STABLE MAPS AND HURWITZ SCHEMES IN MIXED CHARACTERISTIC

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1. The problem

The Hurwitz scheme was originally conceived as a parameter space for simply branched covers of the projective line. A variant of this is a parameter spaec for simply branched covers of the projective line, up to automorphisms of $\mathbb{P}^1$ - the so called unparametrized Hurwitz scheme. Other variants involve fixing branching types which are not necessarily simple (see [D-D-H]). A rigorous algebraic definition of the Hurwitz scheme in characteristic 0 was given by Fulton in [F] - the unparametrized version is its quotient by the $\mathbf{PGL}(2)$ action on $\mathbb{P}^1$. A natural compactification of the unparametrized Hurwitz scheme in characteristic 0 was given by Harris and Mumford [H-M] - the space of admissible covers. A rigorous treatment using logarithmic structures can be found in [Mo], who puts some order in the zoo of variants one can think of (parametrized vs. unparametrized, ordered branch points vs. unordered branch points, stack vs. coarse moduli scheme etc.). A new treatment using twisted principal bundles and stable maps into stacks can be found in [ℵ-V, ℵ-C-V]. In [P] the space of admissible covers is identified as a closed subscheme in the space of stable maps into $\overline{\mathcal{M}}_{0,n+1}$.

We know of no existing treatment of Hurwitz schemes in positive or mixed characteristic when the degree exceeds the characteristic of the residue fields. In this note we follow Pandharipande's idea in [P] and define the compactified Hurwitz scheme as a subscheme of the space of stable maps into $\overline{\mathcal{M}}_{0,n+1}$. Variants with target curves of higher genus are defined as well.

In order to follow this idea we have to treat moduli of stable maps in mixed characteristic. This is implicit in [B-M] but has not been available in the litterature in this generality. We show (Theorem 2.8) that for any base scheme S, and any integers g, n, r and d there exists an Artin algebraic stack with finite stabilizers, denoted $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$, which is proper over S, parametrizing stable maps of n-pointed curves of genus g into $\mathbb{P}^r$ over S. This stack admits a projective coarse moduli scheme $\overline{\mathbf{M}}_{g,n}(\mathbb{P}^r_S, d)$. One immediately derives the existence of a similar stack $\overline{\mathcal{M}}_{g,n}(X, d)$ parametrizing stable maps of degree d into a projective S-scheme of finite presentation $X \subset \mathbb{P}^r_S$.

Denote by $\mathcal{C}_{0,n} \to \overline{\mathcal{M}}_{0,n}$ the universal family of stable n-pointed curves of genus 0. We propose to define the (unparametrized) Hurwitz stack (with ordered simple branch points) to be the closure in $\overline{\mathcal{M}}_{g,n}(\mathcal{C}_{0,n}/\overline{\mathcal{M}}_{0,n}, d)$ of the locus of Hurwitz covers over $\mathbb{Q}$: one identifies an admissible cover $C \to D$ with ordered simple branchings $P_1, \ldots, P_n$ as a stable map from the n-pointed curve $(C, Q_1, \ldots Q_n)$ to the stable n-pointed rational curve $(D, P_1, \ldots P_n)$, where Q_i are the ramification points. Here the pointed curve $(D, P_1, \ldots P_n)$ is identified uniqly as a fiber of $\mathcal{C}_{0,n} \to \overline{\mathcal{M}}_{0,n}$.

There are some interesting things which happen in this construction. For the sake of comparison, recall that in characteristic 0

1. the compactified Hurwitz scheme is **irreducible;**
2. it maps **finitely** to $\overline{\mathcal{M}}_{0,n}$;
3. it parametrizes only **finite morphisms** $C \to D$;
4. all the morphisms are **simply branched away from the nodes;** and
5. the generic point corresponds to a morphism where C is **irreducible.**

We study in some detail the reduction in characteristic 2 of the compactified stack of double covers $C \to D$ of $\mathbb{P}^1$ branched in 4 points $0, 1, \infty, \lambda$. It has the following features:

1. this stack is **reducible**: it has four irreducible components;
2. it **does not map finitely** to $\overline{\mathcal{M}}_{0,n}$; specificatlly, one component is entirely devoted to maps with j invariant 0 and arbitrary λ invariant, whereas the other components parametrize maps with arbitrary j invariant and λ invariant $0, 1$ or ∞;
3. all geometric points correspond to maps which are **not finite;**

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4. all geometric points correspond to maps whose finite part is **inseparable**;
5. the generic points correspond to morphisms where C_1 is **reducible.**

2. Stable maps

2.1. **Definitions.** Fix a base scheme S, and integers g, n, r and d.

