TOWARD A PROOF OF THE MORDELL-LANG CONJECTURE IN CHARACTERISTIC p

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1. Introduction

This paper is concerned with an analogue in positive characteristic of the conjecture known as the Mordell - Lang conjecture.

1.1. The picture in characteristic 0. In a series of papers [Lng 65, Lng 74, Lng 86] S. Lang raised the following conjecture:

> *Let $X \subset A$ be a closed subvariety of a semi abelian variety over a field K of characteristic 0, and let $\Sigma \subset A$ be a subgroup of finite rank of the group of $A(K)$, that is,* $\dim_{\mathbf{Q}} \Sigma \otimes \mathbf{Q} < \infty$. *Then there are finitely many cosets of algebraic subgroups, $B_1, \ldots, B_m \subset X$ such that $X \cap \Sigma \subset B_1 \cup \cdots \cup B_m$.*

It should be noted that this conjecture is very strong. In particular, it implies the Mordell conjecture. The case where Σ is a torsion group and X a projective curve is known as the Manin - Mumford conjecture.

By now this conjecture has been almost fully proven. Faltings [Fal 83, Fal 91, Fal 92] proved the Mordell conjecture and its higher dimensional analogue, that is, A abelian and Σ finitely generated. Vojta [Voj 91] extended this to semi abelian varieties. This was preceded by results over function fields by Manin [Man 63] and Grauert [Gra 65]. The Manin - Mumford conjecture was proven by Raynaud [Ray 83a, Ray 83b, Ray 83c], again including the higher dimensional case, using some previous results of Bogomolov [Bog 81], and extended to commutative algebraic groups by Hindry [Hin 88] (following [Lng 65]). The two directions can be combined to give a proof of the Mordell - Lang conjecture in many cases, with the semi abelian case still open ([Hin 88], [McQ 91], with some results already in [Bog 81] and [Ray 83c]). In the so called "geometric case", that is, X not defined over a number field, stronger results were proven (under mild and probably removable assumptions)

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by Buium [Bui 91]. A differential - geometric analogue was tackled by Kawamata [Kwm 80], Noguchi [Nog 81], Vojta [Voj DA], and many others. A general survey, complete up to 1990, can be found in [Lng NT].

1.2. Positive characteristic. The picture in positive characteristic is very much different. For one thing, both the Mordell conjecture and the Manin - Mumford conjecture fail. This happens if X is isotrivial, that is, can be defined over a finite field, or even if X admits a purely inseparable rational map to an isotrivial variety. However, with these cases excluded, we believe that the conjecture should still hold, at least in the following sense:

Definition 1. *We say that an integral scheme X over a field of characteristic $p > 0$ is* weakly isotrivial, *if after some base extension there is a rational map $X_1 \to X$, which induces a purely inseparable extension on the function fields, and such that X_1 can be defined over a finite field. That is: there is $Y/\mathbf{F}_q$ and a common extension L of K and $\mathbf{F}_q$ such that $X_1 \otimes_K L$ is isomorphic to $Y \otimes_{\mathbf{F}_q} L$.*

Using this definition we can state:

> ***Characteristic p analogue of the Mordell - Lang conjecture:*** *Let $X \subset A$ be a closed, integral subvariety of a semi abelian variety over an algebraically closed field of characteristic $p > 0$. Let $\Sigma \subset A$ be a subgroup of closed points such that $rank_{\mathbf{Z}_p} \Sigma \otimes \mathbf{Z}_p < \infty$. Let $Stab(X)$ be the stabilizer of X. If $\Sigma \cap X$ is Zariski dense in X then $X/Stab(X)$ is weakly isotrivial.*

Remark. The condition on the subgroup Σ means that it is contained in the prime-to-p division hull of a finitely generated group (see below). We do not include p-division because we do not have evidence for the validity of such an extension other than in trivial cases, but we also do not have any evidence to the contrary.

In the sense above, the Mordell conjecture for curves was proven by Grauert and Samuel (see [Sam TL]). Some results about torsion, when X is a projective curve, were obtained by Coleman [Col 87]. The conjecture for X a projective curve and its jacobian A ordinary was proven by the second author in [Vol 90].

In this paper we give some extensions of the paper [Vol 90] in nonordinary cases and higher dimensions of X, and A permitted to be semi abelian. The methods here are an extension of the above mentioned paper, with a considerable amount of added technicality. An easy case showing some of the ideas is proven in section 3.

