n \to \mathcal{C}_n$ such that the restriction of ϕ_n is ϕ_{n-1}; for each n call C''_n the fiber product $C'_n \times_{\mathcal{C}_n} C'_n$; then the groupoid $C''_n \rightrightarrows C'_n$ is a flat presentation of the stack $\mathcal{C}_n$. There are flat schemes C' and C'' over A, whose associated formal schemes are precisely $\{C'_n\}$ and $\{C''_n\}$. For each n there is a family $\mathcal{Y}_n \to \mathcal{C}_n$ of ν-pointed stable curves of genus γ on the stack $\mathcal{C}_n$; by pullback we get families of stable curves $Y'_n \to C'_n$ and $Y''_n \to C''_n$, which yield families of pointed stable curves $Y' \to C'$ and $Y'' \to C''$.

The induced flat groupoid $C'' \rightrightarrows C'$ has a quotient Deligne–Mumford stack $\mathcal{C}$ over Spec A; the groupoid $Y'' \rightrightarrows Y'$ gives a flat family of pointed stable curves $\mathcal{X} \to \mathcal{C}$, or, in other words, a morphism $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$. Let us check that the moduli space of $\mathcal{X}$ is X. In fact, the moduli space of $\mathcal{X}_n$ is X_n, and if we are given a morphism $\mathcal{X} \to Z$, where Z is an algebraic space over S, we get a compatible sequence of morphisms $\mathcal{X}_n \to Z$, and from this a compatible sequence of morphisms $X_n \to Z$. The graphs of these morphisms give a formal subscheme of $X \times_S Z$, which corresponds to a subscheme of $X \times_S Z$, the graph of a morphism $X \to Z$. The composition of this with the morphism $\mathcal{X} \to X$ is the given morphism $\mathcal{X} \to Z$.

Call V the locus of C' where the morphism $C' \to \mathcal{C}$ is étale, $U = V/\Gamma$, and call Y the pullback of the family $\mathcal{X} \to \mathcal{C}$ to V. Then $(U, Y \to V, \Gamma)$ is a chart for the coarse fibered surface $X \to C$; it is easy to check that it is balanced, using the fact that the central fiber is balanced.

Now we have to define an obstruction theory $\mathcal{O}$ for the stack $\mathcal{F}^d_g(\gamma,\nu)$. This can be done, using the results of [Illusie]. A direct more elementary approach is also possible, along the lines of [Vistoli3].

Consider a reduced ring A_0, and a surjective homomorphism $A' \to A$ of infinitesimal extensions of A_0, whose kernel M is an A_0-module. Consider a fibered surface $\mathcal{X} \to \mathcal{C}$ over Spec A; to give a lifting of $\mathcal{X}$ to Spec A' is equivalent to giving a flat lifting of the morphism $\mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$ to Spec A'.

This induces a representable morphism of stacks $f\colon \mathcal{C} \to \overline{\mathcal{M}}_{\gamma,\nu}$; we consider the cotangent complex $\mathcal{L} = \mathrm{L}_{\mathcal{C}/\overline{\mathcal{M}}_{\gamma,\nu} \times_{\text{Spec } \mathbf{Q}} \text{Spec } A}$ over A. Because $\overline{\mathcal{M}}_{\gamma,\nu} \times_{\text{Spec } \mathbf{Q}} \text{Spec } A$ is smooth over A and $\mathcal{C}$ is a local complete intersection, it follows that the complex $\mathcal{L}$ is quasi-isomorphic to the two-step complex having $f^*\Omega_{\overline{\mathcal{M}}_{\gamma,\nu}/\mathbf{Q}}$ in degree -1 and $\Omega_{\mathcal{C}/A_0}$ in degree 0, the differential $f^*\Omega_{\overline{\mathcal{M}}_{\gamma,\nu}/\mathbf{Q}} \to \Omega_{\mathcal{C}/A_0}$ being the obvious homomorphism. In what follows we will consider all sheaves over $\mathcal{C}$ as sheaves over the étale site of $\mathcal{C}$, and the various cohomology and Ext groups will be considered with respect to this site.

It follows from [Illusie], Proposition 2.1.2.3., that, in the notation of [Artin], the A_0-module $\mathrm{D}_{\mathcal{X}}(M)$ is $\mathrm{Ext}^1_{\mathcal{O}_{\mathcal{C}}}(\mathcal{L}, \mathcal{O}_{\mathcal{C}} \otimes_A M)$, and that we get an obstruction theory by setting

$$\mathcal{O}_{\mathcal{X}}(M) = \mathrm{Ext}^2_{\mathcal{O}_{\mathcal{C}}}(\mathcal{L}, \mathcal{O}_{\mathcal{C}} \otimes_A M).$$

We have to check that these A_0-modules are finite, and that they satisfy the conditions of [Artin] (4.1).

If $\mathcal{K}$ is the complex $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\mathcal{L}, \mathcal{O}_{\mathcal{C}_0})$ of sheaves over $\mathcal{C}_0$, which is a bounded complex with coherent cohomology on $\mathcal{C}_0$, then we have

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{\mathcal{C}}}(\mathcal{L}, \mathcal{O}_{\mathcal{C}_0} \otimes_{A_0} M) = \mathcal{K} \overset{\mathrm{L}}{\otimes}_{A_0} M$$

and $\mathrm{D}_{\mathcal{X}}(M)$ and $\mathcal{O}_{\mathcal{X}}(M)$ are respectively the first and second hypercohomology group of the total tensor product complex $\mathcal{K} \overset{\mathrm{L}}{\otimes}_A M$.

Consider the projection $\pi\colon \mathcal{C} \to C$. The functor π_* carries sheaves of $\mathcal{O}_{\mathcal{C}}$-module to sheaves of $\mathcal{O}_C$-modules.

5.5. **Lemma.** *The functor π_* carries quasicoherent sheaves to quasicoherent sheaves, coherent sheaves to coherent sheaves, and is exact.*

Proof. The question is local in the étale topology on C_0, so we may pass to a chart, and assume that $\mathcal{C}$ is of the form $[V/\Gamma]$, where V is a scheme and Γ a finite group. Then sheaves on $\mathcal{C}$ correspond to equivariant sheaves on V, and the statement is standard. ♣

The lemma implies that the hypercohomology of the complex $\mathcal{K} \overset{\mathrm{L}}{\otimes}_A M$ is isomorphic to the hypercohomology of the comples of sheaves $\pi_*(\mathcal{K}\overset{\mathrm{L}}{\otimes}_A M) = (\pi_*\mathcal{K})\overset{\mathrm{L}}{\otimes}_A M$ on $C_0 \subseteq C$, which is finite over A_0. To prove the condition [Artin] 4.1.(ii) we can simply apply the cohomological form of Zariski's theorem on formal functions, which can easily be extended to bounded complexes with coherent cohomology.

For condition [Artin] 4.1.(iii) we can localize A_0 and assume that the cohomology sheaves of the bounded complex $(\pi_*\mathcal{K}) \overset{\mathrm{L}}{\otimes}_A M$ are locally free on Spec A_0, because A_0 is reduced. Then the complex $(\pi_*\mathcal{K})\overset{\mathrm{L}}{\otimes}_A M$ is quasi-isomorphic to a complex with zero differentials, and the statement is clear.

Condition [Artin] 4.1.(i) is straightforward.

Thus we have proven that $\mathcal{F}_g^d(\gamma, \nu)$ is a Deligne-Mumford stack.

