Extending Triangulations and Semistable Reduction

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0. INTRODUCTION

In the past three decades, a strong relationship has been established between convex geometry, represented by convex polyhedra and polyhedral complexes, and algebraic geometry, represented by toric varieties and toroidal embeddings. In this note we exploit this relationship in the following manner. We address a basic problem in algebraic geometry: a certain version of semistable reduction. We translate a local case of the problem into a basic problem about polyhedral complexes: extending triangulations. Once we solve the second problem, the first follows. We have taken the opportunity with this note to try to extend some bridges between the terminologies of these two theories.

0.1. **Semistable Reduction.** We work over the field of complex numbers $\mathbb{C}$. Let $f : X \to B$ be a proper morphism of algebraic varieties, whose generic fiber is reduced and absolutely irreducible. Thus there exists a Zariski dense open set $U \subset B$ such that the fiber $f^{-1}(b)$ over any point in $b \in U$ is a compact complex algebraic variety.

Loosely speaking, semistable reduction for a morphism like f is a meta-problem of "desingularization of morphisms," where the goal is to "change f slightly" so that it becomes "as nice as possible". Of course, we need to specify more precisely what we mean by the clauses in quotation marks.

0.1.1. *What do we mean by a morphism being "as nice as possible?"* First of all, X and B should be as nice as possible, namely nonsingular. Moreover, we want f to have a nice, explicit local description, so that the fibers of f have the simplest possible singularities.

Such a wonderful morphism will be called **semistable**. Here is the definition:

Definition 0.1. Let $f : X \to B$ be a flat projective morphism, with connected fibers, of nonsingular varieties. We say that f is **semistable** if for each point $x \in X$ with $f(x) = b$ there is a choice of formal coordinates $B_b = \text{Spec } \mathbb{C}[[t_1, \ldots, t_m]]$ and $X_x = \text{Spec } \mathbb{C}[[x_1, \ldots, x_n]]$, such that f is given by:

$$t_i = \prod_{j=l_{i-1}+1}^{l_i} x_j,$$

where $0 = l_0 < l_1 \cdots < l_m \leq n$, $n = \dim X$, and $m = \dim B$.

We must state right up front that in this note we will **not** end up with a semistable morphism, but we will get very close. In particular, our results here form an additional step in recent work on semistable reduction [ℵ-dJ96, ℵ-K97].

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0.1.2. *What do we mean by "changing f slightly?"* First we define two types of operations necessary for semistable reduction:

Definition 0.2. An **alteration** $B_1 \to B$ is a proper, generically finite, surjective morphism. A **modification** $Y \to X$ is a birational proper morphism (equivalently, a birational alteration).

Given a morphism $X \to B$ as before, and an alteration $B_1 \to B$, we call the component of $X \times_B B_1$ dominating B_1 the **main component** and denote it by $X \widetilde{\times}_B B_1$.

We are now ready to state the semistable reduction problem in its ultimate form:

Problem 0.3. Let $X \to B$ be a flat projective morphism, with connected fibers, of nonsingular varieties. Find an alteration $B_1 \to B$, and a modification $Y \to X \widetilde{\times}_B B_1$, such that $Y \to B_1$ is semistable.

0.1.3. *Nearly Semistable Morphisms.* We will need some terminology in order to state the weaker version of semistable reduction we actually address here. We will follow [KKMS73] for the basic definitions.[1]

Definition 0.4.

1. A **toric variety** is a normal[2] variety X with an open embedded copy T of $(\mathbb{C}^*)^n$, such that the natural $(\mathbb{C}^*)^n$-action on T extends to all of X. We sometimes call the pair (X,T) a **torus embedding**.
2. More generally, suppose Y is a normal variety with a smooth open subvariety U_Y satisfying the following condition: locally analytically at every point, (Y, U_Y) is isomorphic to a local analytic neighborhood of some torus embedding (X,T). We then call Y a **toroidal variety** and (Y, U_Y) a **toroidal embedding**.[3]
3. A dominant morphism $f : (X, U_X) \to (B, U_B)$ of toroidal embeddings is called a **toroidal morphism**, if locally analytically near every point on X it is isomorphic to a torus equivariant morphism of toric varieties.

