A linear lower bound on the gonality of modular curves

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0. INTRODUCTION

0.1. **Statement of result.** In this note we prove the following:

Theorem 0.1. *Let $\Gamma \subset PSL_2(\mathbb{Z})$ be a congruence subgroup, and X_Γ the corresponding modular curve. Let $D_\Gamma = [PSL_2(\mathbb{Z}) : \Gamma]$ and let $d_{\mathbb{C}}(X_\Gamma)$ be the $\mathbb{C}$-gonality of X_Γ. Then*

$$\frac{7}{800} D_\Gamma \leq d_{\mathbb{C}}(X_\Gamma).$$

For $\Gamma = \Gamma_0(N)$ we have that $d_{\mathbb{C}}(X_{\Gamma_0(N)})$ is bounded below by $\frac{7}{800} \cdot N$. Similarly, we obtain a quadratic lower bound in N for $d_{\mathbb{C}}(X_{\Gamma_1(N)})$.

0.2. **Remarks.** The proof, which was included in the author's thesis [ℵ], follows closely a suggestion of N. Elkies. In the exposition here many details were added to the argument in [ℵ].

We utilize the work [L-Y] of P. Li and S. T. Yau on conformal volumes, as well as the known bound on the leading nontrivial eigenvalue of the non-euclidean Laplacian $\lambda_1 \geq \frac{21}{100}$ [L-R-S]. If Selberg's eigenvalue conjecture is true, the constant 7/800 above may be replaced by 1/96.

Since, by the Gauss - Bonnet formula, the genus $g(X_\Gamma)$ is bounded by $D_\Gamma/12+1$ (indeed the difference is $o(D_\Gamma)$), we may rewrite the inequality above in the slightly weaker form

$$\frac{21}{200}(g(X_\Gamma) - 1) \leq d_{\mathbb{C}}(X_\Gamma).$$

For an analogous result about Shimura curves, see theorem 1.1 below.

It should be noted (as was pointed out by P. Sarnak) that the gonality has an *upper* bound of the same type. For the $\mathbb{C}$-gonality, by Brill-Noether theory [K-L] we have $d_{\mathbb{C}}(X_\Gamma) \leq 1 + \left[\frac{g+1}{2}\right]$. If, instead, one is interested in the gonality over the field of definition of X_Γ, one can use the canonical linear series to obtain the upper bound $2g-2$ if $g > 1$, and in the few cases where $g = 1$ one can use the morphism to $X(1)$ and get the upper bound D_Γ.

0.3. **Acknowledgements.** As mentioned above, I am indebted to Noam Elkies for the main idea. The question was first brought to my attention in a letter by S. Kamienny. The result first appeared in my thesis under the supervision of Prof. J. Harris. Thanks are due to David Rohrlich and Glenn Stevens who set me straight on some details, and to Peter Sarnak for helpful suggestions.

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1. Setup and proof

1.1. **Gonality.** Let C be a smooth, projective, absolutely irreducible algebraic curve over a field K. Define the K-**gonality** $d_K(C)$ of C to be the minimum degree of a finite K-morphism $f : C \to \mathbb{P}^1_K$. Clearly if $K \subset L$ then $d_K(C) \geq d_L(C \times_K L)$, and equality must hold whenever K is algebraically closed.

1.2. **Congruence subgroups and modular curves.** By a **congruence subgroup** $\Gamma \subset PSL_2(\mathbb{Z})$ we mean that for some N, Γ contains the principal congruence subgroup $\Gamma(N)$ of 2×2 integer matrices congruent to the identity modulo N.

Since $PSL_2(\mathbb{R})$ acts on $\mathbb{H} = \{z = x + iy | y > 0\}$ via fractional linear transformations, we may let $Y_\Gamma = \Gamma \setminus \mathbb{H}$. It is well known that Y_Γ may be compactified by adding finitely many points, called **cusps**, to obtain a compact Riemann surface X_Γ, which we call the **modular curve** corresponding to Γ.

1.3. **The Poincaré metric.** The upper half plane $\mathbb{H}$ carries the Poincaré metric $ds^2 = \frac{dx^2+dy^2}{y^2}$, which is $PSL_2(\mathbb{R})$ - invariant. The corresponding area form is given by $\frac{dx\,dy}{y^2}$. Away from a finite set T consisting the cusps and possibly some elliptic fixed points, the metric descends to a Riemannian metric on $X_\Gamma \setminus T$, of finite area. We denote the area measure by $d\mu$.

We will accordingly call a quadratic differential ds^2 a **singular metric** if it is a Riemannian metric away from finitely many points, and has finite area. Thus the Poincaré metric gives rise to a singular metric on X_Γ.