6. The weak valuative criterion

In this section we prove that $\mathcal{F}_g^d(\gamma, \nu)$ satisfies the weak valuative criterion for properness. Let R be a discrete valuation ring, $S = \text{Spec } (R)$. Let $\eta \in S$ be the generic point, $s \in S$ the special point. For a finite extension of discrete valuation rings $R \subset R_1$, we denote by S_1, η_1, s_1 the corresponding schemes.

6.1. **Proposition.** *Let $\mathcal{X}_\eta \to \mathcal{C}_\eta \to \eta$ be a balanced stable fibered surface. Then there is a finite extension of discrete valuation rings $R \subset R_1$ and an extension*

$$\begin{array}{ccc} \mathcal{X}_\eta \times_S S_1 & \subset & \mathcal{X}_1 \\ \downarrow & & \downarrow \\ \mathcal{C}_\eta \times_S S_1 & \subset & \mathcal{C}_1 \\ \downarrow & & \downarrow \\ \{\eta_1\} & \subset & S_1, \end{array}$$

such that $\mathcal{X}_1 \to \mathcal{C}_1 \to S_1$ is a balanced stable family of fibered surfaces. The extension is unique up to a unique isomorphism, and its formation commutes with further finite extensions of discrete valuation rings.

Proof. We will proceed in steps.

Step 1: extension of C. Let $X_\eta \to C_\eta \to \eta$ be the coarse fibered surface, and let $f_\eta : C_\eta \to \overline{\mathbf{M}}_{\gamma,\nu}$ be the coarse moduli morphism. By [F-P] and [B-M], there is a complete Deligne - Mumford stack $\mathcal{K}_g^d\big(\overline{\mathbf{M}}_{\gamma,\nu}\big)$. By the weak valuative criterion for $\mathcal{K}_g^d\big(\overline{\mathbf{M}}_{\gamma,\nu}\big)$, it follows that there is a finite extension of discrete valuation rings $R \subset R_1$ and a unique extension

$$\begin{array}{ccccc} C_\eta \times_S S_1 & \subset & C_1 & \overset{f_1}{\to} & \overline{\mathbf{M}}_{\gamma,\nu} \\ \downarrow & & \downarrow & & \\ \{\eta_1\} & \subset & S_1 & & \end{array} \tag{3}$$

such that $f_1 : C_1 \to \overline{\mathbf{M}}_{\gamma,\nu}$ is a family of Kontsevich stable maps. The extension 3 is unique up to a unique isomorphism and commutes with further base changes.

We now replace R by R_1. Let $\pi \in R$ be a uniformizer.

STEP 2: EXTENSION OF $\mathcal{C}_\eta \to \overline{\mathcal{M}}_{\gamma,\nu}$ OVER $C_{\rm sm}$. We may assume that over $(C_\eta)_{\rm sm}$ we have a chart with $V_\eta = U_\eta = (C_\eta)_{\rm sm}$ and trivial Γ: such a chart is compatible with any other chart, so we may add it to the atlas. We may extend this via $V = U = C_{\rm sm}$. However the map $\mathcal{C}_\eta \to \overline{\mathcal{M}}_{\gamma,\nu}$ does not necessarily extend over $C_{\rm sm}$, so we may need a further base change.

Let C_i be the components of the special fiber C_s, and let $\xi_i \in C_i$ be the generic points. Consider the localization C_{ξ_i}. This is the spectrum of a discrete valuation ring. By the weak valuative criterion for $\overline{\mathcal{M}}_{\gamma,\nu}$, there is a finite cover $\tilde{C}_i \to C_{\xi_i}$ and a map $\tilde{C}_i \to \overline{\mathcal{M}}_{\gamma,\nu}$ lifting the map on $(C_\eta)_{\rm sm}$.

We proceed to simplify these schemes $\tilde{C}_i$. Denote $S_n = \text{Spec } R[\pi^{1/n}]$. By Abhyankar's lemma [SGA I, exp. XIII section 5] we may assume that $\tilde{C}_i$ is étale over $C_{\xi_i} \times_S S_{n_i}$. By descent we already have a lifting $C_{\xi_i} \times_S S_{n_i} \to \overline{\mathcal{M}}_{\gamma,\nu}$. Taking n divisible by all the n_i, we have an extension $C_{\xi_i} \times_S S_n \to \overline{\mathcal{M}}_{\gamma,\nu}$.

We now replace R by $R[\pi^{1/n}]$. Thus there is an extension $C_{\xi_i} \to \overline{\mathcal{M}}_{\gamma,\nu}$. Note that this extension is unique, since $\overline{\mathcal{M}}_{\gamma,\nu}$ is separated. For the same reason it commutes with further base changes.

By [dJ-O], there is a maximal open set $U \subset C_{\rm sm}$ with an extension of the morphism, $U \to \overline{\mathcal{M}}_{\gamma,\nu}$. Since U contains both $(C_\eta)_{\rm sm}$ and ξ_i, we have $U = C_{\rm sm} \smallsetminus P$ for a finite set of closed points P. By the purity lemma (Proposition 2.1) we have that $U = C_{\rm sm}$, and there is an extension $C_{\rm sm} \to \overline{\mathcal{M}}_{\gamma,\nu}$.

The uniqueness of the lifting in the purity lemma guarantees that this extension is unique and commutes with further base changes.

STEP 3: EXTENSION OF $\mathcal{C}_\eta \to \overline{\mathcal{M}}_{\gamma,\nu}$ OVER NON-GENERIC NODES. Let $p \in \text{Sing}(C)$ be a node, and assume p is not in the closure of $\text{Sing}(C_\eta)$.

To build up a chart near p, we first choose the étale neighborhood $U \to C$ of p, as follows. Let U_0 be a Zariski neighborhood such that $U_0 \cap \text{Sing}(C) = \{p\}$. We already have a morphism $U_0 \to \overline{\mathcal{M}}_{\gamma,\nu}$. We take an étale neighborhood U of p over U_0 so that p is a split node on U. Thus we can find elements s_1, s_2 in the maximal ideal of p, such that $\mathfrak{m}_p = (s_1, s_2, \pi)$, satisfying the equation $s_1 s_2 = \pi^r$ for some $r > 1$. In other words, U is also an étale neighborhood of the closed point $\{s_1 = s_2 = \pi = 0\}$ in $U' = \text{Spec } R[s_1, s_2]/(s_1 s_2 - \pi^r)$.

We now find a nodal curve $V_0 \to S$, which is Galois over U, with an equivariant extension $V_0 \to \overline{\mathcal{M}}_{\gamma,\nu}$. Let $V_0' \to U'$ be defined as follows: $V_0' = \text{Spec } R[t_1, t_2]/(t_1 t_2 - \pi)$, and $s_1 = t_1^r, s_2 = t_2^r$. There is an obvious *balanced* action of the group of r-th roots of unity $\boldsymbol{\mu}_r$ on V_0' via $(t_1, t_2) \mapsto (\zeta t_1, \zeta^{-1} t_2)$, and $V_0' \to U'$ is the associated quotient morphism. Let $V_0 = V_0' \times_{U'} U$. The action of $\boldsymbol{\mu}_r$ clearly lifts to V_0. We write $q_0 : V_0 \to U$ for the quotient map.