Roughly speaking, a toric variety is "monomial:" an affine toric variety is always defined by binomial equations, and any toric variety can always be covered by affine charts in such a way that every overlap isomorphism is a monomial map. Similarly, a toroidal variety is "locally monomial" and a toroidal morphism is a "locally monomial morphism."

If $U_B \subset B$ is a toroidal embedding, then we may write $B \setminus U_B$ as a union of divisors $D_1 \cup \cdots \cup D_k$. More precisely, recall that $B \setminus U_B$ can be decomposed into strata of varying dimensions (see [KKMS73] or [GM88]). In particular, let us define $U_B^{(2)}$ to be the union of U_B and the codimension 0 strata of $B \setminus U_B$. This notation makes sense since we've actually only removed pieces of codimension ≥ 2 from B to construct $U_B^{(2)}$.

We now detail the type of morphisms we will treat:

Definition 0.5. A proper toroidal morphism $f : (X, U_X) \to (B, U_B)$ is said to be **nearly semistable** if the following conditions hold:

1. There are no horizontal divisors in X, namely: $U_X = f^{-1}(U_B)$.
2. The base B is nonsingular.
3. The morphism f is equidimensional.

[1] Also, mimicking standard notation from algebraic topology, $f : (X, A) \longrightarrow (Y, B)$ will be understood to mean that A and B are subvarieties of X and Y respectively; and that f is a morphism from X to Y satisfying $f(A) \subset B$.

[2] Although normality is not assumed in some contexts, all toric varieties will be normal in this paper.

[3] We will sometimes follow [KKMS73] and also refer to the inclusion $U_Y \subset Y$ as a toroidal embedding.

4. All the fibers of f are reduced.
5. The restriction of f to $U_B^{(2)}$ is semistable, i.e., "f is semistable in codimension ≤ 1."
6. The singularities of variety X are at worst finite quotient singularities.

One may ask how far a nearly semistable morphism is from a semistable one. The answer is simple: every toroidal semistable morphism is nearly semistable; and a nearly semistable morphism $X \to B$ is semistable if and only if X is nonsingular (see [ℵ-K97]).

0.1.4. *The Result.* The problem addressed in this paper is a special (local) case of nearly semistable reduction:

Theorem 0.6. *Set $B = \mathbb{A}^n_{\mathbb{C}}$ and let U_B be the natural open subscheme of B whose underlying complex variety is $(\mathbb{C}^*)^n$. Note that the inclusion $U_B \subset B$ is a toroidal embedding, and let $f : X \to B$ be a proper morphism satisfying:*

1. *$U_X := f^{-1}(U_B) \subset X$ is a toroidal embedding, and $f : (X, U_X) \to (B, U_B)$ is a toroidal morphism;*
2. *f is equidimensional, with smooth and absolutely irreducible generic fiber;*
3. *every fiber of f is reduced.*

Then there exists a **finite** *toric morphism $(B_1, U_{B_1}) \to (B, U_B)$ and a toroidal modification $Y \to X \times_B B_1$, such that $Y \to B_1$ is nearly semistable.*

One may ask what right we have to make all these assumptions on the morphism f we start with. In [ℵ-K97] it is shown that given any morphism f, as in Problem 0.3, we can reduce it to a toroidal morphism f as in Theorem 0.6. Such morphisms are called **weakly semistable** in [ℵ-K97].

The methods of [ℵ-K97] are quite different from what we do here. In short, they involve:

1. Making $X \to B$ toroidal. This follows easily from the methods of [ℵ-dJ96].
2. Making a toroidal $X \to B$ satisfy the conditions in the theorem. Locally this can be done easily using toroidal modifications and finite base changes. To do it globally one uses a covering trick of Kawamata (see [Kaw81, Theorem 17]).