The results may be summarized as follows:

THEOREM A. *The analogue of the Mordell Lang conjecture holds in the following cases:*

(1) *A is isogenous to an isotrivial semi abelian variety;*
(2) *X is a curve;*
(3) *p-rank$(A/Stab(X)) \geq \dim(A/Stab(X)) - 1$.*
(4) *X is a hypersurface in A and p-rank$(A/Stab(X)) > 0$.*

1.3. Plan of the paper. In section 2 we recall some preliminaries needed for the paper. Section 3 gives a relatively easy proof of case 1 in theorem A, exposing the main ideas in this paper. In section 4 we recall some examples showing why the usual Mordell - Lang conjecture cannot hold in positive characteristic. Section 5 contains the main reduction steps and the proof of the rest of the theorem A.

1.4. Notation. We use the following notation throughout:

$X \subset A$**:** X is a subvariety of a semi abelian variety, usually geometrically integral.
Γ**:** A finitely generated subgroup of A.
Γ'**:** the prime-to-p division hull of Γ: $\Gamma' = \{x \in A(\bar{K}) \mid \exists n : nx \in \Gamma, p \not| n\}$.
$\mathbf{Tr}_{K/K^{(p)}} A$**:** the K-over-$K^{(p)}$ trace of A.
$\bar{K}$**:** an algebraic closure of K.
$K^{\mathbf{sep}}$**:** a separable closure of K.
$X^{(p)}$**:** the Frobenius of X.
$K^{(p)}$**:** the field of $p-$th powers of K.
$\mathbf{F}_q$**:** the finite field of q elements.

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2. Preliminaries

2.1. Semi abelian varieties. These are sometimes also called quasi abelian varieties, because in many senses they behave like abelian varieties, although they are not complete. A semi abelian variety A is a commutative group variety over a field K, which fits in an exact sequence

$$0 \to T \to A \to B \to 0$$

of group varieties, where B is an abelian variety and T is a torus, that is, $T \otimes_K \bar{K} \simeq \mathbb{G}^t_{m_{\bar{K}}}$ (here $t = \dim T$).

For such a semi abelian variety A, the kernel of multiplication by n:

$$0 \to {}_n A \to A \stackrel{n_A}{\to} A \to 0$$

is a group scheme of degree n^{2b+t}, where $b = \dim B$.

In case $p \not| n$, the group ${}_nA$ is geometrically reduced. If $n = p$ then the degree $\deg({}_nA \otimes_K \bar{K})_{\mathrm{red}} = p^r$, where $0 \le r \le b$. We call $r + t$ the p-rank of A. It coincides with the dimension over $\mathbf{F}_p$ of $Hom(\mu_p, A)$ when the field is algebraically closed. A is called *ordinary* when $r = b$. The main differences arising between the positive and zero characteristic cases come from the existence of non ordinary abelian varieties.

We will use the fact that every semi-abelian variety has an equivariant projective compactification. In fact, the embedding in [Cho 57] does that for all homogeneous spaces. See [Hin 88] and [Voj 91] for a specific description for semi abelian varieties.

2.2. Reduction to a finitely generated field. Let $X \subset A$ be a subvariety of a semi abelian variety, let Γ be a finitely generated subgroup of A, and let $\gamma_1, \ldots, \gamma_n$ be generators for Γ. We can embed A as a locally closed subvariety of a projective space, in which A with the group law, X and γ_i are defined by finitely many equations and inequalities. Combining together all the coefficients of these equations and inequalities, we see that there is a finitely generated field over which all our objects are defined. Therefore, from now on, we will assume all the objects involved to be over this finitely generated field, which we call K.

The following (well known) fact will be very useful in this paper:

Lemma 1. *Let X be a geometrically reduced variety over a field K. Then $X(K^{sep})$ is Zariski dense in X.*

Conversely: suppose K/L is a purely inseparable extension, $X \subset Y \otimes_L K$ is a closed reduced subscheme, and $X \cap (Y(L^{sep}) \otimes_L K)$ is scheme theoretically Zariski dense in X. Then X can be defined over L and is geometrically reduced.

Proof. The first statement may be shown as follows: pick an embedding of some affine open of X into a projective space $\mathbf{P}^r$, pick a smooth point $x \in X$ and observe that a projection from a point outside the projective tangent space of X at x is an étale isomorphism onto the image near x. Therefore a projection from a general $r - \dim X - 1$ plane is generically etale onto a projective space. (The projection can be chosen from a plane defined over a separable extension.) Now every rational point in an open set in projective space pulls back to points defined over separable extensions.