Notice that V_0', and therefore V_0, is nonsingular. By the purity lemma we have a lifting $V_0 \to \overline{\mathcal{M}}_{\gamma,\nu}$ of the morphism $f \circ q_0 : V_0 \to \overline{\mathbf{M}}_{\gamma,\nu}$. Notice that for $g \in \boldsymbol{\mu}_r$ we have $f \circ q_0 = f \circ q_0 \circ g$. Thus by the uniqueness of the lifting, the map $V_0 \to \overline{\mathcal{M}}_{\gamma,\nu}$ commutes with the action of $\boldsymbol{\mu}_r$.

Let $Y_0 \to V_0$ be the associated family of stable pointed curves and let $F_0 \subset Y_0$ be the fiber over p. The group $\boldsymbol{\mu}_r$ acts on Y_0, stabilizing F_0. Let $\boldsymbol{\mu}_{r_0} \subset \boldsymbol{\mu}_r$ be the subgroup acting trivially on F_0, and $\Gamma = \boldsymbol{\mu}_r / \boldsymbol{\mu}_{r_0}$ the quotient. Denote by $Y \to V$ the quotient of $Y_0 \to V_0$ by the action of $\boldsymbol{\mu}_{r_0}$. By lemma 3.2 we have: $Y \to V$ is a family of stable nodal curves, Γ acts on $Y \to V$, and this action is essential.

Let $X_U = Y/\Gamma$. By étale descent, there is an algebraic space $X \to C$ such that $X_U \simeq X \times_C U$. The existence of an ample sheaf (lemma 4.30) guarantees that X is a scheme.

Thus we have a diagram

$$\begin{array}{ccccc} Y & \to & X\times_C U & \to & X \\ \downarrow & & \downarrow & & \downarrow \\ V & \to & U & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ S & = & S & = & S \end{array}$$

giving a chart for $\mathcal{X} \to \mathcal{C} \to S$ near p. This is easily seen to be unique and to commute with further base changes.

STEP 4: EXTENSION OF $\mathcal{C}_\eta \to \overline{\mathcal{M}}_{\gamma,\nu}$ OVER GENERIC NODES. Let $p_\eta \in \mathrm{Sing}(C_\eta)$, and let $p \in C_s$ be in the closure of p_η. The construction of a chart here is similar to step 2. We choose a Zariski neighborhood U_0 of p such that $U_0 \cap \mathrm{Sing}(C) = \{p_\eta, p\}$. As before, we may choose an étale neighborhood U over U_0 such that p_η is a split node, thus U is étale over $\{s_1s_2 = 0\}$. We may assume that V_η is given by $t_1t_2 = 0$, where $t_i^r = s_i$ for an appropriate integer r, with the group $\Gamma = \boldsymbol{\mu}_r$. Otherwise we can add a compatible chart with such a V to our atlas. There is an obvious extension $V_\eta \subset V$ via the same equation $\{s_1s_2 = 0\}$. We already have a lifting $V \smallsetminus \{p\} \to \overline{\mathcal{M}}_{\gamma,\nu}$. Since V is normal crossings, the purity lemma applies, and guarantees that there is a unique equivariant extension to V. It is easy to check that this gives an extension of the chart, which is essential (as the Isom scheme of stable curves is finite unramified) and balanced.

The uniqueness and base change properties are straightforward. ♣

7. Boundedness of the category of fibered surfaces

The following lemma gives the basic fact behind boundedness:

7.1. **Lemma.** *Assume that S is the spectrum of an algebraically closed field. A morphism $C \to \overline{\mathbf{M}}_{\gamma,\nu}$ can come from at most finitely many fibered surfaces $\mathcal{X} \to \mathcal{C}$, up to isomorphism.*

Proof. First of all observe that given a family of stable pointed curves $X' \to C_{\mathrm{sm}}$ there are at most finitely many fibered surfaces $\mathcal{X} \to \mathcal{C}$ whose restriction to C_{sm} is isomorphic to X'; so it is enough to prove that any given map $f\colon T \to \overline{\mathbf{M}}_{\gamma,\nu}$, where T is a smooth curve, comes from at most finitely many families on T.

Over $\mathbf{C}$, it is easy to deduce this finiteness result analytically: given a base point $t \in T$ and a fixed lifting $\tilde{f}\colon T \to \overline{\mathcal{M}}_{\gamma,\nu}$, it is easy to associate, in a one to one manner, to any lifting $\tilde{f}'\colon T \to \overline{\mathcal{M}}_{\gamma,\nu}$ a monodromy representation $\phi_{f'} : \pi_1(T) \to \mathrm{Aut} X_t$. Since $\pi_1(T)$ is finitely generated there are only finitely many such representations.

We now give a short algebraic argument using the language of stacks.

Call $\mathcal{T}$ the normalization of the pullback of $\overline{\mathcal{M}}_{\gamma,\nu}$ to T; if there is at least one map $T \to \overline{\mathcal{M}}_{\gamma,\nu}$ giving rise to f then this induces a section $s\colon T \to \mathcal{T}$. It is proved in [Vistoli1] that this section is étale; the argument is similar to that of the purity lemma: by passing to the strict henselization one can lift s to the universal deformation space. The image of this lifting is then evidently étale, and therefore s is étale. This strongly restricts the structure of T.

Call R the fiber product $T \times_{\mathcal{T}} T$; then the étale finite groupoid $R \rightrightarrows T$ is a presentation of $\mathcal{T}$. Each lifting $T \to \overline{\mathcal{M}}_{\gamma,\nu}$ of f comes from a section $T \to \mathcal{T}$, so it is enough to show that up to isomorphisms there are only finitely many such sections. But such a section $\sigma\colon T \to \mathcal{T}$ is determined by the pullback $U$of $T \xrightarrow{s} \mathcal{T}$ to T via σ, and a morphism of groupoids from $U \times_T U \rightrightarrows U$ to $R \rightrightarrows T$ over T. But up to isomorphism there are only finitely many étale coverings $U \to T$ of bounded degree, because the fundamental group of T is topologically finitely generated. For each $U \to T$ there are only finitely many liftings $U \times_T U \to R$. ♣

Continuing the proof of Thorem 5.1, we need to show that $\mathcal{F}_g^d(\gamma,\nu)$ is proper, and that it admits a projective coarse moduli space $\mathbf{F}_g^d(\gamma,\nu)$.

Since we have poven the weak valuative criterion, in order to prove that $\mathcal{F}_g^d(\gamma,\nu)$ is proper, it suffices to show that it is of finite type.

The projectivity of $\mathbf{F}_g^d(\gamma,\nu)$ then follows, since we have a map $\mathbf{F}_g^d(\gamma,\nu) \to \mathbf{K}_g^d\big(\overline{\mathbf{M}}_{\gamma,\nu}\big)$ with finite fibers (Lemma 7.1), which by properness is finite, and since $\mathbf{K}_g^d\big(\overline{\mathbf{M}}_{\gamma,\nu}\big)$ is projective we have that $\mathbf{F}_g^d(\gamma,\nu)$ is projective.

We begin by adressing the geometric points of $\mathcal{F}_g^d(\gamma,\nu)$.

7.2. **Proposition.** *There exists a scheme T of finite type over $\mathbf{Q}$, and a morphism $T \to \mathcal{F}_g^d(\gamma,\nu)$ such that every geometric point $\bar{x} \to \mathcal{F}_g^d(\gamma,\nu)$ is the image of a geometric point $\bar{x} \to T$ on T.*

Sketch of proof. Step 1: a family of maps to $\overline{\mathbf{M}}_{\gamma,\nu}$. We will build on the fact that $\mathcal{K}_g^d\big(\overline{\mathbf{M}}_{\gamma,\nu}\big)$ is a stack of finite type over $\mathbf{Q}$.