Moreover, once the local results here are established, we can go back to [ℵ-K97] and, using Kawamata's covering trick, extend it to prove nearly semistable reduction in general.

0.2. **Extending Triangulations.** We now wear our polyhedral glasses.

For the concepts of a **compact polyhedral complex** Δ and a **conical polyhedral complex** Σ see [KKMS73, pg. 69, Definition 5]. An **integral structure** on a compact or conical polyhedral complex is defined in [KKMS73, pg. 70, Definition 6]. We will always assume that our complexes come equipped with an integral structure. From here on, we will simply say **polyhedral complex**, when we mean a compact polyhedral complex with integral structure.

Remark 0.7. A useful example of a polyhedral complex to consider is a finite collection $\mathcal{P}$ of integral polyhedra in $\mathbb{R}^n$. (Recall that a polyhedron in $\mathbb{R}^n$ is **integral** iff all its vertices lie in $\mathbb{Z}^n$.) If $\mathcal{P}$ is closed under intersection and taking faces, then $\mathcal{P}$ is a polyhedral complex. Note, however, that **not** all polyhedral complexes arise this way. This accounts for some of the geometric richness of toroidal varieties.

Again, in [KKMS73, pg. 70], it is shown that for any compact polyhedral complex Δ, one can construct a conical polyhedral complex, which we denote $\Sigma(\Delta)$ — namely the cone over Δ. To reverse the process, define a **slicing function** $h : \Sigma \to \mathbb{R}$ to be a nonnegative continuous function, whose restriction to every cone $\sigma \in \Sigma$ is linear, which vanishes only at the origin $\mathbf{O} \in \Sigma$. Then the **slice** $h^{-1}(1)$ of Σ defines a compact polyhedral complex $\Delta(\Sigma, h)$.

We denote by $\mathrm{Sk}^k(\Delta)$ the k-skeleton of Δ. We will also use $\#S$ for the cardinality of a set S, and $\mathrm{Cone}(V)$ for the set of all nonnegative linear combinations of a set of vectors $V \subset \mathbb{R}^n$.

By a **subdivision** Δ' of Δ (resp. Σ' of Σ) we will mean a finite partial polyhedral decomposition of Δ (resp. Σ), as in [KKMS73, pg. 86, Definition 2], with the **completeness** property: $|\Delta'| = |\Delta|$. (Recall that the notation $|\Delta|$ simply means the topological space consisting of the union of all the cells of Δ.) A subdivision Δ' is called a **triangulation** or a **simplicial subdivision** if every cell of Δ' is a simplex.

A **lifting function** (or **order function**) $f : \Delta \to \mathbb{R}$ on a polyhedral complex is a continuous function, convex and piecewise linear on each cell of Δ, respecting the integral structure. In the conical case ($f : \Sigma \to \mathbb{R}$) we add the requirement that f be homogeneous: $f(\lambda x) = \lambda f(x)$, for all $\lambda \geq 0$ and all $x \in |\Delta|$ [KKMS73, pg. 91, condition $(*)$].

Remark 0.8. We follow the convention in [KKMS73], where one requires a lifting function to be "convex down" on each cell, namely $f(\lambda x + \mu y) \geq \lambda f(x) + \mu f(y)$. Also, all our lifting functions take rational values on the lattices in the cells. This is in contrast with the polyhedral convention, as in [Zie95], where lifting functions are "convex up" and real values are allowed.