Definition 2.1. Let T be an S scheme. A *stable, n-pointed map $(C \to T, s_i, f)$ of genus g and degree d into $\mathbb{P}^r_S$ over T* consists of a diagram

$$\begin{array}{ccc} C & \stackrel{f}{\to} & \mathbb{P}^r \\ \pi \downarrow \uparrow s_i & & \\ T & & \end{array}$$

such that

1. the morphism $\pi : C \to T$ is a projective flat family of curves;
2. all the geometric fibers of $C \to T$ are reduced with at most nodes as singularities;
3. the sheaf $\pi_* \omega_{C/T}$ is locally free of rank g;
4. the n morphisms $s_i : T \to C$ are sections of π which are disjoint and land in the smooth locus of π;
5. the degree of $f^*\mathcal{O}(1)$ on geometric fibers of $C \to T$ is d; and
6. the group scheme $\mathrm{Aut}_{\mathbb{P}^r_T}(f : C \to \mathbb{P}^r)$ is finite over T.

Remark 2.2. Denote by S_i the image of s_i. The *stability* condition 6 on the automorphisms is equivalent to the condition that $\omega_{C/T}(\sum S_i) \otimes f^*\mathcal{O}(3)$ be ample (see [Kon], [F-P]). This is also equivalent to the usual "three point condition" for components of C mapping via f to a point.

Stable maps form category:

Definition 2.3. Given two S schemes T, T' and stable maps $(C \to T, s_i, f)$ and $(C' \to T', s'_i, f')$, a *morphism* α of stable maps is a commutative diagram

$$\begin{array}{ccc} C & \stackrel{\alpha_C}{\to} & C' \\ \downarrow & & \downarrow \\ T & \stackrel{\alpha_T}{\to} & T' \end{array}$$

inducing an *isomorphism* $C \to C' \times_T T'$, compatible with f and s_i, namely: $\alpha_C \circ s_i = s'_i \circ \alpha_T$, and $f = f' \circ \alpha_C$.

Denote the category of stable, n-pointed maps of genus g and degree d into $\mathbb{P}^r_S$ by $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$.

2.2. **Embedded stable maps and smooth parametrization of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$.** We follow the axioms of [A] for algebraic stacks. One can easily show directly that $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$ is a stack, and that the $\mathcal{I}som$ functor is representable. However, since we want to show that it is an *algebraic* stack, it is convenient to exhibit it as a quotient stack. This is done in a standard manner using embeddings in projective space.

In the rest of this section the discrete parameters g, n, r, d of stable maps are fixed. Choose two positive integers M, D.

Definition 2.4. An *embedded stable map* $(C \subset \mathbb{P}^M_T, s_i, f)$ of embedding degree D is a subscheme $C \subset \mathbb{P}^M_T$, flat over T of degree D, and a stable map $(C \to T, s_i, f : C \to \mathbb{P}^r)$. The embedded stable map is *nondegenerate* if the embedding of all the geometric fibers of $C \to T$ are nondegenerate (nemely not contained in proper linear projective subspaces).

Embedded stable maps form a category: a morphism $(C \subset \mathbb{P}^M_T, s_i, f) \to (C' \subset \mathbb{P}^M_{T'}, s'_i, f')$ is *the pullback morphism*

$$C = \mathbb{P}^M_T \underset{\mathbb{P}^M_{T'}}{\times} C' \to T$$

inducing a morphism of stable maps $(C \to T, s_i, f) \to (C' \to T', s'_i, f')$.

Lemma 2.5. *Given a fixed projective space $\mathbb{P}^M$ and an integer D, the category $\overline{\mathcal{M}}_{g,n}((\mathbb{P}^r \times \mathbb{P}^M)_S, (d, D))^{nd}_{\subset \mathbb{P}^M}$ of non-degenerately embedded stable maps $(C \subset \mathbb{P}^M_T, s_i, f)$ with embedding degree D is a stack representable by a quasi-projective S-scheme H. There is a natural action of* $\mathbf{PGL}(M+1)$ *on this scheme, by translation of the embedding $C \subset \mathbb{P}^M$.*

Proof. First consider the Hibert scheme $\mathcal{H}_{P,M}$ of subschemes of $\mathbb{P}^M$ having Hilbert polynomial $P(T) = D \cdot T - g + 1$. Let $\mathcal{U}_{P,M} \to \mathcal{H}_{P,M}$ be the universal family. There is a natural action of $\mathbf{PGL}(M+1)$ on $\mathcal{U}_{P,M} \to \mathcal{H}_{P,M}$.