Let T be a scheme of finite type over $\mathbf{Q}$ and $T \to \mathcal{K}_g^d\big(\overline{\mathbf{M}}_{\gamma,\nu}\big)$ a morphism which is surjective on geometric points. We have an associated family of stable maps:

$$\begin{array}{ccc} C & \to & \overline{\mathbf{M}}_{\gamma,\nu} \\ \downarrow & & \\ T. & & \end{array}$$

Step 2: Straightenning up C. Passing to a stratification of T, we may assume that

- T is nonsingular, and
- the family $C \to T$ is of locally constant topological type.

Let $\tilde{C} \to C$ be the normalization, and let $\Sigma \subset \tilde{C}$ be the locus lying over the nodes $Sing(C/T)$. Since the topological type is locally constant, $\tilde{C} \to T$ is smooth and $\Sigma \to T$ is étale. Passing to a finite cover of T, we may assume that

- $\tilde{C} = \cup \tilde{C}_i$, where $\tilde{C}_i \to T$ has irreducible geometric fibers, and
- Σ is the union of the images of (disjoint) sections $T \to \tilde{C}$.

Step 3: A finite cover with a map to $\overline{\mathcal{M}}_{\gamma,\nu}$. There exists a projective variety M over $\mathbf{Q}$ and a finite surjective morphism $M \to \overline{\mathcal{M}}_{\gamma,\nu}$. Let $D = \tilde{C} \times_{\overline{\mathbf{M}}_{\gamma,\nu}} M$. Replacing T by a quasi finite scheme mapping onto it, we may assume that the normalization $\tilde{D}$ of D satisfies:

- $\tilde{D} \to S$ has geometrically connected smooth fibers;
- $\tilde{D} \to \tilde{C}$ is flat, branched along the union Δ of disjoint sections; and
- each of the branching sections is either disjoint from Σ or contained in Σ.

Let $\tilde{C}_0 = \tilde{C} \smallsetminus (\Sigma \cup \Delta)$. Denote by $\tilde{D}_0$ the inverse image of $\tilde{C}$ in $\tilde{D}$.

Interlude: discussion of descent data on the open set $\tilde{C}_0$. Let $X \to \tilde{D}$ be the family of pointed curves associated with the morphism $\tilde{D} \to \overline{\mathcal{M}}_{\gamma,\nu}$. Let $\bar{t} \to T$ be a geometric point. Fix any irreducible component $C' \subset (\tilde{C}_0)_{\bar{t}}$ and an irreducible component D' above it. Denote the projection map $\gamma : D' \to C'$. Suppose we have a family of stable pointed curves $Y \to C'$ such that the associated moduli map $C' \to \overline{\mathbf{M}}_{\gamma,\nu}$ is the given one. Then there is a finite étale cover $\alpha : D'' \to D'$ and an isomorphism $\alpha^*\gamma^*Y' \to \gamma^* X_{D'}$. We can choose α so that its degree d is bounded in terms of γ and ν (any bound on the degree of $\mathrm{Isom}_{D'}(X_{D'}, X_{D'})$ will do). Conversely, any such family of curves $Y \to C'$ comes from descent data on $\gamma^* X_{D'}$ over C'. Thus it will be useful to put together such finite étale covers $\alpha : D'' \to D'$.

Note that any such D' has only finitely many étale covers of bounded degree, since its algebraic fundamental group is topologically finitely generated.

Step 4: Choosing families for basing descent data. Replacing T by a surjective quasi finite cover, we may assume that we have a finite collection of finite maps $D^{(i)} \to \tilde{D}$, which are étale over $\tilde{D}_0$, and such that any connected finite étale cover $\alpha : D'' \to D'$ of a component of a geometric fiber of $\tilde{D}_0$, of degree bounded by d as above, appears in one of the $D^{(i)}$. We may assume as usual that $D^{(i)} \to T$ are all smooth. Let $E^{(i)}$ be the normalization of $D^{(i)} \times_{\tilde{C}} D^{(i)}$. It admits two projections $p_1, p_2 : E^{(i)} \to D^{(i)} \to \tilde{D}$.

Step 5: Constructing descent data for the open set. Descent data on $D_0^{(i)}$ for a family of pointed curves over $\tilde{C}_0$ consist of a section of the scheme

$$\underset{E_0^{(i)}}{\mathrm{Isom}}(p_1^*X, p_2^*X)$$

over the open set $E_0^{(i)}$ lying over $\tilde{C}_0$, satisfying the cocycle condition. Note that by our smoothness assumptions we have that any such section extends to a section on $E^{(i)}$. Since $E^{(i)} \to T$ as well as the Isom scheme are projective over T, we have a projective scheme $T' \to T$ (which is in fact finite over T) parametrizing such sections. The cocycle condition puts a closed condition on T'. We replace T' by this closed subscheme.

Thus, there exists a finite scheme $T' \to T$ parametrizing descent data for families of curves over geometric fibers of $\tilde{C}_0$. We have the pulled back family of curves $\tilde{C}' \to T'$, and on the open set $\tilde{C}'_0$ we have a lifting $\tilde{C}'_0 \to \overline{\mathcal{M}}_{\gamma,\nu}$. Moreover, all liftings coming from geometric fibered surfaces appear in this family.

From here on, we can forget about all the D's and E's.

We may again pass to a stratification and assume that T' is nonsingular.

Step 6: Extending the family over a stack. Applying semistable reduction and the purity lemma, it follows that there is a diagram as follows, unique up to étale equivalence:

$$\begin{array}{ccccc} & & \tilde{\mathcal{C}} & \to & \overline{\mathcal{M}}_{\gamma,\nu} \\ & \nearrow & \downarrow & & \downarrow \\ \tilde{C}_0 & \subset & \tilde{C} & \to & \overline{\mathbf{M}}_{\gamma,\nu} \end{array}$$

where $\tilde{\mathcal{C}}$ is a Deligne - Mumford stack, smooth over T; $\tilde{\mathcal{C}} \to \tilde{C}$ is the coarse moduli space, which is isomorphic over $\tilde{C}_0$; and the map $\tilde{\mathcal{C}} \to \overline{\mathcal{M}}_{\gamma,\nu}$ is representable, and extends the given lifting on $\tilde{C}_0$.

There is a subset $T'' \subset T'$, which is open and closed, over which the extension above has the property that $\tilde{\mathcal{C}} \to \tilde{C}$ is an isomorphism over $\pi : \tilde{C} \smallsetminus \Sigma$. Similarly, there is a subset $T''' \subset T''$ over which the stack $\tilde{\mathcal{C}}$ has a chance to be glued together over the nodes, namely: for any pair of sections $s_1, s_2 : T' \to \Sigma$ lying over a family of nodes in C, the automorphism groups of the points in $\mathcal{C}$ over them are of constant orders m_1, m_2, and T''' is the locus where $m_1 = m_2$ for all such pairs.

Step 7: Gluing over nodes. We might as well replace T by T'''. By stratifying T, we might as well assume that the automorphism groups of the families of curves over Σ are locally constant over T. We may also assume that the sections of Σ lift to a universal deformation space for $\tilde{\mathcal{C}}$, which, we may assume, is locally near every node s, a cyclic branched cover of $\tilde{C}$ with Galois group Γ_s.