The converse follows from the fact that Zariski closure commutes with field extensions[1]. In our case it can be show directly as follows: it is enough to work on an affine open subset, with ring of functions R. Since the extension is flat, the ideal of the inverse image of a point is the pulled back ideal. All that is needed is that the intersection of the pulled back ideals is the pullback of the intersection of the ideals. Use the exact sequence

$$0 \to \cap \mathcal{I}_\alpha \to R \to \prod R/\mathcal{I}_\alpha$$

and the fact that finite flat extensions commute with arbitrary products to get the required condition.

To show that it is geometrically reduced, use the same argument for an extension field K_1 such that $(X \otimes_K K_1)_{\text{red}}$ is geometrically reduced. $\square$

2.3. The Frobenius and the verschiebung. Recall that for every variety X in characteristic p there is a canonical finite inseparable morphism $\mathrm{Fr} : X \to X^{(p)}$ called the (geometric) Frobenius. If X is defined over K then $X^{(p)}$ is defined over $K^{(p)}$. Moreover, if X is geometrically reduced then $X^{(p)}$ is the scheme theoretic image of X. If X is defined over $K^{(p)}$ then there is a Frobenius morphism $\mathrm{Fr} : X^{(1/p)} \to X$ defined over K.

For any commutative group scheme A, the Frobenius morphism is a group homomorphism. The kernel of the Frobenius is a group subscheme of the kernel of multiplication by p in A. For a semi abelian variety there is a finite complementary homomorphism $V : A^{(p)} \to A$, called the *verschiebung*, such that $V \circ \mathrm{Fr} = p_A$ and $\mathrm{Fr} \circ V = p_{A^{(p)}}$. We have $\deg V = p^b$, where b is the dimension of the abelian quotient B of A in the defining exact sequence.

Just as in [Lng AV], this allows us to prove a rationality result for semi abelian subvarieties:

Lemma 2 (Chow's theorem). *Let A/F be a semi abelian variety, and let $F \subset K$ be a finite purely inseparable extension. If $B \subset A_K$ is a semi abelian variety, then B is defined over F.*

Proof. The image $\mathrm{Fr}(B)$ is defined over $K^{(p)}$. Since V is defined over F, the image $B = V(\mathrm{Fr}(B))$ is defined over $FK^{(p)}$. For some i, $F \supset K^{(p^i)}$. Repeat the argument i times and obtain the result. $\square$

An alternative argument may be given using $l-$torsion points and lemma 1. Since there are no families of semi abelian subvarieties (the $l-$torsion points are Zariski dense) one can extend this, allowing transcendental extensions as well.

[1] More generally, Zariski closure commutes with strictly faithful extensions in the sense of [Abr 91], in particular essentially free extensions [SGA 3].

2.4. Trace maps. A useful construction on seni abelian varieties is the trace construction, which follows as in [Lng AV] using Chow's theorem:

Let A/K be a semi abelian variety. There exists an isogeny of semi abelian varieties $tr_{K/K^{(p)}} : A_1 \to A$ whose order is a power of p, such that A_1 is defined over $K^{(p)}$, and for any other $f : A_2 \to A$ semi abelian variety defined over $K^{(p)}$ mapping to A, the map factors through the trace map: $f = tr_{K/K^{(p)}} \circ g$ where g is defined over $K^{(p)}$.

In fact, following [Lng AV], in the product of several semi abelian varieties defined over $K^{(p)}$ mapping to A, the semi abelian part of the kernel is defined over $K^{(p)}$, and therefore they all factor through a variety of dimension bounded by that of A.

2.5. The stabilizer. Recall that as in [SGA 3], if $X \subset A$ is a subvariety of a semi abelian variety (or of any essentially free scheme on which an essentially free group scheme acts), then there is a well defined group subscheme of A, $Stab(X)$, called the stabilizer of X, which satisfies an obvious universal property. It may be defined as being the maximal closed subscheme S of A such that $S \times X \subset m^{-1}X$, where $m : A \times A \to A$ is the group law. A key ingredient in this paper is the following:

Lemma 3. *The formation of $Stab(X)$ commutes with field extensions.*

Proof. This follows from the fact that the stabilizer represents a certain functor, but we give here a direct proof. Recall the construction of the stabilizer [SGA 3]: for each open affine U in X one forms the subscheme $S_U \subset A$ maximal with respect to the property $S_U \times U \cap m^{-1}X = S_U \times U$, and then $Stab(X) = \cap_U S_U$. The subscheme S_U is defined as follows: pick a basis for $\mathcal{O}(U)$ and generators for the ideal of $m^{-1}X \cap A \times U$ in $A \times U$, write each of these generators in terms of the basis, and intersect the vanishing loci in A of the coefficients. Each step clearly commutes with field extension.