1.4. **The Laplacian.** It is natural to consider the Hilbert space $L_2(\Gamma \setminus \mathbb{H}) = L_2(X_\Gamma)$, where the L_2 pairing is taken with respect to the Poincaré metric. The Laplace-Beltrami operator associated with the metric

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

gives rise to a self adjoint unbounded operator on $L_2(X_\Gamma)$, which is in fact positive semidefinite.

The kernel of Δ consists of the constant functions. In contrast with the case of a genuine Riemannian metric on a compact manifold, the spectrum of Δ is not discrete (see e.g. [H], VI§9, VII§2, VIII§5). The continuous spectrum is $\{\lambda \geq 1/4\} \subset \mathbb{R}$, and is fully accounted for by an integral formula involving Eisenstein series $E(z,s)$ for $Re(s) = 1/2$. The discrete part of the spectrum is given by $\lambda_0 = 0$ corresponding to the constants, and $0 < \lambda_1 < \lambda_2 ...$ corresponding to the so called **cuspidal** eigenvectors.

1.5. **Selberg's conjecture.** The question, what is λ_1 turns out to be a fundamental one. Selberg [S] has shown that $\lambda_1 \geq 3/16$ and conjectured that $\lambda_1 \geq 1/4$. Recently, Luo, Rudnick and Sarnak [L-R-S] showed that $\lambda_1 \geq 0.21$ (note that $3/16 < 0.21 < 1/4$).

Since the continuous spectrum is known to be $\lambda \geq 1/4$, denote by $\lambda'_1 = \min(\lambda_1, 1/4)$. The value of λ'_1 has the following characterization:

Let g be a nonzero continuous, piecewise differentiable function on X_Γ such that ∇g is square integrable with respect to μ, and $\int_{X_\Gamma} g\,d\mu = 0$. Then (identifying X_Γ with $\Gamma \setminus \mathbb{H}$) we have

$$\int_{\Gamma\setminus\mathbb{H}} \left(\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 \right) dx\,dy \geq \lambda'_1 \int_{\Gamma\setminus\mathbb{H}} g^2 \frac{dx\,dy}{y^2}.$$

This is, in fact, the way Selberg originally stated his result.

1.6. **Conformal area.** Let C be a compact Riemann surface. Following [L-Y], we define the **conformal area**, or the first conformal volume $A_c(C)$ to be the infimum of $\int_C f^*d\mu_0$, where $f : C \to \mathbb{P}^1_{\mathbb{C}}$ runs over all nonconstant conformal mappings, and where $d\mu_0$ is the SO_3-invariant area element on the Riemann sphere. Using the conformal property of homotheties in $\mathbb{P}^1$, Li and Yau show easily that

$$A_c(C) \leq 4\pi \cdot d_{\mathbb{C}}(C).$$

On the other hand, given a Riemannian metric on C, let $A(C)$ be the area of C. Using an elegant fixed point argument, Li and Yau obtain ([L-Y], Theorem 1)

$$\lambda_1 A(C) \leq 2A_c(C).$$

Their proof works word for word in the case of our singular metric on X_Γ, once we replace λ_1 by λ_1'. All that is needed is, first, the characterization of λ_1' discussed above, and second, the fact that differentiable functions on X_Γ have a square-integrable gradient. The latter follows since $\int_{X_\Gamma} |\nabla g|^2 d\mu$ is invariant under conformal change of the metric, therefore it may be calculated using a regular metric, and thus is finite.

1.7. **Conclusion of the proof.** Since the Poincaré metric on X_Γ is pulled back from $X_{PSL_2(\mathbb{Z})} = X(1)$, we have $A(X_\Gamma) = D_\Gamma \cdot A(X(1)) = D_\Gamma \cdot \pi/3$. Combine this with the inequalities of Li and Yau, and obtain the first part of the theorem. Now note that $[PSL_2(\mathbb{Z}) : \Gamma_0(N)]$ is at least N, and similarly $[PSL_2(\mathbb{Z}) : \Gamma_1(N)]$ is quadratic in N (between $6(N/\pi)^2$ and N^2), and obtain the second part. $\square$

1.8. **An analogous result for Shimura curves.** As was pointed out by P. Sarnak, we have the following:

Theorem 1.1. *Let D be an indefinite quaternion algebra over $\mathbb{Q}$, and let G be the group of units of norm 1 in some order of D. Let $\Gamma \subset G$ be a subgroup of finite index, and let $X_\Gamma = \Gamma\backslash\mathbb{H}$ be the corresponding Shimura curve. Then*

$$\frac{21}{200}(g(X_\Gamma) - 1) \leq d_{\mathbb{C}}(X_\Gamma).$$

Proof. Since X_Γ is compact, every automorphic form g appearing in $L^2(X_\Gamma)$ is cuspidal. It follows from the Jacquet - Langlands correspondence (see [G], Theorem 10.1 and Remark 10.4) that unless g is the constant function, there exists a cuspidal automorphic form for some congruence subgroup in $SL_2(\mathbb{Z})$ which has the same eigenvalue with respect to the non-euclidean Laplacian. Therefore $\lambda_1 \geq 0.21$ holds for X_Γ. The results of Li and Yau give $\lambda_1 A(X_\Gamma) \leq 8\pi \cdot d_{\mathbb{C}}(X_\Gamma)$, and the Gauss - Bonnet formula gives $4\pi(g(X_\Gamma)) - 1) \leq A(X_\Gamma)$ (the difference coming from elliptic fixed points). Combining the three inequalities we obtain the result. $\square$

The author was informed that the results of [L-R-S] were generalized by Rudnick and Sarnak to cuspidal automorphic forms on GL_2 over an arbitrary number field F. Therefore Theorem 1.1 holds for D a quaternion algebra over a totally real field, which is indefinite at exactly one infinite place.

2. Applications and remarks

2.1. **$\mathbb{Q}$-gonality and rational torsion on elliptic curves.** Let C be a curve as in 1.1. Recall [ℵ-H] that a point $P \in C$ is called **a point of degree** d if $[K(P) : K] = d$. Suppose C has infinitely many points of degree d. By taking Galois orbits on the d-th symmetric power of

C we have that $\mathrm{Sym}^d(C)(K)$ is infinite. Let $W_d(C) \subset Pic^d(C)$ be the image of $\mathrm{Sym}^d(C)(K)$ by the Abel-Jacobi map. In [ℵ-H] it was noted that in this situation either $d_K(C) \leq d$, or $W_d(C)(K)$ is infinite.

Now assume K is a number field. By a celebrated theorem of Faltings [Fa], if $W_d(C)(K)$ is infinite then $W_d(C) \subset Pic^d(C)$ contains a positive dimensional translate of an abelian variety, and the simple lemma 1 of [ℵ-H] implies that $d_K(C) \leq 2d$ ([ℵ], theorem 9). The latter conclusion was also obtained by G. Frey in [Fr1].

We now restrict attention to the case where $K = \mathbb{Q}$ and $C = X_0(N)$. In [ℵ], Theorem 12, as well as in [Fr1], it was noted that a lower bound on the $\mathbb{Q}$-gonality, such as given by theorem 0.1, implies that there exists a constant $m(d)$ (in fact, $m = 230d$ will do), such that if $N > m(d)$ then $X_0(N)$ (and thus also $X_1(N)$) has finitely many points of degree d. In section 1 of [K-M], Kamienny and Mazur showd that this reduces the uniform boundedness conjecture on torsion points on elliptic curves to bounding rational torsion of prime degree. The conjecture was finally proved by L. Merel in [M].

It should be remarked that, since for this application one only needs a lower bound on the $\mathbb{Q}$-gonality of $X_0(N)$, one can use other methods, such as Ogg's method [O]. This is indeed the method used by Frey in [Fr1], although the bound obtained is not linear. For points of low degree, one can use the main results of [ℵ-H] with Ogg's method to slightly improve the bound on N (see [H-S] and [ℵ], 2.5).

For another arithmetic application of the lower bound on teh $\mathbb{C}$-gonality, regarding pairs of elliptic curves with with isomorphic mod N representations, see Frey [Fr2].

2.2. **Torsion points: the function field case.** Recently, there has been renewed interest in the question of $\mathbb{C}$-gonality of modular curves. In their paper [N-S], K. V. Nguyen and M.-H. Saito used algebraic techniques to give a lower bound on the gonality. Although their bound is a bit weaker than ours, their methods are of interest on their own right: they combine Ogg's method with a Castelnuovo type bound. They pointed out that given any such bound, one obtains a function field analogue of the strong uniform boundedness theorem about torsion on elliptic curves, namely: given a non-isotrivial elliptic curve over the function field of a complex curve B, the size of the torsion subgroup is bounded solely in terms of the gonality of B. This result is strikingly analogous to a recent result of P. Pacelli ([P], Theorem 1.3): assuming Lang's conjecture on rational curves on varieties of general type, the number of non-constant points on a curve C of genus > 1 over the function field of B is bounded solely in terms of the genus of C and the gonality of B.

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