Gluing data for $\tilde{\mathcal{C}}$ over the nodes now consists of a choice of isomorphism of the objects $s_1, s_2 \in \overline{\mathcal{M}}_{\gamma,\nu}(T)$, plus a choice of isomorphism $\Gamma_{s_1} \to \Gamma_{s_2}$. This data is clearly finite over T.

If we want, we can pass to an open and closed subset over which the isomorphism makes the gluing stack $\mathcal{C}$ balanced. The resulting base is S. ♣

To finish the proof of the theorem: let $p \in S$ be a point. The family $\mathcal{X}_S \to \mathcal{C}_S \to S$ gives an object $\mathcal{X}_p \to \mathcal{C}_p \to p$ in $\mathcal{F}_g^d(\gamma, \nu)(p)$. We have constructed a universal deformation space $V \to \mathcal{F}_g^d(\gamma, \nu)$ for any such object, possessing a family $\mathcal{X}_V \to \mathcal{C}_V \to V$. Using

$$\underset{V \times S}{\mathrm{Isom}}(V \times \mathcal{X}_S \to V \times \mathcal{C}_S, \mathcal{X}_V \times S \to \mathcal{C}_V \times S)$$

we see that there is a nonempty open set U in S such that all geomertric fibers of $\mathcal{X}_U \to \mathcal{C}_U \to U$ appear in $\mathcal{X}_V \to \mathcal{C}_V \to V$. Continuing by noetherian induction we have a scheme V' of finite type over $\mathbf{Q}$ and an étale surjective map $V' \to \mathcal{F}_g^d(\gamma, \nu)$, which is what we need. ♣

8. Fibered surfaces and Alexeev stable maps

8.1. **Semi-log-canonical surfaces.** Semi-log-canonical surfaces are introduced in [K-SB] and further studied in [Al2] and [Al3]. We now review their definition. First, let us define log-canonical pairs.

8.2. **Definition.** Let (X, D) be a pair consisting of a normal variety X and an effective, reduced Weil divisor D. Denote by X_{sm} the nonsingular locus of X. We say that (X, D) is a *log canonical pair* if the following conditions hold:

1. (X, D) is log-$\mathbf{Q}$-Gorenstein, namely: for some positive integer m, we assume that the invertible sheaf $(\omega_{X_{\mathrm{sm}}}(D))^m$ extends to an invertible sheaf $(\omega_{X_{\mathrm{sm}}}(D))^{[m]}$ on X.

2. Let $r : Y \to X$ be a desingularization, such that the proper transform of D together with the exceptional locus of $Y \to X$ form a normal crossings divisor. Call this divisor, taken with reduced structure, D'. Then, we assume that the natural pullback of rational differentials gives a morphism of sheaves: $r^*(\omega_{X_{\mathrm{sm}}}(D))^{[m]} \to (\omega_Y(D'))^m$. That is, a logarithmic differential on X pulls back to a logarithmic differential on Y.

A complete description of log-canonical singularities of surface pairs is given in [Al1].

Semi-log-canonical surfaces are a natural generalization of the above to the case of non-normal surfaces.

8.3. **Definition.** Let X be a surface and D a reduced, pure codimension 1 subscheme such that no component of D lies in $Sing(X)$. We say that (X, D) is a semi-log-canonical if the following conditions hold.

1. X is Cohen Macaulay and normal crossings in codimension 1. Let X_{nc} be the locus where X is either nonsingular or normal crossings.

2. (X, D) is log-$\mathbf{Q}$-Gorenstein, namely: for some positive integer m, we assume that the invertible sheaf $(\omega_{X_{nc}}(D))^m$ extends to an invertible sheaf on X.

3. Let $X' \to X$ be the normalization, D' the Weil divisor corresponding of D on X', and $C \subset X'$ the conductor divisor, namely the divisor where $X' \to X$ is not one-to-one, taken with reduced structure. Then the pair $(X', D' + C)$ is log-canonical.

The unique locally free extension of $(\omega_{X_{nc}}(D))^m$ is denoted by $(\omega_X(D))^{[m]}$

In case D is empty, we just say that X is semi-log-canonical (or "has semi-log-canonical singularities").

We will use the following lemma about quotients of semi-log-canonical surfaces:

8.4. **Lemma.** *Let (Y, D) be a semi-log-canonical surface pair. Let $\Gamma \subset \operatorname{Aut}(Y, D)$ be a finite subgroup, and let $X = Y/\Gamma$, $D_X = D/\Gamma$. Then the following conditions are equivalent:*

1. *The quotient (X, D_X) is semi-log-canonical;*
2. *The pair (X, D_X) is log-$\mathbf{Q}$-Gorenstein.*

The proof is straightforward, see [Kollar2].

The reader is referred to [K-SB] for the refined notions of semismooth, semi-canonical and semi-log-terminal singularities.

8.5. **Nodal families and semi-log-canonical singularities.** A family of nodal curves over a nodal curve is always semi-log-canonical:

8.6. **Lemma.** *Let V be a nodal curve and $f : Y \to V$ a nodal family. Then Y has Gorenstein semi-log-canonical singularities. If, furthermore, $D \subset Y$ is a section which does not meet* $\operatorname{Sing}(f)$*, then (Y, D) is a semi-log-canonical pair.*

Proof. Since any family of nodal curves is a Gorenstein morphism, we have that both $V \to \operatorname{Spec} k$ and $Y \to V$ are Gorenstein morphisms, therefore Y is Gorenstein.

Let $y \in Y$ and $p = f(x) \in V$. It is convenient to replace Y and V by their respective formal completions at y and p. The situation falls into one of the following cases:

1. Both $p \notin \operatorname{Sing}(V)$ and $y \notin Sing(f)$. Then $y \in Y_{ns}$, and there is nothing to prove.
2. $p \notin \operatorname{Sing}(V)$ and $y \in Sing(f)$. Choose a regular parameter t at p. By the deformation theory of a node, we have the description

$$Y \simeq \operatorname{Spf}\ k[[u, v, t]]/(uv - h(t)),$$

with $h(0) = 0$. If $h(t) \not\equiv 0$ then we may write $uv = \mu t^k$ for some unit $\mu \in \mathcal{O}_{V,p}$, in which case (Y, y) is a canonical singularity. If $h(t) \equiv 0$ then (Y, y) is a normal crossing point, which is semismooth.
3. $p \in \operatorname{Sing}(V)$ and $y \notin Sing(f)$. Then (Y, y) is a normal crossings point again.
4. $p \in \operatorname{Sing}(V)$ and $y \in Sing(f)$. We have

$$V \simeq \operatorname{Spf}\ k[[t_1, t_2]]/(t_1 t_2) \quad \text{and} \quad Y \simeq \operatorname{Spf}\ O_V[[u, v]]/(uv - h(t_1, t_2)).$$

We can write $h(t_1, t_2) = h_1(t_1) + h_2(t_2)$, with $h_i(0) = 0$.
(a) If neither h_i is 0, write $h_i(t_i) = \mu_i t_i^{k_i}$, with μ_i units, then (Y, y) is a degenerate cusp with exceptional locus a cycle of rational curves with exactly $\max(k_1 - 1, 1) + \max(k_2 - 1, 1)$ components.
(b) If, say, only $h_2 \equiv 0$, then we have a degenerate cusp with $\max(k_1 - 1, 1) + 2$ exceptional components.
(c) If $h_1 \equiv 0 \equiv h_2$, then we have a degenerate cusp with 4 components.
(See [K-SB] §4, or [SB] for (4a) - (4c).)

