# Fibrations by nodal curves and arithmetic applications

My preprints available/linked in my home page:
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**Outline**:

1. Smooth and mild introduction.

2. Statement of theorem.

3. Motivation for the theorem.

4. Something about the proof.

5. Problems.

**ALL VARIETIES OVER C**

# Smooth and Mild Introduction.

"Definition": a dominant morphism of varieties $X \to B$ is **smooth** if for any dominant base change $B_1 \to B$ where $B_1$ is **nonsingular**, the pull back $X \times_B B_1$ is **nonsingular**.

"Corollary": A composition of smooth morphisms is smooth.
"Corollary": Fibered powers of smooth morphisms are smooth

**Definition:** a dominant morphism of varieties $X \to B$ is **mild** if for any dominant base change $B_1 \to B$ where $B_1$ is **r-G**, the pull back $X \times_B B_1$ is **r-G**.

Here r-G = *rational Gorenstein singularities.*

"Corollary": A composition of mild morphisms is mild.
"Corollary": Fibered powers of mild morphisms are mild.

$X$ Gorenstein: There is an invertible sheaf $\omega_X$ satisfying duality
(so $\omega_X^\bullet = \omega_X$).

Rational singularities: for a resolution of singularities $r : X_1 \to X$, one has $r_*(\omega_{X_1}) = \omega_X^\bullet$.

Definition: $X \to B$ is **nodal** if it is a generically smooth family of curves with at most nodes (ordinary double points).

Proposition: A nodal family is mild.
Corollary: a plurinodal family is mild.

Proof of proposition:

1. Nodal families are always Gorenstein morphisms (Rosenlicht).

2. If $B$ is smooth and the discriminant is a d.n.c., the singularities are

$$uv = t_1^{k_1} \cdots t_r^{k_r}$$

which are toric, therefore rational.

3. In general: Let $r : B_1 \to B$ be a good resolution, $X_1 = X \times_B B_1$:

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X \\ \downarrow \pi_1 & & \downarrow \pi \\ B_1 & \xrightarrow{r} & B \end{array}$$

It's enough to show $f_*\omega_{X_1} = \omega_X$. Now:

$$\begin{aligned} f_*\omega_{X_1} &= f_*\left(\omega_{\pi_1} \otimes \pi_1^*\omega_{B_1}\right) \\ &= f_*\left(f^*\omega_\pi \otimes \pi_1^*\omega_{B_1}\right) \\ &= \omega_\pi \otimes f_*\pi_1^*\omega_{B_1} \\ &= \omega_\pi \otimes \pi^* r_*\omega_{B_1} \\ &= \omega_\pi \otimes \pi^*\omega_B \\ &= \omega_X \end{aligned}$$

**Theorem 0.1** (Fibered power theorem)**.** *Let $X \to B$ be a smooth family of positive dimensional varieties of general type, with $B$ irreducible. Then there exists an integer $n > 0$, a positive dimensional variety of general type $W_n$, and a dominant rational map $X^n_B \dashrightarrow W_n$.*

**Theorem 0.2.** *Let $X \to B$ be a smooth family of positive dimensional varieties of general type, with $B$ irreducible, and $\mathrm{Var}(X/B) = \dim B$. Then there exists an integer $n > 0$ such that the fibered power $X^n_B$ is of general type.*

Arithmetic motivation

**Lang's Conjecture:** Let $X$ be a variety of general type defined over a number field $K$. Then the set of rational points $X(K)$ is not Zariski-dense in $X$.

Caporaso - Harris - Mazur: Lang's conjecture implies strong uniformity results for curves.

**Theorem 0.3** (Pacelli)**.** *Assume that Lang's conjecture is true. Then the number of $C(K)$ is bounded solely in terms of the genus and the degree of $K$.*

Theorem 0.1 gives analogues for higher dimensional veriaties of Pacelli's theorem, as well as some other applications for curves (joint work with Felipe Voloch).

Sketch of proof in case $X \to B$ is complete and mild

May assume $B$ nonsingular.

The big guns provided by Kollár, Viehweg:

**Theorem 0.4.** *Assume* $\mathrm{Var}(X/B) = \dim B$. *Then* $\omega_{X/B}$ *is big.*

Need to take care of $\omega_B$.

Bigness implies: for large $k$, we have $\omega_{X/B}^{\otimes k} \otimes \omega_B$ is big as well (Kodaira's lemma).

Write $X_k = X_B^k$, $\pi_k : X_k \to B$, and projections $p_i : X_k \to X$.

We have $\omega_{X_k/B} = \otimes p_i^* \omega_{X/B}$ - so:

$$\otimes(p_i^* \omega_{X/B}^{\otimes k} \otimes \pi^* \omega_B) = \omega_{X_k/B}^{\otimes k} \otimes \pi_k^* \omega_B^{\otimes k} = \omega_{X_k}^{\otimes k}$$

IS BIG!

If $\pi : X \to B$ is mild, then $X_k$ is r-G. Let $r : X_k' \to X_k$ be a resolution of singularities.

Since $\omega_{X_k} = r_* \omega_{X_k'}$ is big, we get that $\omega_{X_k'}$ is big, therefore $X_k$ is of general type, Q.E.D!!

Moral: after taking fibered powers, the bigness of $\omega_{X/B}$ can swallow negativity of anything from $B$.

But in general $X \to B$ is not mild!

Question: Does it always have a mild model after base change?

Partial answer: Yes if one allows to cover $X$ as well:

**Galois pluri-nodal reduction lemma.** There exists a diagram

$$\begin{array}{ccccc} Y & \to & X_1 & \to & X \\ & \searrow & \downarrow & \tilde{\square} & \downarrow \\ & & B_1 & \to & B \end{array},$$

and a finite group $G \subset \mathrm{Aut}_{X_1} Y$ such that $B_1 \to B$ is an alteration, $Y/G \to X_1$ is birational and $Y \to B_1$ is a pluri-nodal fibration.

This follows de Jong.

From here there is a technical part about comparing $\omega_{Y_k}$ with $\omega_{X_k}$.

# Ramification

Suppose $V$ is r-G. Given $G \subset \mathrm{Aut}(V)$, we have a diagram

$$\begin{array}{ccc} & & V \\ & & \downarrow q \\ r : W_1 & \to & W \end{array},$$

where $q : V \to W$ is the quotient map, and $r : W_1 \to W$ a resolution.

There is an ideal sheaf $J_n(V, G)$ on $V$ satisfying

$$J_n(V, G) \otimes \omega_V^n \cong q^* r_* \omega_{W_1}^n.$$

Lemma [~CHM]: $J_n(V, G) \supset q^{-1} I_B^n$, where $B \subset W$ is the branch locus.

In our case:

$$\begin{array}{ccccc} & & Y & & \\ & & \downarrow q & & \\ X_2 & \xrightarrow{r} & X_1 & \xrightarrow{g_0} & X \\ & \searrow^{\pi_2} & \downarrow & & \downarrow \\ & & B_1 & \to & B \end{array}$$

Any locus where $\pi_2$ is mild is no problem with the products. The lemma shows that the complement (contained in the discriminant locus of $\pi_2$) is controlled by $I_\Delta^n$, for an arbitrary fibered power of $X_2$.