Smoothness, Semistability, and Toroidal Geometry

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March 7, 1997

0. INTRODUCTION

0.1. **Statement.** We provide a new proof of the following result:

Theorem 0.1 (Hironaka)**.** *Let X be a variety of finite type over an algebraically closed field k of characteristic 0, let $Z \subset X$ be a proper closed subset. There exists a modification $f : X_1 \to X$, such that X_1 is a quasi-projective nonsingular variety and $Z_1 = f^{-1}(Z)_{\rm red}$ is a strict divisor of normal crossings.*

Remark 0.2. Needless to say, this theorem is a weak version of Hironaka's well known theorem on resolution of singularities. Our proof has the feature that it builds on two standard techniques of algebraic geometry: semistable reduction for curves, and toric geometry.

Remark 0.3. Another proof of the same result was discovered independently by F. Bogomolov and T. Pantev [B-P]. The two proofs are similar in spirit but quite different in detail.

Hironaka's theorem was recently revisited by E. Bierstone and P. Milman [B-M1], [B-M2] and by O. Villamayor [V]. They have given proofs of Hironaka's theorem in full power, making it both constructive and canonical.

Remark 0.4. Our method of proof can be applied when one attempts to control the singularities of a morphism. In [ℵ-K] it is used as a first step towards semistable reduction in characteristic 0 over a base of arbitrary dimension.

0.2. **Structure of the proof.**

1. As in [dJ], we choose a projection $X \dashrightarrow P$ of relative dimension 1, and apply semistable reduction to obtain a model $X' \to P'$ over a suitable Galois base change $P' \to P$, with Galois group G.
2. We apply induction on the dimension to P , therefore we may assume that P is smooth, and that the discriminant locus of $X'/G \to P$ is a G-strict divisor of normal crossings.
3. A few auxiliary blowups make the quotient X'/G toroidal.
4. Theorem 11* of [KKMS] about toroidal resolutions finishes the argument .

0.3. **What we do not show.**

0.3.1. *Canonicity.* Our proof has the drawback that the resolution is noncanonical. Some of the steps are not easily carried out in practice, and in fact we almost always blow up in the smooth locus. However, it follows from the proof, that if $C \subset X_{ns}$ is a curve, then we can guarantee that $X' \to X$ is an isomorphism in a neighborhood of C. One can use our line of

[1]Partially supported by an Alfred P. Sloan Research Fellowship and NSF grant DMS-9503276.

argument to give equivariant resolution of singularities in the case where a finite group acts, see [ℵ-W].

0.3.2. *Positive characteristic.* There are several points in the argument below where our proof fails in positive characteristic. Some of these can be circumvented, but there is one crucial point where the proof decisively fails if char $k = p > 0$. This is at the point where we claim that we can make the quotient X'/G toroidal. Given $x \in X'$ and $g \in \mathrm{Stab}\ x$ of order p, the action of g on $\mathcal{O}_{X',x}$ is unipotent. Thus even if X' is made toroidal, we cannot guarantee that the quotient is toroidal as well. It might happen that the quotient is toroidal by accident, and it would be interesting to see to what extent such accidents can be encouraged to happen.

0.4. **Acknowledgements.** We would like to thank B. Hassett, Y. Karshon, S. Katz, P. Milman, T. Pantev, D. Rohrlich, T. Scanlon, and M. Spivakovsky for discussions relevant to the subject of this paper.

0.5. **Terminology.** We recall some definitions; we restrict ourselves to the case of varieties over the algebraically closed field k.

A **modification** is a proper birational morphism of irreducible varieties.

An **alteration** $a : B_1 \to B$ is a proper, surjective, generically finite morphism of irreducible varieties, see [dJ, 2.20]. The alteration a is a **Galois alteration** if there is a finite group $G \subset \mathrm{Aut}_B(B_1)$ such that the associated morphism $B_1/G \to B$ is birational, compare [dJ2, 5.3].

Let a finite group G act on a (possibly reducible) variety Z. Let $Z = \cup Z_i$ be the decomposition of Z into irreducible components. We say that Z **is G-strict** if the union of translates $\cup_{g\in G} g(Z_i)$ of each irreducible component Z_i is a normal variety. We simply say that Z is **strict** if it is G-strict for the action of the trivial group, namely every irreducible component Z_i is normal.

A divisor $D \subset X$ is called a **divisor of normal crossings** if étale locally at every point it is the zero set of $u_1 \cdots u_k$ where $u_1, \dots, u_k$ is part of a regular system of parameters. Thus a strict divisor of normal crossings is what is usually called a divisor of strict normal crossings, i.e., all components of D are nonsingular.

