Fibrations by nodal curves and applications

De Jong's papers on
`ftp://ftp.math.harvard.edu/pub/AJdeJong`
My preprints available/linked in my home page:
`http://math.bu.edu/INDIVIDUAL/abrmovic`

Let X be a variety over a field k. An **alteration** of X is a dominant proper generically finite morphism $X' \to X$.

Theorem 0.1 (A. J. de Jong)**.** *Any variety has a regular alteration.*
Moreover, given a closed subset $Z \subset X$, the alteration may be chosen so that the reduced inverse image Z' is a strict divisor of normal crossings.

Main tool:

Definition: A projective family $X \to B$ is **nodal** if it is generically smooth and connected, and every fiber is a curve with at most nodes.

Theorem 0.2. *Let $X \to B$ be a proper morphism between varieties, whose geometric generic fiber is a smooth irreducible curve, and $Z \subset X$ a divisor, generically étale over B. Then there exists a diagram*

$$\begin{array}{ccc} X_1 & \stackrel{\psi}{\to} & X \\ \pi_1 \downarrow & & \downarrow \pi \\ B_1 & \stackrel{\phi}{\to} & B \end{array},$$

a finite group $G \subset \mathrm{Aut}_B B_1$, and disjoint sections $s_1 \ldots, s_n : B_1 \to X_1$, where

1. *G lifts into $\mathrm{Aut}_X X_1$, permuting the sections s_i,*
2. *ϕ and ψ are Galois alterations with Galois group G,*
3. *π_1 is nodal,*
4. *$X_1 \to X \times_B B_1$ is birational,*
5. *the sections s_i land in the smooth locus of π_1,*
6. *The generic fiber of $\psi^{-1} Z$ coincides with $\cup s_i(\eta_{B_1})$.*

Remark: Can start with a group H acting on $X \to B$ and extend it by G.

Sketch of proof of the alteration theorem.

Proof is inductive. For dimension 1 apply normalization after field extension

1. May assume $k = \bar{k}$, X projective, Z divisor;
2. project $X \to P$ with relative dimension 1, so that $Z \to P$ generically étale;
3. apply nodal reduction theorem (with G trivial);
4. resolve by induction (B_1, Δ) where $\Delta = Discr(\pi_1)$.
5. The singularities are now awfully simple:
$$uv = t_1^{k_1} \cdots t_r^{k_r}.$$
Either use toroidal geometry, or resolve explicitly!

Remark: Can do this equivariantly. (So every variety is birational to a projective variety with only finite quotient singularities.)

Semistable reduction problems

Question: can you generalize the nodal reduction theorem to higher dimensional fibers?

namely: Given $X \to B$, find an alteration $B_1 \to B$ and a model X_1 of $X \times_B B_1$ such that $X_1 \to B_1$ is *nice*.

Now work over $\mathbb{C}$.

[KKMS]: yes if $\dim B = 1$.

Kollár - Shepherd-Barron: yes if $\dim X_\eta = 2$.

(also $\aleph$-Hassett: a result for surfaces.)

Very very weak semistable reduction theorem:

Galois pluri-nodal reduction lemma. There exists a diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & X_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\tilde{\square}"] & X \arrow[d] \\ & B_1 \arrow[r] & B \end{tikzcd},$$

and a finite group $G \subset \mathrm{Aut}_{X_1} Y$ such that $B_1 \to B$ is an alteration, $Y/G \to X_1$ is birational and $Y \to B_1$ is a pluri-nodal fibration.

Is this useful?

Definition: a dominant morphism of varieties $X \to B$ is **mild** if for any dominant base change $B_1 \to B$ where B_1 is **r-G**, the pull back $X \times_B B_1$ is **r-G**.

Here r-G = *rational Gorenstein singularities.*

"Corollary": A composition of mild morphisms is mild.
"Corollary": Fibered powers of mild morphisms are mild.

Proposition: Nodal fibrations are mild.