The statement about the pair (Y, D) follows easily, since D meets $\operatorname{Sing}(Y)$ transversally at normal crossing points: locally in the étale topology it is isomorphic to the pair

$$(\operatorname{Spec}\ k[t_1, t_2, u]/(t_1 t_2), \operatorname{Spec}\ k[t_1, t_2, u]/(t_1 t_2, u)),$$

which is semi-log-canonical. ♣

For the following lemma assume that the base field is algebraically closed. Here we are interested in balanced quotients of families of nodal curves over nodal base. Studying the resulting singularities becomes easy when we show that the situation can be deformed, which is shown in this lemma.

8.7. **Lemma.** *Let V be the formal completion of a nodal curve and $f : Y \to V$ the formal completion of a nodal family at a closed point. Let $y \in Y$ be the closed point and assume $f(y) = p$ is a node. Suppose a finite cyclic group $\Gamma_y \subset \mathrm{Aut}(Y \to V)$ of order r fixes y, acts faithfully on V stabilizing the two branches of V at p with complementary eigenvalues (in other words, the action is balanced). Then there exists a smoothing*

$$\begin{array}{ccc} Y & \subset & Y' \\ \downarrow & & \downarrow \\ V & \subset & V' \\ \downarrow & & \downarrow \\ \mathrm{Spec}\ k & \subset & \mathrm{Spf}\ k[[s]] \end{array}$$

and a lifting of the action of Γ_y to $Y' \to V'$.

Proof.

1. Suppose $y \notin Sing(f)$. We may choose a parameter u along the fiber so that $Y_y = \mathrm{Spf}\ k[[t_1, t_2, u]]/(t_1 t_2)$. It is not hard to choose u as an eigenvector for Γ_y. The actions is
$$(t_1, t_2, u) \mapsto (\zeta t_1, \zeta^{-1} t_2, \zeta^a u).$$
This clearly lifts to the family given by $t_1 t_2 = s$.
2. Suppose now $y \in Sing(f)$. We have the local equation $uv = h_1(t_1) + h_2(t_2)$. It is easy to choose u, v so that uv is an eigenvector.
   (a) Suppose neither h_i is 0. After a change of coordinates we may assume our local equation is $uv = t_1^{k_1} + t_2^{k_2}, t_1 t_2 = 0$ (so r divides $k_1 + k_2$). We can analyse the action of Γ_y via its action on the fiber $t_1 = t_2 = 0$. Depending upon whether Γ_y stabilizes the branches of u and v or switches them, the action is either $(u, v) \mapsto (\zeta^a u, \zeta^{k_1 - a} v)$ or $(u, v) \mapsto (v, \zeta^{k_1} u)$. In either case this action lifts to the deformation $uv = t_1^{k_1} + t_2^{k_2}, t_1 t_2 = s$.
   (b) Suppose only $h_2 \equiv 0$. We have $uv = t_1^{k_1}, t_1 t_2 = 0$. Again, the possible actions are either $(u, v) \mapsto (\zeta^a u, \zeta^{k_1 - a} v)$ or $(u, v) \mapsto (v, \zeta^{k_1} u)$. As before, the action lifts to the deformation.
   (c) If we have $uv = t_1 t_2 = 0$, the possible actions are either $(u, v) \mapsto (\zeta^a u, \zeta^{k-a} v)$ or $(u, v) \mapsto (v, \zeta^k u)$ for some integers k and a. For any choice of positive k_1, k_2 such that $k_1 - k_2 \equiv k \mod r$ we have that this action lifts to the family $uv = t_1^{k_1} t_2^{k_2}; t_1 t_2 = s$.

Now we look at quotients. First the case of a fixed point which lies in teh smooth locus of a fiber:

8.8. **Lemma.** *Let V be a nodal curve and $f : Y \to V$ a nodal family, let $y \in Y, y \notin \mathrm{Sing}(f)$. Let $\Gamma \subset \mathrm{Aut}Y$ be a balanced finite subgroup, fixing y. Let $q : Y/\Gamma \to X$ be the quotient map and let $x = q(y)$. Then X has a semi-log-terminal singularity at x. If, furthermore, $D \subset Y$ is a section through y which is Γ-stable, then $(X, D/\Gamma)$ is a semi-log-canonical pair.*

The fact that X has a semi-log-terminal singularity is in [K-SB] 4.23(iii). The statement with D follows as in lemma 8.4.

We are left to deal with a fixed point which is a node:

8.9. **Lemma.** *Let V be a nodal curve and $f : Y \to V$ a nodal family, let $y \in Y, y \in \mathrm{Sing}(f)$. Let $\Gamma \subset \mathrm{Aut}Y$ be a balanced finite subgroup, fixing y. Let $q : Y/\Gamma \to X$ be the quotient map and let $x = q(y)$. Then X has a Gorenstein semi-log-canonical singularity at x.*

By lemma ?? it is enough to show that X is Gorenstein, and for this it suffices to show that the quotient of the smoothing Y' in lemma ?? is Gorenstein. The quotient variety Y'/Γ is clearly Cohen-Macaulay, therefore it suffices to show that its canonical divisor class is Cartier.

We are in case (2) of lemma ??. In the cases (2a) and (2b) (respectively (2c)), the sheaf $\omega_{Y'}$ is generated at y by $\frac{du\wedge dv\wedge dt_2}{t_1^{k_1-1}}$ (respectively, $\frac{du\wedge dv\wedge dt_2}{t_1^{k_1-1}t_2^{k_2}}$). The generator is easily seen to be Γ_p-invariant. ♣

We thus obtained:

8.10. **Proposition.** *Let $X \to C$ with sections $s_i : C \to X$ be a coarse fibered surface, $S_i = Im(s_i)$ and $D = \sum S_i$. Then (X, D) is a semi-log-canonical pair.*

8.11. **Alexeev stable maps.** In [Al3], V. Alexeev defined *surface stable maps*, for which he constructed complete moduli spaces. Our goal here is to compare our moduli of fibered surfaces and coarsely fibered surfaces with Alexeev's moduli spaces.

Let X be a reduced, connected projective surface, $D \subset X$ a reduced subscheme of codimension 1. Let $M \subset \mathbf{P}^r$ be a projective scheme.

A morphism $f : X \to M$ is called a *stable map* of the pair (X, D) to M, if

1. the pair (X, D) has only semi-log-canonical singularities; in particular, for some integer $m > 0$ the sheaf $(\omega_X(D))^{[m]}$ is invertible.

2. For a sufficiently large integer n, the sheaf $(\omega_X(D))^{[m]} \otimes f^*\mathcal{O}_M(mn)$ is ample.

It is easy to see that the property of a morphism $f : X \to M$ being stable is independent of the choice of the projective embedding of M.