Given a lifting function $f : \Delta \to \mathbb{R}$, (resp. $f : \Sigma \to \mathbb{R}$) we define the subdivision Δ_f (resp. Σ_f) **induced by** f, to be the coarsest subdivision such that f is linear on each cell.[4]

Remark 0.9. The subdivision induced by f is clearly determined by the values of f on its vertices $\mathrm{Sk}^0(\Delta_f)$ (resp. its edges $\mathrm{Sk}^1(\Sigma_f)$). In fact one can construct f from its values on $\mathrm{Sk}^0(\Delta_f)$ (resp. $\mathrm{Sk}^1(\Sigma_f)$) as the minimal function which is convex on each cell, having the given values on $\mathrm{Sk}^0(\Delta_f)$ (resp. $\mathrm{Sk}^1(\Sigma_f)$). In particular, a subdivision induced by a lifting function **can** sometimes add new vertices to Δ (resp. new edges to Σ). However, with some care, we can control this behavior.

We will prove the following result:

Theorem 0.10. *Let Δ be a polyhedral complex and $\Delta_0 \subset \Delta$ a subcomplex. Let Δ'_0 be a triangulation of Δ_0 induced by a lifting function. Then there exists a triangulation Δ' of Δ, also induced by a lifting function, which extends Δ'_0 and introduces no new vertices. That is, $\mathrm{Sk}^0(\Delta') = \mathrm{Sk}^0(\Delta) \cup \mathrm{Sk}^0(\Delta'_0)$.*

Applying this to a slice of a conical polyhedral complex we obtain:

Corollary 0.11. *Let Σ be a conical polyhedral complex admitting a slicing function $h : \Sigma \to \mathbb{R}$, and let $\Sigma_0 \subset \Sigma$ be a subcomplex. Let Σ'_0 be a triangulation of Σ_0 induced by a lifting function. Then there exists a triangulation Σ' of Σ, also induced by a lifting function, which extends Σ'_0 and introduces no new edges. That is, $\mathrm{Sk}^1(\Sigma') = \mathrm{Sk}^1(\Sigma) \cup \mathrm{Sk}^1(\Sigma'_0)$.*

One may ask, "Do we really need to assume that Δ'_0 is induced by a lifting function?" The simplest example showing that this is indeed the case was communicated to us independently by R. Adin and B. Sturmfels:

Let $\Delta \subset \mathbb{R}^3$ be the triangular prism $\delta = \mathrm{Conv}\{f_{0,0}, \ldots, f_{1,2}\}$, where:

$$\begin{array}{lllllllll} f_{0,0} & = & (0,0,0); & f_{0,1} & = & (1,0,0); & f_{0,2} & = & (0,1,0) \\ f_{1,0} & = & (0,0,1); & f_{1,1} & = & (1,0,1); & f_{1,2} & = & (0,1,1) \end{array}$$

[4] Here we are poised on the verge of a notational quagmire. In [KKMS73], a subdivision induced by a lifting function is called **projective**, which makes sense from the algebro-geometric point of view, as the corresponding modification of toric varieties $X_{\Sigma'} \to X_\Sigma$ is a projective morphism. However, this kind of projectivity is foreign to the polyhedral world. In [Zie95] such subdivisions are called **regular**, which is fine by us, except that in [KKMS73] the term "regular" is sometimes used for an explicit type of subdivision which is very regular indeed... We will thus simply refer to our subdivisions as "subdivisions induced by lifting functions."

Let $\Delta_0 = \partial\Delta$ be the boundary of our prism.

Let Δ_0' be the subdivision of Δ_0 obtained by inserting the following new edges:

$$\overline{f_{0,0}f_{1,1}}, \overline{f_{0,1}f_{1,2}}, \overline{f_{0,2}f_{1,0}}$$

(So we've "cut" a new edge into each 2-face of Δ_0.) It is an easy exercise to see that there is no extension of Δ_0' (to a triangulation of Δ) without new vertices: in particular, any 3-cell of such an extension must have an edge intersecting the midpoint of some edge of Δ_0' — a contradiction. It is also not hard to see that Δ_0' can not be induced by any lifting function [Ful93, Chapter 3].