An open embedding $U \hookrightarrow X$ is called a **toroidal embedding** if locally in the étale topology it is isomorphic to a torus embedding $T \hookrightarrow V$, (see [KKMS], II§1). (One can replace "étale locally" by "complex analytically" in case $k = \mathbb{C}$, or "formally", obtaining the same class of embeddings.) If $D = X \setminus U$, we will sometimes denote this toroidal embedding by (X, D). A finite group action $G \subset \mathrm{Aut}(U \hookrightarrow X)$ is said to be **toroidal** if the stabilizer of every point can be identified on an appropriate neighborhood with a subgroup of the torus T. We say that a toroidal action is **G-strict** if $X \setminus U$ is G-strict. In particular the toroidal embedding itself is said to be strict if $X \setminus U$ is strict. This is the same as the notion of **toroidal embedding without self-intersections** in [KKMS]. The point of introducing toroidal actions is that, since the quotient of a toric variety by a finite subgroup of the torus is also toric, we have that the quotient of a toroidal embedding by a toroidal action is toroidal as well.

The fundamental theorem about toroidal embeddings we will exploit is the following:

Theorem 0.5 ([KKMS], II §2, Theorem 11*, p. 94)**.** *For any strict toroidal embedding $U \hookrightarrow X$ there exists a canonical sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$ such that the blowup $B_{\mathcal{I}}(X)$ is nonsingular.*

(See [KKMS], II §2, definition 1 for the notion of canonical modification.)

Remark 0.6. Another approach for toroidal embeddings was developed by K. Kato (see [K]) using logarithmic structures. The resulting notions are equivalent.

1. The proof

Our line of proof in 1.1 and 1.2 follows closely the arguments of [dJ].

1.1. **Reduction steps.** We argue by induction on $d = \dim X$. The case $d = 1$ is given by normalizing X.

As in [dJ], 3.6-3.10 we may assume that

1. *X is projective and normal, and*
2. *Z is the support of a divisor.*

Recall that according to [dJ], Lemma 3.11 there is a modification $X' \to X$ and a generically smooth morphism $f : X' \to P = \mathbb{P}^{d-1}$ satisfying the following properties:

3. *The modification $X' \to X$ is an isomorphism in a neighborhood of Z,*
4. *every fiber of f is of pure dimension 1,*
5. *the smooth locus of f is dense in every fiber, and*
6. *the morphism $f_{|Z} : Z \to P$ is finite.*

Since the construction in [dJ] is obtained via a general projection, we can guarantee that

7. *the generic fiber of $f : X' \to P$ is a geometrically connected curve* (this follows from [J], 6.3(4), see [א], 4.2).

We now replace X by X'.

Applying Lemma 3.13 of [dJ], we may choose a divisor $D \subset X$ mapping finitely to P, which meets every component of every fiber of f in at least 3 smooth points. We may replace Z by $Z \cup D$ (see [dJ, 3.9]), and therefore we may assume that

8. *Z meets every component of every fiber of f in at least 3 smooth points.*

Let $\Delta_X \subset P$ be the discriminant locus of the map $X \to P$, and let $\Delta_Z \subset P$ be the discriminant locus of the map $Z \to P$. Let $\Delta = \Delta_Z \cup \Delta_X$. We can think of Δ as the discriminant locus of the pair $Z \subset X \to P$.

The inductive assumption gives us a resolution of singularities of $\Delta \subset P$. Thus we may replace X, Z, P and assume that P is smooth and Δ is a strict divisor of normal crossings. Let $\nu : X^{\mathrm{nor}} \to X$ be the normalization. By [dJ-O], the discriminant locus of $\nu^{-1}Z \subset X^{\mathrm{nor}} \to P$ is a divisor Δ' contained in Δ. We replace X by X^{nor} and Δ by Δ'. Thus we may assume in addition to **1** - **8**, that