Note that, given a stable map $f : X \to M$, one has a well defined triple of rational numbers:

$$A = c_1(\omega_X(D))^2; \quad B = c_1(\omega_X(D)) \cdot c_1(f^*\mathcal{O}_M(n)); \quad C = c_1(f^*\mathcal{O}_M(n))^2.$$

One can define a functor of families of stable maps with fixed invariants A, B and C. This is somewhat subtle, since the sheaf $\omega_X(D)$ is not invertible, and saturation does not commute with base change. This can be resolved either by resticting to "allowable" deformations on which the saturation $(\omega_X(D))^{[m]}$ does commute with base change (this is discussed in an unpublished work by Kollár), or by endowing the surface X with the structure of a Deligne-Mumford stack using the log-Gorenstein covers (this has not been carried out in the litterature). Once the functor is defined, one can look for a moduli space. The following result of Alexeev gives the answer:

8.12. **Theorem. [Alexeev].** *Given rationals A, B and C, there is a Deligne-Mumford stack $\mathcal{A}l_{A,B,C}(M)$ admitting a projective coarse moduli space $\mathbf{Al}_{A,B,C}(M)$ for surface stable maps $f : X \to M$ with invariants A, B, C.*

Now let $\mathcal{X} \to \mathcal{C} \to S$ be a balanced fibered surface, $X \to C \to S$ the associated coarse fibered surface. We have a morphism $C \to \overline{\mathbf{M}}_{\gamma,\nu}$, which we can compose with $X \to C$ and obtain a morphism $f : X \to \overline{\mathbf{M}}_{\gamma,\nu}$. In addition to that, we have ν sections $C \to X$. The union of the images of these sectios gives rise to a divisor $D \subset X$. We have already seen that (X, D) is a semi-log-canonical pair. We now claim:

8.13. **Proposition.** *The morphism $f : X \to \overline{\mathbf{M}}_{\gamma,\nu}$ is a stable map of the pair (X, D) to $\overline{\mathbf{M}}_{\gamma,\nu}$.*

Proof. Fix a projective embedding $\overline{\mathbf{M}}_{\gamma,\nu} \in \mathbf{P}^r$. Since $C \to \overline{\mathbf{M}}_{\gamma,\nu}$ is a stable map, we have that $\omega_{C/S} \otimes f^*\mathcal{O}_{\overline{\mathbf{M}}_{\gamma,\nu}}(n)$ is ample for $n \geq 3$. Moreover, the line bundle $\mathcal{L} = \omega_{X/C}^{[m]}$ defined in lemma is nef and relatively ample for $X \to C$. Therefore $\omega_{X/S}^{[m]} \otimes f^*\mathcal{O}_{\overline{\mathbf{M}}_{\gamma,\nu}}(mn)$ is an invertible sheaf, which is relatively ample for $X \to S$. ♣

It is now easy to see that we have a finite morphism

$$\mathcal{F}_g^d(\gamma,\nu)^{balanced} \to \mathcal{A}l_{A,B,C}(\overline{\mathbf{M}}_{\gamma,\nu})$$

for a suitable A, B, C. Here $\mathcal{F}_g^d(\gamma,\nu)^{balanced}$ is teh open substack of balanced fibered surfaces. The image can be viewed as the stack of coarse fibered surfaces.

Since the proof of Alexeev's theorem is quite involved, especially showing boundedness of the stable maps, it is worthwhile seeing that in this particular case we can deduce the existence of the space of coarse fibered surfaces from our work so far.

Indeed, the existence of $\mathcal{F}_g^d(\gamma,\nu)^{balanced}$ implies that there is a scheme Z and a finite surjective morphism $Z \to \mathcal{F}_g^d(\gamma,\nu)^{balanced}$. Over Z we have a family of fibered surfaces, and in particular a family of coarse fiberes surfaces $X \to C \to Z$. We have constructed a relatively ample line bundle L on X; we may replace L by a suitable power and thus assume that L is relatively very ample. After passing to the frame bundle Z' of sections of L, we may assume that the pushforward of L to Z is a free sheaf V. Since $X \to Z$ is flat and embedded in $\mathbf{P}(V)$, we have a morphism $Z \to Hilb$ to a suitable Hilbert scheme. The projective linear group acts with finite stabilizers, therefore the quotient is a Deligne Mumford stack, parametrizing coarse fibered surfaces.

It is interesting to look for properties of the morphism

$$\mathcal{F}_g^d(\gamma,\nu)^{balanced} \to \mathcal{A}l_{A,B,C}(\overline{\mathbf{M}}_{\gamma,\nu})$$

. For instance, it is clearly birational on the closure of the locus of normal fibered surfaces. As it turns out, this is all that we can say: this morphism fails to be one to one, and moreover, it may be ramified, even on the closure of the locus of normal fibered surfaces. To show this, one simply needs to produce examples of (deformable) coarse fibered surfaces admitting incompatible atlases.

8.14. **Example.** Here we give an example of two incompatible balanced charts for a coarse fibered surface. Fix an algebraically closed field k of characteristic 0, and set $S = \operatorname{Spec} k$. Take a smooth projective curve W over k with an automorphism s of order 2 with a fixed point $p \in W(k)$, and let Y_0 be the curve obtained by attaching two copies of W at p. Consider the action of a cyclic group Γ of order 2 on X_0 where a generator acts like s on each copy of W; also let f be the equivariant automorphism of Y_0 which acts like s on one copy of W and as the identity on the other. The point is that f commutes with s, and the automorphism of Y_0/Γ induced by f is the identity.

Let L be a universal deformation space of X_0; Γ acts on L. Let V' be a small étale neighborhood of 0 in $\mathbf{A}^1$, and let Γ act on V', such that a generator sends t to $-t$. Choose a non constant Γ-equivariant map $V' \to L$; this yields a family Y' of stable curves on V' whose fiber over 0 is exactly X_0; Γ also acts on Y. Let V be the union of two copies of V' glued at 0; there are two families Y_1 and Y_2 on V obtained by attaching two copies of Y' at X_0, one using the identity, the other using f. These will not be isomorphic in general, not after an automorphism of V, and not even after going to an étale neighborhood of 0 in V.

Set $U = V/\Gamma$ and call X the union of two copies of Y'/Γ along X_0/Γ; we claim that the two generically fibered surfaces $X_1 = Y_1/\Gamma \to V/\Gamma = U$ and $X_2 = Y_2/\Gamma \to V/\Gamma = U$ are canonically isomorphic to $X \to U$. In fact the structure sheaf $O_{X_i'}$ fits into an exact sequence

$$0 \longrightarrow O_{X_i'} \longrightarrow O_{Y'} \times O_{Y'} \longrightarrow O_{X_0} \longrightarrow 0,$$

where the map $O_{Y'} \times O_{Y'} \longrightarrow O_{X_0}$ is the difference of the two projections for $i = 1$, and the difference of one projection with the other projection twisted by f for $i = 2$. Now take invariants; we get an exact sequence

$$0 \longrightarrow O_{X_i'/\Gamma} \longrightarrow O_{Y'/\Gamma} \times O_{Y'/\Gamma} \longrightarrow O_{X_0/\Gamma} \longrightarrow 0;$$

where the two maps $O_{Y_1/\Gamma} \times O_{Y_2/\Gamma} \to O_{X_0/\Gamma}$ are equal, because f induces the identity on $O_{X_0/\Gamma}$.

So $(U, Y_1 \to V, \Gamma)$ and $(U, Y_2 \to V, \Gamma)$ are incompatible charts for the surface $X \to U$. If we call $\overline{U}$ the union of two disjoint copies of U, $\overline{V}$ the union of two disjoint copies of V and $\overline{Y} \to V$ the family of stable curves which coincides with Y_1 on one copy and with Y_2 on the other one, we even obtain an example $(\overline{U}, \overline{Y} \to V, \Gamma)$ of a chart that is not compatible with itself. This is not surprising, because we are using the étale topology, and so U is not in general embedded in C; it could not happen if we were working over $\mathbf{C}$ with the analytic topology.