We add the data of points: there is a closed subscheme $H_1 \subset \mathcal{H}_{P,M} \times (\mathbb{P}^M)^n$ parametrizing collections $(X, P_1, \dots P_n)$ where $X \subset \mathbb{P}^M$ has Hilbert polynomial P and the points $P_i \in \mathbb{P}^M$ satisfy $P_i \in X$. There is a natural diagonal action of $\mathbf{PGL}(M+1)$ on H_1.

A small deformation of a nodal curve is nodal; similarly, a small deformation of a pointed curve with distinct points has distinct points, and a small deformation of a non-degenerate embedding is non-degenerate. Thus, there is an open quasi-projective subscheme $H_2 \subset H_1$, stable under the action of $\mathbf{PGL}(M+1)$, parametrizing embedded pointed curves $(X, P_1, \dots P_n)$ such that X is a nodal curve, which spans $\mathbb{P}^M$, and the points P_i are distinct. Denote the pullback of $\mathcal{U}_{P,M} \to \mathcal{H}_{P,M}$ by $U_2 \to H_2$.

We now add the data of the map f: By [G], there is a quasi projective scheme $\mathrm{Hom}_{H_2}(U_2, \mathbb{P}^r)$, of finite presentation, parametrizing morphisms $f : C \to \mathbb{P}^r$ where C is a fiber of $U_2 \to H_2$ and $f^*\mathcal{O}(1)$ has degree d. The action of $\mathbf{PGL}(M+1)$ lifts by composing with f.

Finally, we impose the stability condition: there is an open subscheme $H \subset H_3$ over which the sheaf $\omega_{U/H_3}(\sum S_i) \otimes f^*\mathcal{O}(3)$ is ample. The scheme H satisfies the requirements of the lemma. ∎

We will now choose the integers M and D:

Fixing an integer $\nu \geq 3$, it was shown in [F-P] that the invertible sheaf

$$\mathcal{L}^\nu = \left(\omega_{C/T}(\sum S_i) \otimes f^*\mathcal{O}(3)\right)^\nu$$

is very ample and has no higher cohomology along the fibers. Let $\dim H^0(C_t, \mathcal{L}^\nu) = M+1$ and $\deg_{C_t} \mathcal{L}^\nu = D$. The lemma above provides us with a scheme H parametrizing non-degenerately embedded stable maps with embeding degree D in $\mathbb{P}^M$. By [Mu], there is a closed subscheme V where the embedding sheaf coincides along the fibers with $\mathcal{L}^\nu$. There is a natural $\mathbf{PGL}(M+1)$ action on V.

We claim that the stack quotient $V/\mathbf{PGL}(M+1)$ is equivalent to $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$. Indeed, there is an obvious forgetful functor $V/\mathbf{PGL}(M+1) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$, by leaving out the embedding. On the other hand, given a stable map $(\pi : C \to T, s_i, f)$, the projective frame bundle $P \to T$ associated with the locally free sheaf $\pi_*\mathcal{L}^\nu$ is a principal $\mathbf{PGL}(M+1)$-bundle over which $\mathbb{P}(\pi_*\mathcal{L}^\nu)$ has a canonical trivialization, therefore there is a $\mathbf{PGL}(M+1)$-equivariant morphism $P \to V$, giving rise to a morphism $T \to V/\mathbf{PGL}(M+1)$. This induces a functor $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d) \to V/\mathbf{PGL}(M+1)$. The two compositions of these functors are easily seen to be equivalent to the identity.

Thus $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$ is an Artin algebraic stack.

2.3. **Properness.** Since $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$ is a quotient stack it is already of finite presentation. To check that $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d) \to S$ is proper, we need to verify the valuative criteria for separation and properness.

Let T be an S scheme which is the spectrum of a discrete valuation ring R, with special point s and generic point η. Assume we are given two stable maps $(\pi : C \to T, s_i, f)$ and $(\pi_1 : C^1 \to T, s_i^1, f^1)$ over the same scheme T, and an isomorphism along the generic fiber $\alpha_\eta : (C \to T, s_i, f)_\eta \to (C^1 \to T, s_i^1, f^1)_\eta$. We wish to extend this isomorphism over T.