Theorem 0.3 (Fibered power theorem)**.** *Let $X \to B$ be a smooth family of positive dimensional varieties of general type, with B irreducible. Then there exists an integer $n > 0$, a positive dimensional variety of general type W_n, and a dominant rational map $X^n_B \dashrightarrow W_n$.*

Sketch for the fibered power theorem

1. using moduli can assume $\mathrm{Var}(X/B) = \dim B$.
2. Compactify and resolve $X \to B$.
3. Kollár - Viehweg: $\omega_{X/B}$ is big.
   Idea: bigness of $\omega_{X^k_B/B}$ should grow in k, and swallow possible negativity in ω_B.
   Problem: X^n_B increasingly singular, differentials gain poles on a desingularization.
4. Apply Galois pluri-nodal reduction.

$$\begin{tikzcd} & Y \arrow[d, "q"] & \\ X_1 \arrow[r, "r"] \arrow[dr, "\pi"] & X' \arrow[r, "g_0"] \arrow[d] & X \arrow[d] \\ & B' \arrow[r] & B \end{tikzcd}$$

Over the locus where π is mild the idea does work. Over the complement $F \subset B'$, one has only log-canonical singularities in a suitable sense, so it is enough to take differentials vanishing on F:

$$I^k_F(\otimes_i p_i^* \omega^k_\pi) \hookrightarrow \omega^k_{\pi_k}.$$

Resolution of singularities in characteristic 0

Theorem

Let X be a variety of finite type over an algebraically closed field k of characteristic 0, let $Z \subset X$ be a proper closed subset. There exists a modification $f : X_1 \to X$, such that X_1 is a quasi-projective nonsingular variety and $Z_1 = f^{-1}(Z)_{red}$ is a strict divisor of normal crossings.

This theorem is a weak version of **Hironaka**'s (and **Bierstone - Milman**'s)

Sketch of proof by ℵ - **de Jong.**

1. May assume X projective, Z a divisor.

2. We choose a projection $X \dashrightarrow B$ of relative dimension 1, and apply semistable reduction to obtain a model $X' \to B'$ over a suitable Galois base change $B' \to B$, with Galois group G. Replace X by X'/G and B by B'/G.

3. We apply induction on the dimension to B , therefore we may assume that B is smooth, and that the discriminant locus of $X \to B$ is a strict divisor of normal crossings.

4. A few auxiliary blowups make X toroidal.

5. Theorem 11* of [KKMS] about toroidal resolutions finishes the argument.

Sketch of proof by **Bogomolov - Pantev**

1. May assume X projective, Z a divisor.

2. Project X finitely onto $\mathbb{P}^n, n = \dim X$. Then project from a generic point to get $X' \to P \to \mathbb{P}^{n-1}$ where $P \to \mathbb{P}^{n-1}$ is a $\mathbb{P}^1$ bundle. Make sure $Z \to \mathbb{P}^{n-1}$ **finite**.

3. a repeated application of Belyi's method allows one to map P to another $\mathbb{P}^1$ bundle P_0 such that $X \to P_0$ is ramifies over a union Z_0 of sections, and Z maps to a section.

4. Apply semistable reduction for rational pointed curves - this amounts to a blowup of P_0. We can guarantee that the whole situation is toroidal

5. Now X becomes toroidal as well - resolve toroidally!

De Jong meets Kontsevich

To prove de Jong's semistable reduction theorem:

After finite extension B' may assume that Z is the union of sections.

Let $\eta \subset B'$ be the generic point. Let X_η be the generic fiber. Let n be the number of sections, g the genus, d the degree via some embedding.

Kontsevich et al.: $\overline{\mathcal{M}_{g,n}}(X,d) \to \overline{M_{g,n}}(X,d)$.

Kollár: There is a finite $M \to \overline{\mathcal{M}_{g,n}}(X,d)$, M projective, admitting a **tautological family**:

$(C \to M, s_i, f : C \to X)$.

have a rational map $B' \dashrightarrow \overline{\mathcal{M}_{g,n}}(X,d)$, there is a generically finite $B_1 \to B$ together with $B_1 \to M$.

May assume $B_1 \to B$ Galois, with group G.

Have a *canonical* morphism $X_1 = C \times_M B_1 \to X$, such that X_1 birational to $B_1 \times_B X$.

Since D-M stable reduction over a normal base is unique when exists, G lifts.