2.6. Hilbert points. Let $X \subset A$ be a geometrically integral subvariety of a semi abelian variety. We assume that $Stab(X)$ is finite. By [Abr 91] X has the property known as being "of logarithmic general type" or in Lang's language as "log canonically pseudo polarized":

We give the definition following Z. Luo [Luo 87, Luo 88]. Denote by X' the normalization of any projective compactification of X, and by X'_{ns} the nonsingular locus. For large and divisible enough integer n, there is a finite dimensional linear subseries of $H^0(X'_{ns}, K_{X'_{ns}}^{\otimes n})$, consisting of differentials which are regular at each divisorial valuation of the function field $K(X)$ whose center lies over X and have at most logarithmic singularities at the divisorial valuations whose center does not lie over X (that is, lies over the "boundary divisor" of X). In [Abr 91] it is shown using k-jet Gauss maps (for appropriate k) that a linear subseries induces a rational map $X - \to \mathbf{P}^r$ which has degree 1 onto the image. The map is be called "the logarithmic $n-$canonical map". In case X is projective, this is the $n-$canonical map.

It is important to note the behavior of the linear series under field extensions and certain birational modifications. Over an algebraically closed field, the logarithmic $n-$canonical map is a logarithmic birational invariant. However, it may drop in dimension with descent of base field. We use it only to bound the subseries given by the k-jet Gauss map. The formation of the Gauss map does commute with field extension, and therefore can be descended to any field of definition of $X \subset A$. If $X_1 \subset A_1 \stackrel{f}{\to} A \supset X = f(X_1)$ is such that $f : X_1 \to X$ is birational, then the linear series of the Gauss map of X injects to that of X_1. So if we replace X by such dominating X_1 we only gain more information.

This allows us to speak about fields of moduli, at least in a weak sense: for a large k, the image of the k-jet Gauss map gives a birational model of X. By abuse of notation we denote by $[[X]]$ the Hilbert point of this model as a subscheme of the associated projective space. If there is any geometrically reduced subscheme of the orbit of $[[X]]$ containing $[[X]]$ and defined over a field K, then a birational twist of X can be defined over a finite separable extension of K, by lemma 1 above.

2.7. Warning. In what follows we will sometimes take a finite separable field extension with little advance notice.

3. Warming up

In this section we deal with the case when A is isogenous to an isotrivial variety, in particular completing the supersingular case. This allows us to introduce some of the methods used in the following sections.

The following lemma will be used a few times in the argument:

Lemma 4. *Let $X \subset A$ be a closed subvariety of a semi abelian variety, $\Gamma \subset A$ a finitely generated subgroup, and assume $\Gamma' \cap X$ is Zariski dense in X. Suppose $f : A' \to A$ is an isogeny of order p^n, and $g : A \to A'$ a "complementary" map: $f \circ g = p^m_A$, that is, the multiplication by p^m, for some m. Then there is $\gamma \in \Gamma$, and a geometrically reduced $X' \subset A'$, which is a union of components of $f^{-1}(X - \gamma)$, such that $g(\Gamma') \cap X'$ is Zariski dense in X'.*

Proof: Notice that $\Gamma'/p^m\Gamma' \cong \Gamma/p^m\Gamma$ is finite, therefore a coset of $p^m\Gamma'$ will intersect X densely. Let $\gamma \in \Gamma$ be a representative of the coset. Then $(X - \gamma) \cap p^m\Gamma'$ is Zariski dense in $X - \gamma$. Let X' be the Zariski closure of $g(\Gamma') \cap f^{-1}(X)$. Then X' has the required properties, by lemma 1 above. □

Theorem A Part 1: *Let $X \subset A$ be an integral subvariety of a semi abelian variety over an algebraically closed field of characteristic $p > 0$. Let $\Gamma \subset A$ be a finitely generated group and let Γ' be the prime-to-p division hull. Assume $f : A \to A_0$ is an isogeny to a semi abelian variety A_0 which is defined over a finite field. If $X \cap \Gamma'$ is Zariski dense in X, then $f(X)$ is defined over a finite field.*

Proof: We may assume that the variety $X_0 = f(X)$ and $\Gamma_0 = f(\Gamma)$ are defined over a field K which is a finitely generated extension of $\mathbf{F}_p$. Since A_0 is defined over a finite field, there exists a Frobenius morphism $Fr : A_1 \to A_0$ defined over a finite field $\mathbf{F}_q$, as well as the verschiebung $V : A \to A_0$. We may assume that $\mathbf{F}_q \subset K$.