8.15. **Example.** We now give an example of two incompatible charts on a fibered surface over $S = \text{Spec } k[\epsilon]/(\epsilon^2)$, which coincide modulo ϵ. We then globalize this example to a complete curve. This implies, in particular, that the map $\mathcal{F}_g^d(\gamma, \nu) \to Al(\overline{\mathbf{M}}_{\gamma,\nu})$ is not always unramified.

Let $F = \text{Spec } k[u, w]/(uw)$. We will use F as a constant fiber in charts $Y \to V$, so that $Y = F \times V$. Specifically, consider the following two curves: $V_0 = \text{Spec } k[z, t, \epsilon]/(zt, \epsilon^2)$ and $V_\epsilon = \text{Spec } k[z, t, \epsilon]/(zt - \epsilon, \epsilon^2)$. Define $Y_0 = F \times V_0$ and $Y_\epsilon = F \times V_\epsilon$. We have obvious morphisms $Y_0 \to V_0 \to S$ and $Y_\epsilon \to V_\epsilon \to S$. The fibers over $\text{Spec } k \subset B$ are clearly isomorphic, but Y_0 and Y_ϵ are clearly non-isomorphic.

We now define an action of C_6, the cyclic group of order 6, on these schemes. Let ζ be a primitive sixth root of 1. We choose a generator of C_6 and call it ζ as well; define its action as follows:

$$(u, w, z, t, \epsilon) \mapsto (\zeta^3 u, \zeta^2 w, \zeta z, \zeta^{-1} t, \epsilon)$$

This clearly defines an action of C_6 on both $Y_0 \to V_0 \to S$ and $Y_\epsilon \to V_\epsilon \to S$.

Let $C_0 = V_0/C_6$. Explicitly,

$$C_0 = \text{Spec } k[z^6, t^6, \epsilon]/(z^6 t^6, \epsilon^2).$$

Similarly let $C_\epsilon = V_\epsilon/C_6$. It is easy to see that we similarly have

$$C_\epsilon = \text{Spec } k[z^6, t^6, \epsilon]/(z^6 t^6, \epsilon^2).$$

Thus we can use the notation

$$C = V_0/C_6 = V_\epsilon/C_6.$$

We denote $X_0 = Y_0/C_6$ and $X_\epsilon = Y_\epsilon/C_6$. Clearly $X_0 \to C \to S$ and $X_\epsilon \to C \to S$ are coarse fibered surfaces over S. Our main claim is that they are isomorphic.

We will produce an isomorphism of coarse fibered surfaces by chosing isomorphisms over suitable open sets, showing that these glue together over a large open set, and arguing that an isomorphism on the large open set must extend. Let us first work out the open sets.

We denote by U_0 the localization of Y_0 at u, $\ldots$, T_ϵ the localization of Y_ϵ at t. Explicitly,

$$\begin{aligned}
U_0 &= \text{Spec } k[u, u^{-1}, z, t, \epsilon]/(zt, \epsilon^2) \\
W_0 &= \text{Spec } k[w, w^{-1}, z, t, \epsilon]/(zt, \epsilon^2) \\
Z_0 &= \text{Spec } k[u, w, z, z^{-1}, \epsilon]/(uw, \epsilon^2) \\
T_0 &= \text{Spec } k[u, w, t, t^{-1}, \epsilon]/(uw, \epsilon^2) \\
U_\epsilon &= \text{Spec } k[u, u^{-1}, z, t, \epsilon]/(zt - \epsilon, \epsilon^2) \\
W_\epsilon &= \text{Spec } k[w, w^{-1}, z, t, \epsilon]/(zt - \epsilon, \epsilon^2) \\
Z_\epsilon &= \text{Spec } k[u, w, z, z^{-1}, \epsilon]/(uw, \epsilon^2) \\
T_\epsilon &= \text{Spec } k[u, w, t, t^{-1}, \epsilon]/(uw, \epsilon^2)
\end{aligned}$$

These open sets are clearly stable under the action of C_6.

The expressions above immediately give isomorphisms $Z_0 \simeq Z_\epsilon$ and $T_0 \simeq T_\epsilon$. These canonically induce isomorphisms $Z_0/C_6 \simeq Z_\epsilon/C_6$ and $Z_0/C_6 \simeq Z_\epsilon/C_6$.

Let us address the open sets U_0 and U_ϵ. These are clearly nonisomorphic, but we claim that their quotients by the subgroup of two elemets are isomorphic. Indeed, note that ζ^3 acts trivially on u, u^{-1}. Therefore

$$\begin{aligned} U_\epsilon/C_2 &\simeq \text{Spec } k[u,u^{-1}] \times V_\epsilon/C_2 \\ &= \text{Spec } k[u,u^{-1}] \times \text{Spec } k[z^2,t^2,zt,\epsilon]/(zt-\epsilon,\epsilon^2) \\ &= \text{Spec } k[u,u^{-1}] \times \text{Spec } k[z^2,t^2,\epsilon]/(\epsilon^2) \\ &\simeq \text{Spec } k[u,u^{-1}] \times V_0/C_2 \\ &\simeq U_0. \end{aligned}$$

Taking the quotient by the residual C_3 action, this isomorphism clearly induces an isomorphism

$$U_\epsilon/C_6 \simeq U_0/C_6.$$

Note that on the intersections $U_\epsilon/C_2 \cap Z_\epsilon/C_2$ and $U_0/C_2 \cap Z_0/C_2$ the isomorphism above coincides with the restriction of the isomorphism $Z_\epsilon/C_2 \simeq Z_0/C_2$. Indeed, both are given by identifiying the variables u, z^2 and ϵ. Therefore these isomorphisms coincide on $U_\epsilon/C_6 \cap Z_\epsilon/C_6$ as well. A similar situation is obtained by replacing Z_ϵ by T_ϵ.

We now address the open sets W_ϵ and W_0 in a similar manner, noting that this time C_3 acts trivially on the variable w. Thus

$$\begin{aligned} W_\epsilon/C_3 &\simeq \text{Spec } k[w,w^{-1}] \times V_\epsilon/C_3 \\ &= \text{Spec } k[w,w^{-1}] \times \text{Spec } k[z^3,t^3,zt,\epsilon]/(zt-\epsilon,\epsilon^2) \\ &= \text{Spec } k[w,w^{-1}] \times \text{Spec } k[z^3,t^3,\epsilon]/(\epsilon^2) \\ &\simeq \text{Spec } k[w,w^{-1}] \times V_0/C_3 \\ &\simeq W_0. \end{aligned}$$

Again, it is easy to check that this isomorphism agrees on the interstections with the open sets Z_ϵ and T_ϵ. We note that all these isomorphisms are isomorphisms over the curve C, which coincide with the identity modulo ϵ.

Let X'_ϵ be the open subscheme which is the complement of the image of the point $u = w = z = t = 0$ on Y_ϵ; define X'_0 in an analogous way. Thus far we have obtained an isomorphism $\phi' : X'_\epsilon \to X'_0$. The closure $\Sigma \subset X_\epsilon \times_C X_0$ of the graph of ϕ' is a subscheme supported along the graph of the identity $(X_\epsilon)_{\text{Spec } k} = (X_0)_{\text{Spec } k}$. Therefore the projections $p_\epsilon : \Sigma \to X_\epsilon$ and $p_0 : \Sigma \to X_0$ are finite. Note also p_ϵ and p_0 are isomorphic along a dense open set. Since the schemes X_0 and X_ϵ satisfy Serre's condition S_2, these projection admit sections, giving rise to a morphism $\phi : X_\epsilon \to X_0$, which by the same reason is an isomorphism.

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