The closure C' of the graph of α is proper and flat over T. It admits maps $r : C' \to C$ and $r^1 : C' \to C^1$. Considering the morphisms $f \circ r$ and $f^1 \circ r^1$, we see that their graphs are the closure of the graph of the same morphism on the generic fiber; therefore $f \circ r = f^1 \circ r^1$. Call this map f'. Similarly, the sections s_i and s_i^1 lift to sections $s_i' : T \to C'$. We may replace C' by a desingularization on which the images of the sections S_i' and the central fiber C_s' together form a divisor of normal crossings.

We claim that C is the image of C' under the relative linar series of $(\pi \circ r)_* \left((\omega_{C'/T}(\sum S_i') \otimes f'^*\mathcal{O}(3))^\nu\right)$. Indeed, $\omega_{C'/T}(\sum S_i') \otimes f'^*\mathcal{O}(3))^\nu = \omega_{C/T}(\sum S_i) \otimes f^*\mathcal{O}(3))^\nu(E)$ where E is an effective r-exceptional divisor, and therefore we have

$$(\pi \circ r)_* \left((\omega_{C'/T}(\sum S_i') \otimes f'^*\mathcal{O}(3))^\nu\right) = (\pi)_* \left((\omega_{C/T}(\sum S_i) \otimes f^*\mathcal{O}(3))^\nu\right)$$

Since the same holds for C^1, we have an isomorphism $C \to C^1$. It follows that the Isom scheme is proper. Thus $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d) \to S$ is separated. Since stable maps have finite automorphisms, the Isom scheme is in fact finite.

Continuing to work with T as above, let $(C_\eta \to \eta, (s_i)_\eta, f_\eta)$ be a stable map. We wish to extend it over T, at least after a finite base change $T' \to T$.

There always exists a projective extension $C_0 \to T$ of C_η, by taking the closure in some projective embedding of C_η. Replacing C_0 by the closure of the graph of f_η in $C_0 \times \mathbb{P}^r$ we may assume f_η extends to C_0. The sections $(s_i)_\eta$ are extended by taking the closure of $(S_i)_\eta$.

Lemma 2.6. *There exists a finite extension of discrete valuation rings $R \subset R'$ giving rise to a finite base change $T' \to T$, a diagram*

$$\begin{array}{ccc} C_\eta \underset{T}{\times} T' & \subset & C' \\ \downarrow & & \downarrow \\ \{\eta'\} & \subset & T' \end{array}$$

and an extension $f' : C' \to \mathbb{P}^r$ of f_η and $s_i : T' \to C'$ of $(s_i)_\eta$, such that $C' \to T'$ is a family of pointed nodal curves.

Proof. First assume C_η is smooth. The results of [J] say in particular that there exists a scheme T' as in the lemma, and a diagram

$$\begin{array}{ccc} C' & \to & C_0 \\ \downarrow & & \downarrow \\ T' & \to & T \end{array}$$

such that $C' \to C_0 \times_T T'$ is birational and $C' \to T'$ is a family of pointed nodal curves. This immediately proves the lemma in this case.

For the general case, first take a base change so that the irreducible components of C_η are absolutely irreducible, and all the nodes are split. Apply the argument above to the normalization of each irreducible component of C_η, adding extra sections for the points above the nodes. Finally, glue the resulting schemes together along the sections as in [B-M]. The glued object is an Artin algebraic space.

Now $(C' \to T', s_i, f)$ is not stable - we need to contract the so called "components to be contracted" of [B-M]. One can apply a modified verision of Knudsen's contraction technique directly to carry that through. Alternatively, after a suitable base change one can add enough sections meeting the components whose image in $\mathbb{P}^r$ is nontrivial, and then Knudsen's contraction technique for stable pointed curves applies without modifications.

Thus $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d) \to S$ is proper.

Remark 2.7. A proof of De Jong's result which we used above in this case can be summarized as follows: one first picks a projective completion $X \to T$ of C_η with a morphism $X \to \mathbb{P}^r$. After a base change, one adds enough sections so that every component of every fiber has at least three sections in the non-singular locus of its reduction. After a further base change, one has a stable pointed curve $C' \to T'$ and a rational map $C' \to X$. De Jong's "three point lemma" (which is relatively straightforward in the case of one-dimensional base) implies that this is a morphism.

As noted in [$\aleph$], De Jong's alteration theorems can be deduced from the existence of the moduli stack of stable maps.