By lemma 4 applied to f =identity and $g = p_A$ there is a translate of X_0 by an element γ_0 of Γ_0 which intersects $p\Gamma'_0$ densely. But the points of $p\Gamma'$ are images of K-rational points of A_1 by the Frobenius, therefore they are defined over $K^{(p)}$. Therefore, by lemma 1, $X - \gamma_0$ is defined over $K^{(p)}$. Continuing in this manner, we find that for all i, a translate $X_0 - \gamma_i$ of X_0 is defined over $K^{(p^i)}$. Let $\bar{A}$ be an equivariant compactification of A. We can now choose any ample line bundle on $\bar{A}$ which is defined over a finite field, and obtain an associated Hilbert scheme. Define:

$$Z = \cap_{k \geq 0} \overline{\{[X_0 - \gamma_i]\}_{i \geq k}}$$

that is, the Zariski limit of the associated Hilbert points of the translates of $\bar{X}_0$. The scheme Z will be a subscheme of the Hilbert scheme, describing translates of X_0, and defined over a finite field (since it is defined over all $K^{(p^i)}$). Therefore, by lemma 1 there is a translate of X_0 defined over a finite field. $\square$

Remark: In the general case the semi abelian variety will not be defined over a finite field, and we will not be able to use a Hilbert scheme for subvarieties of a fixed semi abelian variety. This is why we need to use the log canonical and Gauss maps. The utilization of Hilbert points is inspired by [Sam TL].

4. Counterexamples

It is important to note that there are many weakly isotrivial varieties which are not isotrivial.

First, let A_0/K be a product of supersingular elliptic curves and let $X_0 \subset A_0$ be a subvariety defined over $\mathbf{F}_q$ which has trivial stabilizer and has a smooth rational point P. Let $V \subset T_{(A_0),0}$ be a subspace of dimension $\dim V > \dim A - \dim X_0$ which has no nonzero subspaces defined over a finite field. Suppose V intersects the translate of the tangent space of X at P transversally. Let N be the local subgroup-scheme of A_0 generated by V. Define $A = A_0/N$ and let X be the image of X_0 by the quotient map.

Claim: A is not defined over a finite field.

Otherwise the morphism $A_0 \to A$ would be defined over a finite field, by Chow's theorem (see [Lng AV] or [Oor 73]).

Claim: X is not defined over a finite field.

Otherwise the embedding $X \to A$ would factor through the Albanese of X, and after base change, the map from the Albanese would factor through the $K\overline{\mathbf{F}_p}/\overline{\mathbf{F}_p}$-trace of A, call it A' which is dominated by A_0. By construction the morphism $X_0 \to X$ has degree greater than 1, therefore $Ker(A_0 \to A')$ is nontrivial, defined over $\overline{\mathbf{F}_p}$, and contained in the kernel of $A_0 \to A$, contradictory to assumption.

We can use this example to show that the Mordell conjecture does not hold for varieties which are isogenous to isotrivial varieties. We simply follow the example of Samuel [Sam TL]. It is enough to deal with X_0, because then we can take the image in X. Now X_0 is a base extension $X_0 = Y_0 \times_{\mathbf{F}_q} K$. Take K to be the function field of Y_0. Since Y_0 is defined over some $\mathbf{F}_q$ with $q = p^r$, the $r-$th power of the Frobenius morphism is an automorphism of Y_0. Let $x_i \in X_0$ be the generic point of the graph of the $(ir)-$th power of Frobenius of Y_0, for $i \geq 0$. We get infinitely many $K-$rational points on X_0. Their Zariski closure is defined over $\mathbf{F}_q$, as it is stable under the Frobenius. But a proper subvariety of X which is defined over a finite field cannot contain the generic point of the diagonal, therefore these points are Zariski dense in X.

Main Question 1. *Can one construct X as above such that $Stab(X)$ is trivial?*

If the answer is "no" in a slightly stronger sense (that is, for inseparable extensions rather than transcendental extensions), then the analogue of the Mordell Lang conjecture in characteristic p is proved.

5. Proofs

5.1. The main reduction. Let $X \subset A$ be a geometrically integral subvariety of a semi abelian variety over K, which is a finitely generated extension of $\mathbf{F}_p$. Let $\Gamma \subset A$ be a finitely generated subgroup, and let Γ' be the prime-to-p division hull of Γ. Assume $X \cap \Gamma'$ is Zariski dense in X.