9. *P is smooth, and*
10. *Δ is a strict normal crossings divisor.*

1.2. **Stable reduction.** We are now ready to perform stable reduction. We follow [dJ], 3.18-3.21, but see Remark 1.2 below. Let $j : U_0 = P \setminus \Delta \hookrightarrow P$. Let $a_U : U_0' \to U_0$ be an étale Galois cover which splits the projection $Z_{U_0} = j^{-1}(U_0) \cap Z \to U_0$ into n sections, and trivializes the 3-torsion subgroup in the relative Jacobian of $X_{U_0} \to U_0$. Let G be the Galois group of this cover. Let $P^\sharp$ be the normalization of P in the function field of U_0'. Let g be the genus of the generic fiber of $X \to P$. Let us write $\overline{{}_3\mathbf{M}_{g,n}}$ for the Deligne-Mumford compactification of the moduli scheme of n-pointed genus g curves with an abelian level 3 structure, see [DM], [PJ, 2.3.7], [dJ, 2.24] and references therein. There is a 'universal' stable n-pointed curve over $\overline{{}_3\mathbf{M}_{g,n}}$. We take the closure of the graph of the morphism $U_0' \to \overline{{}_3\mathbf{M}_{g,n}}$ in $P^\sharp \times \overline{{}_3\mathbf{M}_{g,n}}$ and obtain a modification $P' \to P^\sharp$ and a family of stable pointed curves $X' \to P'$. Note that the morphism $P' \to P^\sharp$ blows up outside of U_0'. We perform a G-equivariant blow up of P' to ensure that we have a morphism $r : X' \to X \times_P P'$, see [dJ, 3.18, 3.19 and 7.6]. Again this blow up has center outside U_0'. In summary:

Situation 1.1. *There is a Galois alteration $a : P' \to P$, with Galois group G, and a modification $r : X' \to X \times_P P'$ satisfying:*

a. *the morphism a is finite étale over U_0,*
b. *the morphism r is an isomorphism over the open set $U_0 \times_P P'$,*
c. *there are n sections $\sigma_i : P' \to X'$ such that the proper transform Z' of Z is the union of their images, and*
d. *$(X' \to P', \sigma_1, \dots, \sigma_n)$ is a stable pointed curve of genus g.*

Remark 1.2. The results of [dJ-O] imply that the curve X' exists over the variety $P^\sharp$. We suspect that the morphism r exists over $P^\sharp$ as well. If this is true, the following step is redundant. See [ℵ] for a similar construction using Kontsevich's space of stable maps.

We want to change the situation in such a way that we may assume $P' = P^\sharp$. This we can do as follows. Take a blow up $\beta : P_1 \to P$ such that the strict transform P_1' of P' with respect to β is finite flat over P_1, see [RG]. By our inductive assumption, we may assume that P_1 is nonsingular and that the inverse image Δ_1 of Δ in P_1 is a divisor with normal crossings. The Galois covering $U_0' \to U_0$ pulls back to a Galois covering of $P_1 \setminus \Delta_1$, and we get a ramified normal covering $P_1^\sharp$ of P_1. It follows that $P_1^\sharp$ maps to the strict transform P_1'. Hence it is clear that if we replace P by P_1, Δ by Δ_1, X' by $X' \times_P' P_1^\sharp$, then the morphism r of 1.1 exists over $P_1^\sharp$.

We now have in addition to (**a**) - (**d**),

e. *P is smooth, Δ is a strict divisor of normal crossings, and $a : P' \to P$ is finite.*

Since stable reduction over a normal base is unique (see [dJ-O], 2.3) we have that the action of G lifts to X'. Note that the G-action does not preserve the order of the sections.

1.3. **Local description.** Denote by $q : P' \to P$ the quotient map. Let $p \in P$ and let $p' \in q^{-1}(p)$. Let $s_1, \dots, s_{d-1} \in \mathcal{O}_{P,p}$ be a regular system of parameters on P at p such that $\Delta_x = V(s_1 \cdots s_r)$. Note that the stabilizer of p' is abelian; this actually follows from **(a)-(e)** above, as the morphism $P' \to P$ is ramified only along the divisor of normal crossings Δ: the local fundamental groups of $P \setminus \Delta$ are abelian, generated by loops around the components of Δ. Writing $t_i^n = s_i$ for suitable n, we identify a formal neighbourhood of p in P as a quotient of the smooth formal scheme $P'' = \operatorname{Spf} k[[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}]]$ by the finite group $(\mathbb{Z}/n\mathbb{Z})^r$ acting by n-th roots of unity on the t_i. Thus a formal neighbourhood of p' in P' is the quotient of P'' by a suitable subgroup $H \subset (\mathbb{Z}/n\mathbb{Z})^r$. This implies that $(\mathbb{Z}/n\mathbb{Z})^r/H$ is identified with the stabilizer of p' in G. We will use the formal scheme P'' only as a convenient device for describing the situation in terms of local coordinates. All the modifications we will perform below will have to occur over P' and X', which are global objects.

