Compactifying the space of stable maps

- `http://math.bu.edu/people/abrmovic`
- work with Angelo Vistoli (+ Alessio Corti, Johan de Jong)
- Work over $\mathbb{C}$

if one could do all characteristics one would probably be famous, because of likely relation to resolution of singularities.

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(1) As every graduate student of algebraic geometry knows, nice moduli problems are typically:

Deligne-Mumford stacks admitting projective coarse moduli spaces.

since most likely some of the audience are not graduate students in algebraic geometry, I'll tell you how to think about them so you can pretend you know what they are (going into details would not be the right thing to do in a talk)

Think about it as locally like a scheme V modulo the action of a finite group G; but not the scheme V/G, but V/G with an atlas of charts like (V, G) with suitable gluing data.

This are like orbifolds in Thurston's terminology.

(2) Examples:

- $\overline{\mathcal{M}}_{g,n}$; $\quad \overline{\mathcal{M}}_{g,n}(X, d =: \mathcal{K}^{d}_{g,n}(X) \rightsquigarrow$ stable maps (here X projective)
- V/G as a stack
- $BG = \{pt\}/G =$ moduli of principal homogeneous G-spaces

In geometry we draw pictures. I can only draw this by going back to kindergarten:

$$\begin{array}{rll} \{pt\}: & \bullet & \\ & \downarrow & \text{degree } 6 \\ B\mathbb{Z}/6: & \blacktriangleleft & \\ & \downarrow & \\ \{pt\}: & \bullet & \text{"coarse moduli space"} \end{array}$$

$$\begin{array}{ccc} G & & \{pt\} \\ \downarrow & \xrightarrow{/G} & \downarrow \\ \{pt\} & & \mathbf{B}G \end{array}$$

(3) Stacks have been accepted as "moduli objects" for schemes

we are part of a conspiracy to force you to

now accept them as basic algebraic geometry objects!

Mumford initiated this with their cohomological study. If you want quantum cohomology you may need space of stable maps

(4) Is $\mathcal{K}^d_{g,n}(\mathcal{X})$ likely to be interesting for a DM stack $\mathcal{X}$?

examples:

$\mathcal{X} = \overline{\mathcal{M}}_{\gamma,\nu} \rightsquigarrow \mathcal{K}^d_{g,n}(\mathcal{X})$ = moduli of fibered surfaces: families of stable curves over stable curves

$\mathcal{X} = \mathbf{B}G \rightsquigarrow$ moduli of principal G bundles on stable curves!

(5) Naive def. of stable maps (n=0):

$$f : C \to \mathcal{X}$$

where $C =$ nodal projective connected curve

s.t. $\mathrm{Aut}_{\mathcal{X}}(f : C \to \mathcal{X})$ finite (equivalently - three point condition)

PROBLEM: NOT COMPLETE!!

(6) e.g. consider degeneration of a smooth curve of genus 2, $G = (\mathbb{Z}/l)^4$:

central fiber has geom. genus 1 with one node

smooth fiber C_η has connected principal G bundle

one for each isomorphism $H_1(C_\eta, \mathbb{Z})/l \to G$

C_s has $H_1(C_\eta, \mathbb{Z}) = \mathbb{Z}^3$, so no connected G bundle!!

We asked ourselves: is Nature blocking us?

We learned that we need to patiently observe what solution Nature is giving us, since it is beautiful, and yes, most natural!

$$\begin{array}{ccc} P_\eta & \subset & ? \\ \downarrow & & \downarrow \\ C_\eta & \subset & C \\ \downarrow & & \downarrow \\ \{\eta\} & \subset & S \end{array}$$

If we try to extend directly (e.g. normalization of C in P_η) we typically get ramification index l over C_s.

(i) after base change of order l may assume P_η extends over C_{sm}:

$$\begin{array}{ccccc} P_\eta & \subset & P_{\mathrm{sm}} & & \\ \downarrow & & \downarrow & & \\ C_\eta & \subset & C_{\mathrm{sm}} & \subset & C \end{array} \qquad \text{(Purity of Branch Locus)}$$

(ii) at node x: $C_x : uv = t^l$

find a "uniformization" $V_x \to C_x$

$V_x : \{u_1 v_1 = t\} \quad u = {u_1}^l, v = {v_1}^l$

V_x nonsingular $\Longrightarrow$ cover extends!! Purity...