Step 1. We may assume that X generates A by taking the subgroup generated by X. We may assume $Stab(X)$ is trivial, by taking the quotient.

Step 2. We may assume that $X \subset A$ is not dominated by $X_1 \subset A_1$, defined over $K^{(p)}$ with X_1 birational to X: assume it were, and let $T : A_0 \to A$ be the trace map: $A_0 = \mathrm{Tr}_{K/K^{(p)}} A$ as in [Lng AV]. Then the map $A_1 \to A$ factors through A_0. Therefore an open set in X lifts to be a subvariety of A_0. The image of $X \cap \Gamma'$ by this lifting generates a group which is contained in the prime-to-p division hull of a (finitely generated) subgroup Γ_0 of $T^{-1}(\Gamma)$. This subgroup is generated by elements defined over the separable closure K^{sep}. After a finite separable extension of K, this group is generated by rational points.

Let X_0 be the birational image of X in A_0. By lemma 4 there is a translate $X_0 - \gamma$ such that $(X_0 - \gamma) \cap p\Gamma_0'$ is Zariski dense in $X_0 - \gamma$. Now the generators of $p\Gamma_0$ are also defined over $K^{(p)}$, because they are the image of rational points on $A_0^{(1/p)}$ by the Frobenius. The situation is now completely reduced to $K^{(p)}$, and we can replace K by $K^{(p)}$.

Now we can use the argument of 2.6 to show that we can repeat this process only finitely many times: otherwise X would be weakly isotrivial, because the k-jet Gauss maps of the varieties we obtain are defined over $K^{(p^i)}$. Finally we arrive at $X_n \subset A_n$ and $\Gamma_n \subset A_n$ such that $X_n \cap \Gamma_n'$ is Zariski dense in X_n, and X_n is geometrically reduced, and not defined over $K^{(p)}$. One easily checks that $Stab(X_n)$ is trivial as well.

Step 3. Let $V : A^{(p)} \to A$ be the Verschiebung. Let $X_V = V^{-1}X_{\mathrm{red}}$. As usual, there is a subgroup Γ_V in $A^{(p)}$ which intersects densely with a component of X_V. Replace X_V by this component. Now X_V has a dense subset of separable points over $K^{(p)}$, therefore X_V is defined over $K^{(p)}$ and geometrically reduced. We can make a finite separable extension of K, and assume X_V is geometrically integral.

Lemma 5. *$Stab(X_V) \subset Ker(V)$.*

Proof: Let $N = Stab(X_V) \cap Ker(V)$. Then the image of $Stab(X_V)/N$ in A stabilizes X, and therefore has to be trivial. □

Lemma 6. *Let $X_1 = X_V/Stab(X_V)$ and let $A_1 = A^{(p)}/Stab(X_V)$. Then the map $f_1 : X_1 \to X$ is generically purely inseparable.*

Proof. Assume $f_1 : X_1 \to X$ has a separable part. Let K_1 be an extension field over which $Ker(V)/Stab(X)$ splits as a sum of an étale group and a local group scheme. The separable part of f_1 generates a subgroup of the étale part which stabilizes X_1. By the rationality of the stabilizer (lemma 3), X_1 must have a nontrivial stabilizer before the field extension, which contradicts the construction.

Conclusion of reduction: Repeating the above steps, we arrive at the following situation:

$X_i \subset A_i; X = X_0 \subset A = A_0; \gamma_i \in \Gamma_i \subset A_i$;

X_i are geometrically integral and $Stab(X_i)$ is trivial;

$f_i : A_i \to A_{i-1}$ an isogeny such that the restriction $f_{i|X} : X_i \to X_{i-1} - \gamma_{i-1}$ is generically purely inseparable;

The verschiebung $A_i^{(p)} \to A_i$ factors through f_{i+1}.

$\Gamma_i \subset f^{-1}(\Gamma_{i-1})$ a finitely generated subgroup;

$X_i \cap \Gamma_i'$ is Zariski dense in X_i;

A_i, X_i and the generators of Γ_i are defined over $(K^{(p^i)})^{\mathrm{sep}}$.

In this situation we say that "X admits an inseparable isogeny tower".

Main Question 2. *Are there varieties which are not weakly isotrivial but admit inseparable isogeny towers?*

The analogue of the Mordell - Lang conjecture is proven if the answer is "no". We will now show several cases in which this holds.