2.4. **Projectivity of the coarse moduli scheme.** By [Ke-Mo] there exists a coarse moduli morphism $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d) \to \overline{\mathbf{M}}_{g,n}(\mathbb{P}^r_S, d)$ such that $\overline{\mathbf{M}}_{g,n}(\mathbb{P}^r_S, d) \to S$ is a proper Artin algebraic space. We endow the category $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$ with a canonical semipositive polarization ([Kol], 2.3 and 2.4). This implies that $\overline{\mathbf{M}}_{g,n}(\mathbb{P}^r_S, d) \to S$ is projective by [Kol], Theorem 2.6. Note that the assumption on tame automorphisms in [Kol] was used only to guarantee the existence of the algebraic space $\overline{\mathbf{M}}_{g,n}(\mathbb{P}^r_S, d)$; however, by [Ke-Mo] it is enough that the automorphism schemes are finite.

In order to define such a canonical polarization, one could modify the proof of [Kol], Proposition 4.7 to apply for the sheaf $\mathcal{L}^\nu$ we constructed above for stable maps. Alternatively, we can reduce to the case treated by Kollár as follows:

First note that since $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)$ is proper, the inseparable degree of any such stable map $(C \to T, s_i, f)$ is bounded from above by some integer $B > 1$.

We pick the canonical polarization $\mathcal{L}^\nu$ as above with $\nu = B$. We claim that this is a semipositive polarization. Let $(C \to T, s_i, f)$ be a stable map over a projective curve T. Choose a general hypersurface H in $\mathbb{P}^r$ of degree $3B$. In particular we may assume that the pullback f^*H is finite over T of inseparable degree $< B$. After a base change we may assume $f^*H = \sum b_i \Sigma_i$ where Σ_i are sections of $C \to T$. By assumption we have $b_i < B$. Therefore $(\omega_{C/T}(\sum \cdot S_i) \otimes f^*\mathcal{O}(3))^\nu = \omega_{C/T}^\nu(\sum \nu \cdot S_i + \sum b_i \Sigma_i)$ satisfies the assumptions of [Kol], Proposition 4.7, and therefore the polarization is semipositive.

We have thus proven:

Theorem 2.8. *The category $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d) \to S$ is aproper Artin algebraic stack with finite stabilizers, admitting a projective coarse moduli scheme $\overline{\mathbf{M}}_{g,n}(\mathbb{P}^r_S, d) \to S$.*

2.5. **Stable maps into a projective scheme.** Let $X \subset \mathbb{P}^r_S$ be a projective scheme of finite presentation over S. Given a stable map $(C \to T, s_i, f : C \to \mathbb{P}^r)$, it follows from [G] that there is a closed subscheme $T' \subset T$ of finite presentation where the maps land inside X. It follows that there exists a proper stack of finite presentation $\overline{\mathcal{M}}_{g,n}(X/S, d) \to S$ admitting a projective coarse moduli scheme. As in [F-P], given an element $\beta \in A_1(X)$, there is an open and closed substack $\overline{\mathcal{M}}_{g,n}(X/S, \beta) \subset \overline{\mathcal{M}}_{g,n}(X/S, d)$ where the maps have image class β.

2.6. **Canonical maps.** Assume that $2g-2+m > 0$ and $n \geq m$. As noted in [F-P] and [B-M] there is a morphism $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d) \to \overline{\mathcal{M}}_{g,m}$ by "forgetting the map and the last markings, and contracting the components to be contracted". More generally, given an S-morphism $X \to Y$ of projective schemes over S, there is a morphism $\overline{\mathcal{M}}_{g,n}(X/S, \beta) \to \overline{\mathcal{M}}_{g,n}(Y/S, f_*\beta)$. The condition $2g-2+m > 0$ can be removed as long as $f_*\beta \neq 0$.

2.7. **Stable maps over a base stack.** Let $S_1 \to S_2$ be an étale morphism. It is easy to see that $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_{S_1}, d) = \overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_{S_2}, d) \times_S S_1$. Thus relative stable maps are well-behaved in the étale site. We now apply this to Deligne-Mumford stacks.