Denote by $X'' = P'' \times_{P'} X' \to P''$, the pulled back stable pointed curve. As $(\mathbb{Z}/n\mathbb{Z})^r/H \subset G$ acts on X', equivariantly over P', we get an induced action of $(\mathbb{Z}/n\mathbb{Z})^r$ on X'', equivariantly over P''. Let $x \in X''$ be a closed point lying over p', and let G_x be the stabilizer of x in $(\mathbb{Z}/n\mathbb{Z})^r$. There are two cases to be considered:

i. (Smooth case) Here x is a smooth point of the morphism $X'' \to P''$. In this case the completion of X'' at x is isomorphic to the formal spectrum of the $k[[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}]]$-algebra

$$k[[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}]][[u]].$$

Here we chose some coordinate u along the fiber such that G_x acts by a character ψ_u on u. This can easily be done by lifting an appropriate eigenvector in the cotangent space of X'' at x. There are two cases with respect to the position of the sections $Z'' \subset X''$:

(a) The point $x \in Z''$. In this case, since Z'' is invariant under the action of G_x, we can choose the coordinate u so that $Z'' = V(u)$.
(b) The point $x \notin Z''$. Note that the coordinate u is not uniquely chosen, and therefore the locus $u = 0$ is not uniquely determined. However, in case ψ_u is nontrivial, if we want to see G_x as acting through the torus for some toroidal structure on X'' at x it is necessary to include a locus like $V(u)$ in the boundary. Since globally we do not have such a locus, we must somehow modify this situation.

ii. (Node case) Here x is a node of the fiber of $X'' \to P''$ over p''. In this case the completion of X'' at x is isomorphic to the formal spectrum of the $k[[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}]]$-algebra

$$k[[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}]][[u, v]]/(uv - t_1^{k_1} \cdots t_r^{k_r}).$$

Again, by lifting appropriate eigenvectors in the cotangent space, we may choose the coordinates u, v so that there is a subgroup G_d of G_x of index at most 2 such that G_d acts by characters on u, v, but elements of G_x not in G_d switch the fiber components $u = 0$ and $v = 0$.

We would like to have the stabilizers acting toroidally on X'', in such a way that the quotient X becomes a strict toroidal embedding.

1.4. **Making the group act toroidally.** In this section we construct a two step G equivariant toroidal modification $Y_1 \to Y \to X'$ of X' such that (Y_1, D_1) is a G-strict toroidal situation. Here D_1 is the union of the inverse image of Z and Δ in Y_1.

There are two issues we need to resolve: in case (**ii**) above, we want to modify so that G_x/G_d disappears from the picture. In case (**i**b) we need to modify so that the stratum $u = 0$ is not necessary for the toroidal description of the G-action.

In order to describe a global modification we go back to our stable pointed curve $f : X' \to P'$.

1.4.1. *Separating branches along nodes.* Let $S = \operatorname{Sing} f$ be the singular scheme of the projection f. Let $Y' = B_S(X') \to X'$ be the blowup of X' along S. Let Y'' be the fiber product $Y' \times_{P'} X''$, and let $y'' \in Y''$. We remark that neither Y' nor Y'' is normal in general.

We want to give a local description of Y' and Y''. We use the notation $X_{/x}$ to denote the completion of X at the closed point $x \in X(k)$, and similar for the other varieties occuring below. We have $P'_{/p'} = \operatorname{Spf} R$ where R is the ring $(k[[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}]])^H$, with H as in 1.3 and we have $X'_{/x'} = \operatorname{Spf} R[[u, v]]/(uv - h)$ for some monomial $h \in R \subset k[[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}]]$. Finally $S = V(u, v)$ scheme theoretically.

From this it is easy to read off the following local description, using that blowing up commutes with completion in a suitable manner.

i'. (Smooth case) $Y''_{/y''} \simeq X''_{/x''}$ as above.
ii'. (Node case) $Y''_{/y''} \simeq \operatorname{Spf} k[[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}]][[u, w]]/(wu^2 - t_1^{l_1} \cdots t_r^{l_r})$. The stabilizer $G_{y''}$ acts diagonally on t_i, u, w (i.e., it acts via characters on these elements).
iii'. (Double case) $Y''_{/y''} \simeq \operatorname{Spf} k[[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}]][[u, w]]/(u^2 - t_1^{l_1} \cdots t_r^{l_r})$. The stabilizer $G_{y''}$ acts diagonally, but as in (**i**b) the coordinate w may not be determined.