Remark. One can show that after a finite number of steps the isogenies f_i themselves are purely inseparable. We will not need this in the proofs.

5.2. The theorems.

Theorem A Part 2: *Let $X \subset A$ be a nonisotrivial geometrically integral curve of genus at least 2 in a semi abelian variety over an field of characteristic p. Let $\Gamma \subset A$ be a finitely generated group. Then $X \cap \Gamma'$ is finite.*

Proof. Before starting the reduction, note that according to [Serre], $X \subset A$ can be replaced by $X_0 \subset A_0$, A_0 purely inseparably isogenous to A and X birational to X_0, with the extra property that the invariant 1-forms on A_0 pull back injectively to X_0. Now apply the reduction to $X_0 \subset A_0$. At the end of the reduction we obtained a map $X_1 \to X_0$ which is generically purely inseparable. By construction, the degree of X_1 over X_0 is positive, therefore the 1-forms on X_0 vanish when pulled back to X_1. By the construction of X_0, the invariant 1-forms on A_0 vanish on X_1. Therefore the kernel of $A_1 \to A_0$ contains the span of all tangent vectors to X_{ns}, say V. The space V is certainly defined over $K^{(p)}$ and we can take the quotient by the group scheme generated by V. Repeating this argument we finally arrive at $X_1 = X_0$ defined over $K^{(p)}$, which contradicts the reduction steps. $\square$

Theorem A Part 3: *Let $X \subset A$ be a geometrically integral subvariety of an abeian variety over a field K of characteristic p such that $X/Stab(X)$ is nonisotrivial, and assume p-rank$(A/Stab(X)) \geq \dim(A/Stab(X)) - 1$. Let $\Gamma \subset A$ be a finitely generated group. Then $X \cap \Gamma'$ is not Zariski dense in X.*

Proof. At the conclusion of the reduction, there are two possibilities:

Case 1: the map $f_1 : X_1 \to X$ is a birational map, which means that X is birationally defined over $K^{(p)}$.

Case 2: The map f_1 is inseparable of degree exactly p. But the degree of the local part $(Ker(V))_l$ is p, therefore this sub-group-scheme must stabilize X_1, contrary to the reduction.

Thus we always end up in case 1. We can now repeat the reduction indefinitely, showing, using 2.6, that X is birationally defined over a finite field. We need to show that X is actually isotrivial. Using the trace map we obtain a birational model X' inside a semi abelian variety A', both defined over a finite field, and dominating a translate of X in A. But all isogenies of a semi abelian variety A' with p-rank$(A') \geq \dim A' - 1$ can be defined over the separable closure of the field of definition (see [Oor 73]), therefore X and A are isotrivial.

Theorem A Part 4: *Let $X \subset A$ be a closed hypersurface in a semi abelian variety over a field of characteristic p. Assume that p-rank$(A/Stab(X)) > 0$. Then the analogue of the Mordell - Lang conjecture holds for $X \subset A$.*

Proof. By the reduction steps, X admits an inseparable isogeny tower. Let $a = \dim A$, let $\bar{A}_i$ be compatible projective compactifications of A_i and let $\bar{X}_i$ be the Zariski closure of X_i in $\bar{A}_i$. Since $Stab(X)$ is trivial, the self intersection number $(\bar{X}^a) \neq 0$. Notice that applying steps (1) and (2) in the reduction does not increase the intersection number.

Set $d_i = \deg f_i$ and $e_i = \deg f_{i|X}$. Now $d_i \leq p^a$ and $pe_i \leq d_i$ (otherwise $Kerf_i$ stabilizes X_i). Combining, we get $e_i^a / d_i^{(a-1)} \leq 1$. Now, by the projection formula:

$$(\bar{X}_i{}^a) = \left(\bar{X}_i \cdot (\frac{e_i}{d_i} f_i^{-1}(\bar{X}_{i-1}))^{a-1}\right) = \frac{e_i^a}{d_i^{a-1}}(\bar{X}_{i-1}^a) \leq (\bar{X}_{i-1}^a).$$

Thus the intersection numbers must stabilize, which means that we have equalities: $d_i = p^a$, therefore f_i is the verschiebung and A is an abelian variety; and $pe_i = d_i$, therefore $e_i = p^{a-1}$. If p-rank$(A) > 0$ we have that the inseparable part of f_i has degree at most p^{a-1}, and therefore it has to stabilize X, contradictory to construction.