Let $\mathcal{S}$ be a Deligne-Mumford algebraic stack. Fix an étale parametrization $S \to \mathcal{S}$, and let $R = S \times_{\mathcal{S}} S$. The remarks above show that the representable morphism $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_R, d) \to \left(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_S, d)\right)^2$ is an étale groupoid in Artin stacks. Denote its quotient by $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^r_{\mathcal{S}}, d) \to \mathcal{S}$. One checks easily that this is a proper Artin algebraic stack admitting a relatively coarse projective moduli stack $\overline{\mathbf{M}}_{g,n}(\mathbb{P}^r_{\mathcal{S}}, d) \to \mathcal{S}$. More concretely, it is the quotient of the following groupoid using non-degenerately embedded stable maps:

$$\overline{\mathcal{M}}_{g,n}((\mathbb{P}^r \times \mathbb{P}^M)_R, (d, D))^{nd}_{\subset \mathbb{P}^M} \times \mathbf{PGL}(M+1) \to \left(\overline{\mathcal{M}}_{g,n}((\mathbb{P}^r \times \mathbb{P}^M)_S, (d, D))^{nd}_{\subset \mathbb{P}^M}\right)^2.$$

It is equivalent to the category of stable maps $(C \to T, s_i, f : C \to \mathbb{P}^r)$ where T runs over schemes over $\mathcal{S}$. Note that, since we work with stable maps *relative to a projective morphism,* there is no need to invoke the setup of [ℵ-V].

Asimilar construction works for $\mathcal{X} \subset \mathbb{P}^r_{\mathcal{S}}$, a projective substack of finite presentation.

3. Complete Hurwitz stacks

Fix integers $d \geq 1$ and $n \geq 3$. Let T be a scheme of pure characteristic 0. Consider a diagram

$$\begin{array}{ccc} C & \longrightarrow & D \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

where $C \to T$ is a smooth projective connected curve admitting n disjoint sections $s_i : T \to C$; where $D \to T$ is a smooth projective connected curve of genus h, admitting n disjoint sections $\sigma_i : T \to D$; and $f : C \to C$ is a Hurwitz cover simply ramified precisely along s_i with corresponding branch points σ_i. The genus g of C is given by the Hurwitz formula $2g - 2 = n + (2h-2)d$. It is shown in [Mo] that such diagrams, with morphisms given by pullback diagrams, form a Deligne-Mumford stack $\mathcal{H}_{d,h,n}$.

As noted in [P], the pointed curve (D, σ_i) is a stable pointed curve of genus h, and therefore it is the pullback of $\mathcal{C}_{h,n}$ along a unique morphism $T \to \overline{\mathcal{M}}_{h,n}$. Composing with the pullback map, we obtain a well defined morphism $f : C \to \mathcal{C}_{h,n}$. We thus have a stable n-pointed map $(C \to T, s_i, f : C \to \mathcal{C}_{h,n})$, of genus g, and image class $\beta = d \cdot F$ (F being the class of a fiber) over $S = \overline{\mathcal{M}}_{h,n}$. We therefore obtained a functor $\mathcal{H}_{d,h,n} \to \overline{\mathcal{M}}_{g,n}(\mathcal{C}_{h,n}/\overline{\mathcal{M}}_{h,n}, \beta)$, which is clearly an embedding of stacks. Denote by $\overline{\mathcal{H}}_{d,h,n}$ the closure. In case $h = 0$ we write $\overline{\mathcal{H}}_{d,n}$ for $\overline{\mathcal{H}}_{d,0,n}$.

Definition 3.1. The stack $\overline{\mathcal{H}}_{d,h,n}$ is called *the complete Hurwitz stack.*

There are morphisms $\overline{\mathcal{H}}_{d,h,n} \to \overline{\mathcal{M}}_{h,n}$ (the base curve morphism) and $\overline{\mathcal{H}}_{d,h,n} \to \overline{\mathcal{M}}_{g,n}$ (forgetting the map and contracting the extraneous components).

4. Double covers of $\mathbb{P}^1$ branched over 4 points.

Consider $\overline{\mathcal{H}}_{2,4}$ - the complete Hurwitz stack of double covers of $\mathbb{P}^1$ branched over 4 points. Over Spec $\mathbb{Z}[1/2]$ it is a $\mathbb{Z}/2$-gerbe over $\overline{\mathcal{M}}_{0,4} = \mathbb{P}^1$, equivalent to the stack of elliptic curves with level 2 structure. Here we are interested in its stucture in characteristic 2.

We will make use of the morphisms $\lambda : \overline{\mathcal{H}}_{2,4} \to \overline{\mathcal{M}}_{0,4}$ and $j : \overline{\mathcal{H}}_{2,4} \to \overline{\mathcal{M}}_{1,1}$. In addition, if $(C \to T, s_i, f : C \to D)$ is a Complete Hurwitz map whose generic fiber is in $\mathcal{H}_{d,n}$, there is a natural involution σ acting on the generic fiber, which extends to C, since C is normal. Consider the associated stable 1-pointed curve $\overline{C} \to T$ of genus 1 obtained by contracting the extraneous components. The induced involution on the smooth locus of $\overline{C} \to T$ coincides with the involution (-1) of the group law, therefore its fixed point locus is the 2-torsion group-scheme.