(iii) $V_x \to C_x$ is a **chart** for an **orbifold structure** $\mathcal{C} \to C$, namely a DM stack structure over C!!

That's all!!

(so the biggest conspirator is Mother Nature herself)

(8) Definition (case n=0):

Let $\mathcal{X}$ be a D-M stack, $\mathcal{X} \to X \subset \mathbb{P}^N$ proj. coarse moduli scheme.

A **stable map** $f : \mathcal{C} \to \mathcal{X}$ to $\mathcal{X}$ of genus g, degree d, is a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ \downarrow & & \downarrow \\ C & \xrightarrow{\overline{f}} & X \end{array}$$ where

(i) $\mathcal{C} \to C$ coarse moduli scheme, isomorphic over C_{sm}

(ii) $\mathcal{C}$ nodal stack (local charts nodal)

(iii) $C \to X$ stable map ($\Leftrightarrow \mathrm{Aut}_{\mathcal{X}} f$ finite) of geus g and degree d

(iv) $\mathcal{C} \to \mathcal{X}$ representable (no unnecessary branchings)

(9) **Theorem** ($\aleph$-Vistoli): the moduli of stable maps of genus g and degree d to $\mathcal{X}$ is a proper D-M stack $\mathcal{K}_g^d(\mathcal{X}, d)$ admitting a projective coarse moduli scheme $\mathbf{K}_g^d(\mathcal{X}, d)$.

In fact, there is a commutative diagram of finite maps (not cartesian)

$$\begin{array}{ccc} \mathcal{K}_g^d(\mathcal{X}, d) & \rightarrow & \mathbf{K}_g^d(\mathcal{X}, d) \\ \downarrow & & \downarrow \\ \mathcal{K}_g^d(X, d) & \rightarrow & \mathbf{K}_g^d(X, d) \end{array}$$

(Behrend-Manin) (Fulton-Pandharipande).

(10) Issues coming in proof:

(i) existence of bottom line in diagram

(ii) valuative criterion for properness like for $\mathbf{B}G$ (purity!! see below)

(iii) annoying technicalities since there's no Hartshorne for stacks

(11) **Purity Lemma**

S a nonsingular surface, $U \subset S$, $\quad U = S \smallsetminus \{pt\}$.

Given a diagram

$$\begin{tikzcd}
 & & \mathcal{X} \arrow[d] \\
U \arrow[r] \arrow[urr, "\tilde{f}_U"] & S \arrow[r, "\overline{f}"] \arrow[ur, dotted, "\exists ?"] & X
\end{tikzcd}$$

Then the $\tilde{f}_U$ extends uniquely to $f : S \to \mathcal{X}$

(12) **Proof.** Problem is local. Replace X by small analytic nbhd of $\overline{f}(pt)$. Pick a chart $\mathcal{X} = V/G$. So $V \to \mathcal{X}$ finite étale. Replace S by a similar nbhd.

$$\begin{array}{ccccc}
U \times_{\mathcal{X}} V & \to & S \times_X V & \to & V \\
 & & & & \downarrow \\
\downarrow & & \downarrow & & \mathcal{X} \\
 & & & & \downarrow \\
U & \subset & S & \to & X
\end{array}$$

$U \times_{\mathcal{X}} V \to U$ finite étale, $\pi_1(U) = \{1\}$

$\implies \exists$ a section $U \to U \times_{\mathcal{X}} V$.

Closure of image in $S \times_X V$ is finite birational over S.

ZMT $\implies$ isomorphic to S, so $\exists$ a section $S \to S \times_X V$.

So $\exists$ a lifting $S \to \mathcal{X}$.

Some applications of $\mathcal{K}_g(\mathbf{B}G)$:

$\mathcal{K}_g(\mathbf{B}G)$ = moduli of twisted principal G bundles on stable curves.
$\supset$ Galois admissible covers.

Galois case avoides problems in Harris - Mumford

(a) $G = (\mathbb{Z}/l)^{2g} \rightsquigarrow$ moduli compactification of Mumford's $\mathcal{M}_g^{(l)}$.

(b) $G = \mathfrak{S}_n$: recall

$$\begin{array}{ccc} \{\mathfrak{S}_n\text{- bundles}\} & \leftrightarrow & \{n\text{- sheeted étale covers}\} \\ P & \mapsto & \mathfrak{S}_{n-1}\backslash P \end{array}$$

Moduli of twisted covers = normalization of Harris-Mumford space of admissible covers.