The descriptions above determine the local structure of Y' also. Indeed, $Y'_{/y'}$ is simply the quotient of $Y''_{/y''}$ by the group H which acts trivially on the coordinate(s) u (and w). We are actually interested in a local description of the normalization $Y = (Y')^{\text{nor}}$ of Y'.

Let $D \subset Y = (Y')^{\text{nor}}$ be the union of the inverse image of Δ and of Z in Y. From the local descriptions given above it is already clear that $Y \setminus D \hookrightarrow Y$ is a strict toroidal embedding.

Moreover, D is G-strict, since Δ is G strict and as the blowup $Y \to X'$ separates fiber components. However, the action of G on $(Y \setminus D) \subset Y$ is not yet toroidal: indeed, in case (**i'**) and in case (**iii'**), if the character ψ_u or ψ_w is nontrivial, we have a problem. More precisely, this is the situation explained in (**i**b).

1.4.2. *Torifying a pre-toroidal action.* Here we show how to do one canonical blow up $Y_1 \to Y$ (analogously to [KKMS], II §2) which makes the action of G toroidal. The situation (Y, D, G) we reached at the end of Section 1.4.1 is summarized by the conditions in Defintion 1.4 below, see remark 1.5.

Remark 1.3. We expect that this discussion should be of interest in a more general setting.

Definition 1.4. Let $U = Y \setminus D \subset Y$ be a toroidal embedding, $G \subset \operatorname{Aut}(U \subset Y)$ a finite subgroup, such that D is G-strict. For any point $y \in Y$ denote the stabilizer of y by G_y. We say that the action of G is **pre-toroidal** if at every point $y \in Y$, either G_y acts toroidally at y, or the following situation holds:

- There exists an isomorphism $\epsilon : Y_{/y} \cong \operatorname{Spf}\ R[[u]]$,
- where $\operatorname{Spf} R$ is the completion of a toroidal embedding $T_0 \hookrightarrow Y_0$ at a point $y_0 \in Y_0$,
- where $D_{/y}$ corresponds to $(Y_0 \setminus T_0)_{/y_0} \times_{\operatorname{Spf}\ k} \operatorname{Spf}\ k[[u]]$,
- where G_y acts toroidally on $T_0 \hookrightarrow Y_0$ fixing y_0 and
- G_y acts on the coordinate u via a character ψ_u such that the isomorphism ϵ of completions is G_y-equivariant.

Remark 1.5. Note that (Y, D, G) is indeed pre-toroidal: the only cases where the action of G is not toroidal are (**i**b) (or (**i'**)) and (**iii'**). In case case (**i**b) we can take $Y_0 = P', T_0 = U'$. In case case (**iii'**) we can take Y_0 to be the normalization of Spec $k[t_1, \dots t_r, s_{r+1}, \dots, s_{d-1}, u]/(u^2 - t_1^{l_1} \cdots t_r^{l_r})$ with its natural toric structure, and use the parameter w in the place of u above.

Analogously to definition 1 of [KKMS], II §2, we define pre-canonical ideals:

Definition 1.6. Let $G \subset \operatorname{Aut}(U \subset Y)$ be a pre-toroidal action. A G-equivariant ideal sheaf $\mathcal{I} \subset \mathcal{O}_Y$ is said to be **pre-canonical** if the following holds:

For any y, y' lying on the same stratum, and any isomorphism $\alpha : O_{Y_{/y}} \to O_{Y_{/y'}}$ preserving the strata and inducing an isomorphism carrying G_y to $G_{y'}$, we have $\alpha(\mathcal{I}_{/y}) = \mathcal{I}_{/y'}$.

If $\mathcal{I}$ is pre-canonical, we say that the normalized blowup $b : (B_{\mathcal{I}}Y)^{\text{nor}} \to Y$ is a pre-canonical blowup.

Definition 1.7. We say that a pre-canonical blowup $b : \tilde{Y} \to Y$ **torifies** Y if G acts toroidally on $(b^{-1}U \subset \tilde{Y})$.