We also have a natural action of the symemtric group of 4 letters on $\overline{\mathcal{H}}_{2,4}$, permuting the markings.

Finally, it is easy to check that for any object $(C \to T, s_i, f : C \to D)$ of the complete Hurwitz stack, the source curve C is a stable 4-pointed curve (and in particular $\overline{\mathcal{H}}_{2,4}$ is a Deligne-Mumford stack). This follows since for any component C_0 of C mapping finitely to a component D_0 of D, and any marked point P of D_0, there is a unique point Q of C_0 which is either marked or possibly a node.

Let $(C_s, s_i, f : C_s \to D_s)$ be a geometric point of the Complete Hurwitz stack in characteristic 2. We can choose a discrete valuation ring R with spectrum T, with generic point of characteristic 0 and residue characteristic 2, and a stable map $(C \to T, s_i, f : C \to D)$ specializing to the given one. Denote by j_s and λ_s the j invariant and the λ invariant of the special fiber.

Case 1: $\lambda_s \neq 0, 1, \infty$.

The curve C_η admits a Legendre equation $y^2 = x(x-1)(x-\lambda)$. This equation has a unique singular point in characteristic 2 corresponding to $x^2 = \lambda$. It is easy to see that this is the unique point invariant under the symmetries of $(0, 1, \infty, \lambda)$: the permutation $0 \leftrightarrow \infty, 1 \leftrightarrow \lambda$ is given by the transformation $x \mapsto \lambda/x$, whose unique fixed point is $\sqrt{\lambda}$. The permutation $0 \leftrightarrow 1 \infty \leftrightarrow \lambda$ is given by the transformation $x \mapsto (x-1)/(\lambda^{-1}x - 1)$ with the same fixed point! In any case, in the stable map f_s the singular point $x^2 = \lambda$ must be blown up (it is not a node!) and for stability reasons a component of genus 1 must be attached. The reduction of the 2-torsion locus on $\overline{C}$ consists precisely of the attaching point, therefore the elliptic curve is supersingular, namely $j_s = 0$.

The resulting picture is as follows: C_s has exactly 2 components: C_0, which has genus 0, and C_1, which has genus 1 and j invariant 0. The morphism f_s maps C_1 to $x = \sqrt{\lambda}$. The component C_0 is the normalization of $y^2 = x(x-1)(x-\lambda)$, mapping 2-to-1 to D_s, purely inseparably. It is attached over $x = \sqrt{\lambda}, y = \lambda + x$ with the elliptic curve C_1.

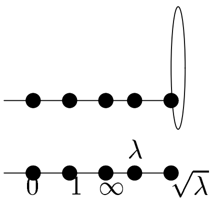

Figure 1. Case 1: $\lambda \neq 0, 1, \infty$.

Case 2: $\lambda_s = 0, j_s \neq 0, \infty$.

Consider $\overline{C}$. It has 4 sections, only 2 of which reduce to the origin and 2 reduce to another point. The associated stable 4-pointed curve of genus 1, denoted C_s, has rational components C_0^1 and C_0^2 attached at these two points, through which the sections pass. These rational components are branced over the rational components of D_s at the 2 sections. Moreover, the two attaching points of $\overline{C}$ are fixed points of the involution, therefore they must be branch points of f_s on C_0^1 and C_0^2. This means that C_0^i are ramified at 3 points, which implies that they map purely inseparably onto the components of D. The component C_1 maps to the node of D.

Case 3: $\lambda_s = 0, j_s = \infty$. Considering $\overline{C}$, we can take 2 of the sections, one passing through the origin and one through the node. The stabilization as a 2-pointed curve is a "banana curve" of two rational components meeting in two points, with one marking on each component. Adding the two other sections and arguing as in case 2 we arrive at the following picture:

The curve C_s has 4 components, denoted C_1^1, C_1^2, C_0^1 and C_0^2. The components C_1^i are attached to each other at 2 points, and f_s maps them to the node of D_s. The component C_1^i is also atatched at one point to C_0^i. The morphism f_s maps C_0^i purely inseparably to the components of D.