1. Reduction of Theorem 0.6 to 0.10

Let $f : X \to B$ be as in Theorem 0.6 and $f_\Sigma : \Sigma_X \to \Sigma_B$ the associated morphism of rational conical polyhedral complexes. Note that Σ_B is a nonsingular cone (a simplicial cone of index 1): it is simply the nonnegative orthant in $\mathbb{R}^n$, generated by the standard basis vectors $\{\hat{e}_i\}$. Let τ_i be the edges of Σ_B, namely $\tau_i = \mathrm{Cone}(\hat{e}_i)$. We identify the lattice of τ_i with $\mathbb{Z}\hat{e}_i$.

Let $\Sigma_B^1 = \bigcup \tau_i$ be the 1-skeleton of Σ_B and $\Sigma_X^1 = f_\Sigma^{-1}(\Sigma_B^1)$. Also let $\Sigma_{X,i} = f_\Sigma^{-1}(\tau_i)$. For an integer k_i let $N_i(k_i)$ be the integral structure on $\Sigma_{X,i}$ obtained by intersecting the lattices in $\Sigma_{X,i}$ with $f_\Sigma^{-1}(\mathbb{Z}k_i \cdot \hat{e}_i)$.

By [KKMS73, Chapter III, Theorem 4.1 pg. 161], as interpreted in [KKMS73, Chapter II, §3], there exists an integer k_i and a simplicial subdivision $\Sigma_{X,i}'$ of $\Sigma_{X,i}$, **which is induced by a lifting function**, having index 1 with respect to the integral structure $N_i(k_i)$.

Let $B_1 \simeq \mathbb{A}^n_{\mathbb{C}}$ be complex affine space with coordinates $s_1, \dots, s_n$. The substitution $s_i^{k_i} = t_i$ gives a homomorphism $\mathbb{C}[t_1, \dots, t_n] \to \mathbb{C}[s_1, \dots, s_n]$, giving rise to a finite morphism $B_1 \to B$. Then Σ_{B_1} is the same as Σ_B but taken instead with the lattice $N_{B_1} = \prod \mathbb{Z}k_i\hat{e}_i$. Let $X_1 = X \times_B B_1$. Since the fibers of X are reduced, it follows that X_1 is normal and $X_1 \to B_1$ is again toroidal. Likewise, Σ_{X_1} is just Σ_X with integral structure given by intersecting the lattices in Σ_X with $f_\Sigma^{-1}(N_{B_1})$.

Putting the triangulations $\Sigma_{X,i}'$ of $\Sigma_{X,i}$ together, there exists a triangulation $\Sigma_X^{1'}$ of Σ_X^1 (induced by a lifting function) of index 1 with respect to the integral structure on Σ_{X_1}!

Let us verify that Σ_X admits a slicing function: let $h_B : \Sigma_B \to \mathbb{R}$ be the function defined by $h_b(\sum a_i\hat{e}_i) = \sum a_i$. Then the pullback $h_b \circ f_\Sigma$ is a slicing function on Σ_X.

By Corollary 0.11 of Theorem 0.10, there is an extension of $\Sigma_X^{1'}$ to a triangulation Σ_X' of Σ (induced by a lifting function) without added edges.

Let $Y \to X_1$ be the corresponding toroidal modification and let $f_1 : Y \to B_1$ the resulting morphism.

Note that since all the edges in the triangulation Σ_X' map to the edges τ_i of Σ_{B_1}, we have that f_1 is equidimensional [ℵ-K97]. Since the integral generator of every edge in Σ_X' maps to the generator of the image edge in Σ_{B_1}, and since B is nonsingular, all the fibers of f_1 are reduced [ℵ-K97]. By the construction of [KKMS73], f_1 is semistable in codimension 1. Since Δ_X' is simplicial, Y has at worst quotient singularities. Thus f_1 is nearly semistable. $\square$