Theorem 1.8. *Let $G \subset \operatorname{Aut}(U \subset Y)$ be a pre-toroidal action. Then there exists a canonical choice of a pre-canonical ideal sheaf $\mathcal{I}_G$ such that the pre-canonical blowup $b : (B_{\mathcal{I}_G}Y)^{\text{nor}} \to Y$ torifies Y.*

The theorem follows immediately from the affine case below:

Proposition 1.9. *Let $T_0 \subset X_0$ be an affine torus embedding, $X_0 = \operatorname{Spec} R$. Let $G \subset T_0$ be a finite subgroup of T_0, let $p_0 \in X_0$ be a fixed point of the action of G, and let ψ_u be a character of G. Consider the torus embedding of $T = T_0 \times \operatorname{Spec} k[u, u^{-1}]$ into $X = X_0 \times \operatorname{Spec} k[u]$, and let G act on u via the character ψ_u. Write $p = (p_0, 0) \in X$ and write $D = (X_0 \setminus T_0) \times \operatorname{Spec} k[u]$. There is a canonical T-equivariant ideal $I_G \subset R[u]$, satisfying the following:*

Let $b : X_1 = (B_{I_G}X)^{\text{nor}} \to X$, the normalization of the blowup of X along I_G. Let $U_1 = b^{-1}(T_0 \times \operatorname{Spec} k[u])$. Then $U_1 \hookrightarrow X_1$ is toroidal and G acts toroidally on $U_1 \hookrightarrow X_1$.

If X_0', T_0', G', p_0' and ψ_u' is a second set of such data, and if we have an isomorphism of completions

$$\varphi : X_{/p} \cong X'_{/p'},$$

which induces isomorphisms $G \cong G'$ and $D_{/p} \cong D'_{/p'}$, then φ pulls back I_G to the ideal $I_{G'}$.

Furthermore, if q_0 is any point of X_0 and if $G_q \subset G$ is the stabilizer of $q = (q_0, 0)$ in G, then the stalk of I_G at q is the same as the stalk of I_{G_q} at q.

Remark 1.10. We call the ideal I_G the **torific ideal** of the situation $G \subset \mathrm{Aut}(T_0 \times \mathrm{Spec}\, k[u] \subset X)$.

Proof. For any monomial $t \in R[u]$ let χ_t be the associated character. We restrict these characters to G and obtain a character $\psi_t : G \to k^*$.

Define $M_u = \{t | \psi_t = \psi_u\}$, the set of mononials on which G acts as it acts on u. Notice that the character ψ_u of G is uniquely determined by the data $G \to \mathrm{Aut}(X_{/p}, D_{/p})$. Define the torific ideal $I_G = \langle M_u \rangle$, the ideal generated by M_u. We see that it satisfies the variance property of the proposition with respect to isomorphisms φ. We leave the localization property at the end of the proposition as an exercise to the reader.

Define $b : X_1 \to X$ as in the proposition.

Since G is a subgroup of T_0, we have that M_u contains a monomial in R. Therefore the ideal I_G is generated by u and a number of monomials $t_1, \ldots, t_m \in R \cap M_u$. The blow up has a chart associated to each of the generators $u, t_1, \ldots, t_m$. On the chart associated to u (nicknamed "$u \neq 0$") we have that the inverse image of D contains the inverse image of $D \cup V(u)$, and hence the action of G is toroidal, being toroidal with respect to $T \subset X$ on X. The other charts "$t_i \neq 0$" can be described as the spectra of the rings

$$\tilde{R} = R[u][v, \{s_j\}_{j \neq i}]/(vt_i - u, s_j t_i - t_j, h_\alpha),$$

that is $v = u/t_i$ and $s_j = t_j/t_i$. Since there are no relations between u and t_j we can take the h_α to be certain polynomials in $R[s_j]$. Note that G fixes the element v in this algebra.

Thus it follows that the normalization of the ring $\tilde{R}$ is of the form $R'[v]$, where G acts trivially on v, R' is the ring associated to an affine torus embedding and G acts toroidally on $\mathrm{Spec}\, R'$. □

1.4.3. *Strictness.* We return to the triple (Y, D, G) we obtained at the end of 1.4.1, in particular we have the Galois alteration $P' \to P$ with group G and the G-equivariant map $Y \to P'$. Denote by $b : Y_1 \to Y$ the torifying blowup obtained by normalizing the blowup at the torific ideal of Y, as in Theorem 1.8.

It remains to check the following (but see also Remark 1.14 below):

Proposition 1.11. *The divisor $D_1 = b^{-1}(D) \subset Y_1$ is G-strict.*

Thus (Y_1, D_1) is G-strict toroidal, and therefore the quotient $(Y_1/G, D_1/G)$ is a strict toroidal embedding.