Case 4: $\lambda_s = 0, j_s = 0$. The Legendre equation $y^2 = x(x-1)(x-\lambda)$ is singular at $x = 0$. Applying the transformation $x \mapsto x/\lambda$ we get the Legendre equation $y^2 = x(x-1)(x-\lambda^{-1})$ which is singular at $x = \infty$. It follows that the double cover of the stable 4-pointed curve D_s is singular at the node, and as in case 1 the

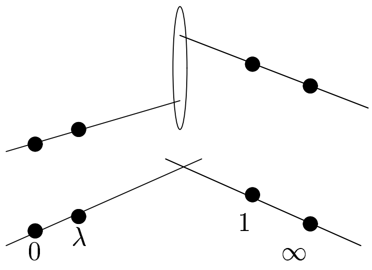

FIGURE 2. Case 2: $\lambda_s = 0, j_s \neq 0, \infty$.

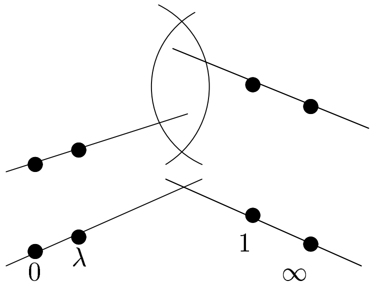

FIGURE 3. Case 3: $\lambda_s = 0, j_s = \infty$.

singularity contributes genus 1. The stable limit C_s thus has to contain a component of genus 1 and j invariant 0, attached at one point to the rest of the curve. The only stable configuration possible is the following:

The curve C_s has 4 components, C_1, C_0', C_0^1 and C_0^2. The component C_1 has genus 1 and j-invariant 0. The other components have genus 0. The component C' is attached at one point to each of C_1, C_0^1 and C_0^2. Both C_1 and C_0' are mapped to the node of D_s, and C_0^1 and C_0^2 map purely inseparably to the two components of D_s.

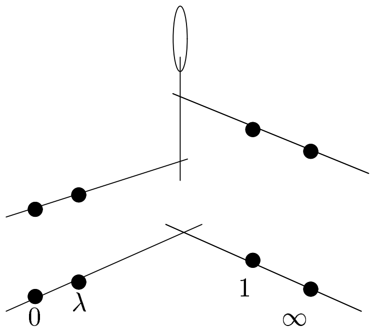

FIGURE 4. Case 4: $\lambda_s = 0, j_s = 0$.

A situation identical to cases 1,3,4 holds for $\lambda_s = 1, \infty$, by permuting the marked points.

4.1. **Speculations about intermediate quotient curves.** One would like to insert an intermediate curve P between C and D so that $C \to P$ is finite. In the present case of $\overline{\mathcal{H}}_{2,4}$ there is a natural involution σ on C, and in characteristic 0 we have $C/\langle\sigma\rangle = D$. Thus in general, given a family $C \to D$ over a scheme T which is an element of $\overline{\mathcal{H}}_{2,4}(T)$, it is natural to consider the quotient curve $P = C/\langle\sigma\rangle$, and we have natural morphisms $C \to P \to D$.

Unfortunately *the formation of P does not commute with base change!* Indeed, if T has generic point of characteristic 0 and special point in characteristic 2 as before, then over the generic points of D_s, the morphism $P \to D$ is an isomorphism. However if T is replaced by s, then σ acts *trivially* on the components C_0^i, therefore $P \to D$ inseparable of degree 2 over the generic points of D!

There is, however, a remedy. In analogy to the methods of [ℵ-V], one could take P to be the *stack* $C/\langle\sigma\rangle$. This stack automatically commutes with base changes. The morphism $C \to P$ is a principal $\mathbb{Z}/2\mathbb{Z}$ bundle, giving rise to a representable morphism to the classifying stack $P \to B(\mathbb{Z}/2\mathbb{Z})$. In a sense yet to be understood, this may be considered a stable map.

In order to introduce intermediate curves for general complete Hurwitz stacks, one might replace a Hurwitz cover $C \to D$ by its associated S_d-cover $\tilde{C} \to D$, where S_d is the symmetric group on d letters. This again is in analogy with [ℵ-V]. One can complete the locus of these S_d-covers inside the appropriate stack of S_d-equivariant stable maps, and obtain a new stack which one might denote $\overline{\mathcal{H}}_{S_d,h,n}$. An object of this stack over a reduced base scheme T is an S_n-equivariant T-morphism $\tilde{C} \to D$, branched over n-marked points in D, whose geometric fibers can be lifted to Horwitz covers in characteristic 0. One can obtain a replacement for C by taking the quotient $\tilde{C}/S_{d-1}$. This quotient should be taken as a Deligne-Mumford stack. The quotient $\tilde{C}/S_d$ is a candidate for P.

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