Remark 1.1. The variety Y may be singular, as the following example shows: let $\Sigma_Y \subset \mathbb{R}^4$ be the nonnegative orthant, generated by the standard basis vectors $\hat{e}_1, \dots \hat{e}_4$. Let $w = (1/2, 1/2, 1/2, 1/2) \in \mathbb{R}^4$ and N_Y the lattice generated by $w, \hat{e}_1, \dots \hat{e}_4$. Also let Y be the corresponding toric variety — the quotient of $\mathbb{A}^4_{\mathbb{C}}$ by the diagonal $\mathbb{Z}/2$ action given by $p \mapsto -p$ — which happens to be singular. Finally, let $\Sigma_B \subset \mathbb{R}^2$ be the first quadrant, generated by the standard basis vectors $\hat{e}_1, \hat{e}_2$, with the standard lattice $N_B = (\{0\} \cup \mathbb{N})^2$. We have a canonical morphism $\Sigma_Y \to \Sigma_B$ via

$$(a, b, c, d) \mapsto (a + b, c + d)$$

which maps N_Y into N_B. The resulting morphism $Y \to \mathbb{A}^2_{\mathbb{C}}$ is nearly semistable, but not semistable.

2. Proof of Theorem 0.10

It is a simple fact, made precise in Lemma 2.1 below, that any **generic** lifting function on a polyhedral complex induces a simplicial subdivision. This fact is used frequently in applications of subdivisions to the computation of mixed volumes, polyhedral homotopies, and **toric** (or **sparse**) resultants [Stu93, HS95, CE95, Roj97]. The last two constructions give effective recent techniques, sometimes more efficient than Gröbner bases, for solving systems of polynomial equations.

However, it should be emphasized that the lifting functions considered here and in [KKMS73] are more general than those in [Stu93, HS95, CE95, Roj97]: via the use of convex hulls, the lifting functions there are completely determined by the values assigned to the vertices of Δ. We will call these more restricted lifting functions **verticial**. The verticial lifting functions are a bit more "economical" in the sense that their corresponding subdivisions never introduce any new vertices.

There is an easy way to resolve this difference by passing to the verticial case from the start. In fact, we will reduce the proof of Theorem 0.10 to finding **any** triangulation (given by a verticial lifting function) in a new, specially constructed, polyhedral complex. The latter problem is then almost trivial to solve.

First recall (see [KKMS73], Corollary 1.12) that induced subdivisions are transitive: if Δ' is a subdivision of Δ induced by a lifting function f on Δ, and Δ'' is a subdivision of Δ' induced by a lifting function f' on Δ', then Δ'' is a subdivision of Δ as well. In fact, Δ'' is induced by $f + \epsilon f'$ for sufficiently small $\epsilon > 0$.

Thus let $f_0 : \Delta_0 \to \mathbb{R}$ be a lifting function which induces the given subdivision Δ'_0 in our theorem. By adding a constant if necessary, we may assume f_0 is positive. Following Remark 0.9, we can take the values of f_0 on $\mathrm{Sk}^0(\Delta'_0)$, extend them by zero to the other vertices $\mathrm{Sk}^0(\Delta) \setminus \mathrm{Sk}^0(\Delta'_0)$, and take the minimal lifting function $f : \Delta \to \mathbb{R}$ which has these values on the vertices $\mathrm{Sk}^0(\Delta) \cup \mathrm{Sk}^0(\Delta'_0)$. Clearly $f|_{\Delta_0} = f_0$. Let Δ_1 be the induced subdivision. Then clearly the restriction of Δ_1 to Δ_0 coincides with Δ'_0. If Δ' is any subdivision of Δ_1 without new vertices, then its restriction to Δ_0 must be Δ'_0, since Δ'_0 is already simplicial: any subdivision of a simplicial complex without new vertices is trivial. Thus all we need to do to prove Theorem 0.10 is find a **verticial** lifting function on Δ_1 giving a triangulation. In summary, by replacing Δ with Δ_1, we can assume that $\Delta_0 = \Delta'_0$ and then conclude by finding **any** triangulation of Δ_1 (given by a verticial lifting function) — a simpler problem than finding a triangulation of one complex extending some other triangulation.