First, we show that D_1 is strict:

Lemma 1.12. *Let $U \subset Y$ be a strict toroidal embedding and let $G \subset \mathrm{Aut}(U \subset Y)$ be a pre-toroidal action. Let $b : Y_1 \to Y$ be a torifying blowup. Then $b^{-1}U \subset Y_1$ is a strict toroidal embedding.*

Proof. Clearly it is enough to check the claim on Zariski open subsets covering Y_1. Fix $y \in Y$. If G acts toroidally at y, then we can pick a Zariski neighborhood of y where it acts toroidally, and the result follows from Theorem 1^* of [KKMS].

Assume G does not act toroidally at y. Now, although a coordinate u for the pre-toroidal action is not globally defined, we may always find such coordinates on Zariski neighborhoods of y, as follows. Let $\epsilon : Y_{/y} \cong \text{Spf } R[[u]]$ be as in Defintion 1.4. Let $u' \in O_{Y,y}$ be an element of the local ring of Y at y that is congruent to $\epsilon^{-1}(u)$ modulo the maximal ideal m_y (or any power of m_y). Then the element

$$u'' = \sum_{g \in G_y} g(u')\psi_u^{-1}(g)/|G_y| \in O_{Y,y}$$

transforms according to ψ_u under the action of G_y and is congruent to $\epsilon^{-1}(u)$ modulo m_y. Thus we have $\epsilon(u'') = \mu u$ for some unit μ. We may then change the isomorphism ϵ so that the element u'' will correspond to the coordinate u. Therefore, we may assume that there is a G_y-invariant Zariski open neighbourhood $W = W_y$ of y and a function $u \in \Gamma(W, \mathcal{O})$ that transforms according to the character ψ_u under the group G_y, and giving rise to the local coordinate for a suitably chosen isomorphism ϵ as in Definition 1.4. After possibly shrinking W, we get that W is strict toroidal with respect to the divisor $D_W = (W \cap D) \cup V(u)$. Now the strictness of $Y \setminus D \subset Y_1$ follows immediately from theorem 1^* of [KKMS], applied to (W, D_W). ☐

Remark 1.13. Note that if we choose W sufficiently small, we may assume that the conical polyhedral complex of (W, D_W) consists of a single cone σ_W (see [KKMS], II §1 definition 5, through p. 71).

Remark 1.14. At this point it should be remarked that we could get away without G-strictness: using the constructions in [KKMS], I §2 lemmas 1-3 on pages 33-35, and [dJ], 7.2, it is not difficult to construct a G-equivariant blowup which is a G-strict toroidal embedding, namely the blowup corresponding to the barycentric subdivision. This is discussed in detail in [ℵ-W]. Still it is of interest to know that $b^{-1}(D) \subset Y_1$ is already G-strict.

Proof of Proposition 1.11. Let $E \subset b^{-1}(D)$ be an irreducible component and let $g \in G$ be an element such that $g(E) \cap E \neq \emptyset$. We need to show that $g(E) = E$. In the case that $b(E)$ is a component of D, this follows immediately from the G-strictness of D, see end of 1.4.1; the difficult case is that of a component which is collapsed under b.

Let $y_1 \in Y_1$ be a general point of the intersection $g(E) \cap E$, and let $y = b(y_1) \in Y$. Let $D_1, \ldots, D_s$ be the components of D at y. The components $g^{-1}(D_i)$ are the components of D at $g^{-1}(y)$. After reordering, we may assume that $D_1, \ldots, D_r$ are the components which contain $b(E)$, for some $r \leq s$. As $y_1 \in g(E) \cap E$ we get $y \in g(b(E)) \cap b(E)$, hence $y \in g(D_i) \cap D_i$ for all $i = 1, \ldots, r$. By G-strictness of D, we get $g(D_i) = D_i$ for $i = 1, \ldots, r$.

Let σ_y be the cone corresponding to the toroidal structure (W_y, D_{W_y}) at y from remark 1.13 above. The divisors D_i correspond to rays $\tau_{D_i,y}$ in the boundary of σ_y. The divisor E corresponds to a ray $\tau_{E,y}$ in σ_y, belonging to the polyhedral decomposition associated to the torifying blowup.

Around the point $g^{-1}(y)$ we use the Zariski open neighbourhood $W_{g^{-1}(y)} = g^{-1}(W_y)$ and for the divisor $D_{W_{g^{-1}(y)}}$ we take $g^{-1}(D_{W_y})$. Hence we get an identification $\sigma_y \cong \sigma_{g^{-1}(y)}$ given by g^{-1}. Under this identification the ray $\tau_{E,y}$ is mapped to the ray $\tau_{g^{-1}(E),g^{-1}(y)}$. Note that $b(E)$ passes through $g^{-1}(y)$, hence we have the ray $\tau_{E,g^{-1}(y)}$ in $\sigma_{g^{-1}(y)}$. We have to show that $\tau_{E,g^{-1}(y)} = \tau_{g^{-1}(E),g^{-1}(y)}$.

