Resolution of singularities in characteristic 0

Preprints available/linked in my home page:

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Also on alg-geom

Theorem

Let X be a variety of finite type over an algebraically closed field k of characteristic 0, let $Z \subset X$ be a proper closed subset. There exists a modification $f : X_1 \to X$, such that X_1 is a quasi-projective nonsingular variety and $Z_1 = f^{-1}(Z)_{red}$ is a strict divisor of normal crossings.

This theorem is a weak version of **Hironaka**'s (and **Bierstone - Milman**'s)

Will give proof of $\aleph$ **- de Jong.**

Another (beautiful) proof by **Bogomolov - Pantev.**

0. May assume X projective, Z a divisor.

1. Following de Jong, we choose a projection $X - - \to P$ of relative dimension 1, and apply semistable reduction to obtain a model $X' \to P$ over a suitable Galois base change $P' \to P$, with Galois group G. Replace X by X'/G.

2. We apply induction on the dimension to P , therefore we may assume that P is smooth, and that the discriminant locus of $X \to P$ is a strict divisor of normal crossings.

3. A few auxiliary blowups make X toroidal.

4. Theorem 11* of [KKMS] about toroidal resolutions finishes the argument.

Theorem (de Jong):

Given a family of curves $X \to P$ with a horizontal divisor $Z \subset X$, there is a Galois alteration $P' \to P$, a family of curves $X' \to P'$

and disjoint sections $s_1, ..., s_n : P' \to X'$ such that

$X' \to P'$ is nodal

$X' \to X \times_P P'$ birational

Z is the union of images of the sections

The Galois group G lifts to X', permuting s_i.

Let $\Delta \subset P$ be the discriminant of (P', X, Z).

Induction: resolve (P, Δ) and pull back everything.

Local description of $P' \to P$.

Since P is nonsingular, P' is ramified in codimension 1, inside Δ.

local fundamental groups of $P \setminus \Delta$ are well understood. Every finite covering is locally given by taking roots.

$P'' \to P' \to P$, $Gal(P'' : P) = (Z/nZ)^r$.

$P = Spec\, k[[s_1, \dots, s_{d-1}]]$

$P'' = Spec\, k[[t_1, \dots, t_r, s_{r+1}, \dots, s_{d-1}]]$

where $t_i^n = s_i$.

Both (P, Δ) and (P', Δ') are toroidal. G acts toroidally.

Want: same with X'.

Definition: $U \subset X$ is toroidal if it is etale locally a toric embedding $T \subset V$. An action $G \subset Aut(U \subset X)$ is toroidal if under suitable identification $Stab(x) \subset T$ for any $x \in X$.

Local description of X''.

Let $R = k[[t_1, \ldots, t_r, s_{r+1}, \ldots, s_{d-1}]]$

$X'' \to X' \to X$

$x \in X'', \quad G_x = Stab(x)$

Case **I**: $X'' = SpecR[[u]]$.

Case **II**: $X'' = SpecR[[u,v]]/(uv - t_1^{k_1} \cdots t_r^{k_r})$.

In case (I) can take u so that G_x acts on u via a character. Action almost toroidal: may be missing stratum $u = o$.

In case (II): Action almost toroidal: may switch u, v.

To correct (II): blow up $sing(X' \to P')$. Get equations of type

II': $X'' = SpecR[[u,w]]/(u^2 w - t_1^{k_1} \cdots t_r^{k_r})$.

II": $X'' = SpecR[[u,w]]/(u^2 - t_1^{k_1} \cdots t_r^{k_r})$.

Then normalize.

We look at the open subset of $U \subset X'$ given by the complement of $Z' \cup (f')^{-1}\Delta$.

Everything toroidal, apart from the $u = 0$ in case (I) (and $w = 0$ in case II").

To correct (I):

Theorem: Let $T_0 \subset X_0$ be an affine torus embedding, $X_0 = spec\ R$. Let $G \subset T_0$ be a finite subgroup of T_0, and let ψ_u be a character of G. Consider

$$X = X_0 \times spec\ k[u].$$

We let G act on u via the character ψ_u. There is a **canonical** ideal $I_G \subset R[u]$, such that the normalized blowup

$$m : Y \to X$$

torifies the action of G on X, namely: G acts toroidally on $U_Y \subset Y$, where U_Y is the inverse image of T_0.

Proof: let $M \subset R[u]$ be the monomials. For $t \in M$ the group G acts via ψ_t. Define the **torific ideal** to be

$$I_G = (\{t|\psi_t = \psi_u\}).$$

Example: $X_0 = Spec\ k[x]$; $G = Z/nZ$,

$$g(x, u) = (\zeta \cdot x, \zeta^a \cdot u)$$

$$I_G = (x^a, u)$$

Conclusion of the proof of the main theorem

We blew up X' to obtain $\tilde{U} \subset \tilde{X}'$ on which G acts toroidally.

Since the quotient of a toric variety by a finite subgroup of the torus is again toric, we have that $\tilde{X}'/G$ is toroidal.

We may replace X by this variety.

By theorem 11* of [KKMS], there is a nonsingular toroidal blowup.

The divisor over $Z \cup \Delta$ is then automatically a (strict) divisor of normal crossings.

De Jong meets Kontsevich

To prove de Jong's theorem:

After finite extension P_1 may assume that Z is the union of sections.

Let $\eta \subset P_1$ be the generic point. Let X_η be the generic fiber. Let n be the number of sections, g the genus, d the degree via some embedding.

Kontsevich et al.: $\overline{\mathcal{M}_{g,n}}(X,d) \to \overline{M_{g,n}}(X,d)$.

Kollár: There is a finite $M \to \overline{\mathcal{M}_{g,n}}(X,d)$, M projective, admitting a **tautological family**:

$(C \to M, s_i, f : C \to X)$.

have a rational map $P_1 - - \to \overline{\mathcal{M}_{g,n}}(X,d)$, there is a generically finite $P' \to P$ together with $P' \to M$.

May assume $P' \to P$ Galois, with group G.

Have a *canonical* morphism $X' = C \times_M P' \to X$, such that X' birational to $P' \times_P X$.

Since D-M stable reduction over a normal base is unique when exists, G lifts.

Plan

1. Statement
2. Overview and discussion
3. Proof in some detail
4. De Jong meets Kontsevich
5. The big picture

This theorem is only “the generic fiber” of what we want

Semistable reduction problem. Given a family of varieties $X \to B$, find an alteration $B_1 \to B$ such that there is a model $Y \to B_1$ of the pullback family in which every fiber is “nice”. Here’s a possible direction:

“Definition”: a flat family $X \to B$ is **smooth** if for any dominant base change $B_1 \to B$ where B_1 is **nonsingular**, the pull back $X \times_B B_1$ is **nonsingular**.

Definition: a flat family $X \to B$ is **mild** if for any dominant base change $B_1 \to B$ where B_1 is **r-G**, the pull back $X \times_B B_1$ is **r-G**.

Proposition: Nodal fibrations are mild. Therefore pluri-nodal fibrations are mild.

Galois pluri-nodal reduction lemma: can get a Galois cover of $X \times_B B_1$ which is pluri-nodal over B_1, and hence mild.

Can even make Y nonsingular over some particular B_1. Maybe even everything toroidal.

Would be happy if the generic fiber of $Y/G \to B_1$ were nonsingular (or even canonical)!