To complete the proof of Theorem 0.10, recall the following lemma:

Lemma 2.1. *Supppose Δ is a polyhedral complex. Then*

1. *The set L_Δ of all verticial lifting functions on Δ is a finite-dimensional rational vector space.*
2. *The set of all lifting functions which do* **not** *induce simplicial subdivisions is a finite union of proper subspaces of L_Δ.*

Proof: Note that any verticial lifting function on Δ is uniquely determined by its values on $\mathrm{Sk}^0(\Delta)$, which are assumed to be rational, so part (1) follows immediately.

To prove (2), let $\mathcal{C} := (c_v \mid v \in \mathrm{Sk}^0(\Delta))$ be a vector of rational constants. Let $\Delta_{\mathcal{C}}$ denote the subdivision of Δ induced by the verticial lifting function sending $v \mapsto c_v$ for all $v \in \mathrm{Sk}^0(\Delta)$.

Now suppose that there is a nonsimplicial cell C, with vertex set $V(C)$, in $\Delta_{\mathcal{C}}$. Recall that the coordinates of $d+2$ points lying on a d-flat in $\mathbb{R}^n$ must satisfy a determinant depending only on (d,n).[5] (In particular, this determinant is a nonconstant multilinear function in the coordinates of the points.) Then, by the definition of a cell in a subdivision induced by lifting, there must be a (nontrivial) linear relation satsified by $(c_v \mid v \in V(C))$. Furthermore, this linear relation depends only on Δ and the set of vertices $V(C)$. Since there are only finitely many possible nonsimplicial cells (since, by definition, our polyhedral complexes have only finitely many vertices), (2) follows immediately. □

The following is an immediate corollary of our lemma.

Corollary 2.2. *Recall the notation of the proof of Lemma 2.1, and endow* $\mathbb{Q}^{\#\mathrm{Sk}^0(\Delta)}$ *with the standard Euclidean metric* $\|\cdot\|$. *Let* $\mathcal{C} \in \mathbb{Q}^{\#\mathrm{Sk}^0(\Delta)}$. *Then for sufficiently small* $\varepsilon > 0$,

1. $\Delta_{\mathcal{C}'}$ *is a simplicial subdivision for* **some** $\mathcal{C}' \in \mathbb{Q}^{\#\mathrm{Sk}^0(\Delta)}$ *satisfying* $\|\mathcal{C}' - \mathcal{C}\| < \varepsilon$.
2. *If* $\Delta_{\mathcal{C}}$ *is already a simplicial subdivision, then so is* $\Delta_{\mathcal{C}'}$, **for all** $\mathcal{C}' \in \mathbb{Q}^{\#\mathrm{Sk}^0(\Delta)}$ *satisfying* $\|\mathcal{C}' - \mathcal{C}\| < \varepsilon$.

□

Remark 2.3. Put another way, simplicial subdivisions are a dense (via (1)) and open (via (2)) subset of the space of all subdivisions arising from verticial lifting functions. In fact, we really have the stronger statement that the set of all lifting values giving a **particular** simplicial subdivision forms an open cell within the space of all subdivisions.

Note also two "nearby" subdivisions S_1 and S_2 need not have the same extensions, even if $S_1 = S_2$: for example, consider the unit square S with vector of vertices (ordered clockwise) (a,b,c,d), and the subcomplex E consisting of the edges $\{a,b\}$ and $\{c,d\}$. Then $\mathcal{C} = (0,0,0,0)$ and $\mathcal{C}' = (-1,1,-1,1)$ both generate the same (trivial) subdivision of E. However, these two liftings generate different subdivisions of S, the first being trivial.

Returning to the proof of Theorem 0.10, it follows by Corollary 2.2 that there exists a simplicial subdivision of Δ_1 without new vertices, which is what we needed to prove. □

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[5] For example, (x_1,y_1), (x_2,y_2), and (x_3,y_3) lie on a line iff $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{vmatrix} = 0$. Note also that this determinant is linear in the "last" coordinates $\{y_i\}$.

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