Let M_y be the group of Cartier divisors supported along D_{W_y} (see [KKMS], II §1, definition 3). After shrinking W_y we may assume that M_y is the free abelian group generated by the divisors of rational functions $f_j \in \Gamma(W_y, \mathcal{O}), j = 1, \ldots, m$ and the function u on W_y. We may consider f_j and u as rational functions on Y_1 as well.

The ray $\tau_{E,y}$ is determined by the orders of vanishing of the functions f_j and u along E on Y_1, let us call these orders n_j and n. Similarly, $\tau_{E,g^{-1}(y)}$ (resp. $\tau_{g^{-1}(E),g^{-1}(y)}$) is determined by the order of vanishing of the functions $g^*(f_j)$ and $g^*(u)$ along E (resp. $g^{-1}(E)$), let us call these orders n'_j and n' (resp. n''_j and n'').

It is clear that $n_j = n''_j$ and $n = n''$. Notice that the divisors of the functions $g^*(f_j)$ are supported along $\cup_{i=1}^s g^{-1}(D_i)$ in $W_{g^{-1}(y)}$. Here $g^{-1}(D_i) = D_i$ for $i \le r$, so that f_j/g^*f_j has order 0 along $D_i, i \le r$. Note also that the components D_i, $i > r$ do not contain $b(E)$ so they do not come into play. Since the vanishing along E may be computed on $b^{-1}(W_y \cap W_{g^{-1}(y)})$, we see that $n_j = n'_j$. We are through if we show that $n = n'$; this is equivalent to showing that the order of vanishing of u along E is the same as the order of vanishing of $g^{-1}(u)$ along E.

The component E is exceptional for the morphism b. The torifying blowup b over W_y blows up inside $V(u)$ only. Hence we see that $b(E) \cap W_y$ is contained in $V(u)$, i.e., $n > 0$. The same argument applied to $g^*(u)$ on $W_{g^{-1}(y)}$ works to show that $n' > 0$. For a general point $z \in b(E)$, we see that both u and $g^{-1}(u)$ define a local coordinate that can be used to define the local toroidal structure around z, as in Definition 1.4. Therefore, we need only to show that the valuation of u along E does not depend on the choice of the parameter u as in Definition 1.4.

Let $\epsilon : Y_{/z} \cong \mathrm{Spf}\ R[[u]]$ be as in Definition 1.4. Any other $u' \in R[[u]]$ that gives a local coordinate for some pretoroidal structure is of the form

$$u' = \mu u + r_0 + \sum_{j \ge 2} r_j u^j,$$

with $\mu, r_0, r_j \in R$ and where μ is a unit and r_0 is in the maximal ideal of R. The element μ being a unit, we may divide by it without changing the orders of vanishing on Y_1. Thus we may assume $\mu = 1$.

Consider the automorphism φ of $R[[u]]$ given by $\varphi(r) = r$, $r \in R$ and

$$\varphi(u) = u' = u + r_0 + \sum_{j \ge 2} r_j u^j.$$

This is one of the automorphisms which occur in Proposition 1.9. Hence φ lifts to an automorphism of the formal completion $Y_1^\wedge$ of Y_1 along $b^{-1}(z)$. Thus the order of vanishing of u' along E corresponds to the order of vanishing of u along the transform of E under φ. However, since φ obviousely lies in the connected component of the identity in the group of all these automorphisms, it cannot nontrivially permute the finitely many exceptional components. Hence we get the equality as desired.

1.5. **Conclusion of proof.** If we combine the results of 1.4.1, 1.4.2 and Proposition 1.11, we see that we may assume our Galois alteration (Y_1, D_1, G) of (X, Z) is such that $Y \setminus D_1 \hookrightarrow Y_1$ is a G-strict toroidal embedding. Hence the quotient $(Y_1 \setminus D_1)/G \hookrightarrow Y_1/G$ is a strict toroidal embedding, and $Y_1/G \to X$ is a modification. Hence we replace (X, Z) by the toroidal $(Y_1/G, D_1/G)$. By Theorem 0.5, there exists a (toroidal) resolution of singularities of this pair. This ends the proof of Theorem 0.1. □

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