5.3. Remarks about characteristic 0. It was pointed out by S.Lang [Lng NT] that, given a non-isotrivial curve over a function field of characteristic zero, one could deduce the Mordell conjecture for it from the result of [Vol 90] by reduction mod p, if p could be chosen so that the curve had ordinary reduction modulo p. As now we have removed the ordinariness hypothesis from our result, we can use Lang's argument to give a short proof of Mordell's conjecture for function fields of characteristic zero. It should be noted that Lang's suggestion involves a mixed characteristic analogue of Hilbert Irreducibility, which is supplied by Silverman's theorem (see [Lng NT]).

In order to generalize this method to higher dimensions, several steps should be achieved. First, a rather strong analogue of the irreducibility result is needed: a dense collection of points in characteristic 0, when reduced modulo a prime, might not be dense anymore. The use of the nonisotriviality assumption is very essential. The situation is much simpler if one is interested in torsion points (the Manin-Mumford conjecture). Further, we need to either remove the hypothesis on the p-rank in part

3 of theorem A or prove that the reduction modulo p of a non-isotrivial semi abelian variety satisfies it for infinitely many p. Unfortunately we cannot do the former, except when the abelian part of A has dimension at most two. As for the latter, this is a well-known open problem and can only be done at the moment (to the authors' knowledge) when the abelian part of A is isogenous to a product of non-isotrivial elliptic curves and an Abelian variety with complex multiplication. Note that the case when the abelian part has complex multiplication we can also handle the problem by part 1 of theorem A, when applicable.

We should remark that the function field Mordell conjecture also follows from the number field case, in which case the ultimate known result is Vojta's [Voj 91] . The argument is basically the same, with specialization playing the role of reduction modulo a prime.

Theorem A could have been proved more generally for A a semi abelian scheme over a scheme S of finite type over $\mathrm{Spec}(\mathbf{F}_p)$, interpreting isotrivial to mean that a variety comes from $\mathrm{Spec}(\mathbf{F}_q)$ by base-change and for the proof replacing "not defined over $K^{(p)}$" by "not coming from base change of $S \to S^{(p)}$". This is almost a curiosity, but it gives the following interesting result. Let R be ramified extension of the p-adic integers with algebraically closed residue field and A a semi-abelian scheme over $\mathrm{Spec}(R)$. Define a subvariety X of A to be non-isotrivial if its Kodaira-Spencer class does not vanish (see [CoVo 91] for the Kodaira-Spencer class in this context). Note that the module of differentials of R over Z is non-zero by hypothesis so this condition is not empty. Let k be the residue field of R. Then the set of k-points in the reduction of X that lift to a point of $X(R)$ in $pA(R)$ is not Zariski dense in X. This is proved by applying the above considerations to varieties over $S = \mathrm{Spec}(R/p)$. This is an analogue of proposition of [CoVo 91] which deals with unramified extensions of the p-adics and can be used as in [CoVo 91] to prove the Manin-Mumford conjecture in the rather restrictive case in which it applies.

References

[Abr 91] D. Abramovich, *Subvarieties of semi abelian varieties*, preprint (a modified version of chapter 1 of *Subvarieties of abelian varieties and of jacobians of curves*, P.H.D. Thesis, Harvard University, 1991.

[Bog 80] F. A. Bogomolov, *Sur l'algébricité des représentations l-adique.* C. R. Acad. Sci. Paris Sér A-B 290 no.15 (1980).

[Bog 81] F. A. Bogomolov, *Points of finite order on an abelian variety.* Math. of the U.S.S.R. Izvestya, Vol. 17 no. 1 p. 55-72 (1981).

[Bui 91] A. Buium, *Intersections in jet spaces and a conjecture of S. Lang*, preprint.

[Cho 57] Y. L. Chow, *On the projective embedding of homogeneous varieties.* In: Algebraic Geometry and Topology: a symposium in honor of S. Lefschetz. p. 122-128. Princeton University Press 1957.

[Col 87] R. F. Coleman, *torsion points on curves.* Adv. Stud. Pure Math. 12, 235-247 (1987).

[CoVo 91] R. F. Coleman and J.-F. Voloch.

[Fal 83] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpen.* Invent. Math. 73, 349-366 (1983).

[Fal 91] G. Faltings, *Diophantine approximation on abelian varieties.* Annals of Math. 1991.

[Fal 92] G. Faltings, *The general case of S. Lang's conjecture,* to appear 1992.

[Gra 65] H. Grauert, *Mordells Vermutung über Punkte auf algebraischen Kurven und Funktionenkörper.* I.H.E.S. Publ. Math. 25, 131-149 (1965).

[SGA 3] A. Grothendieck et. al., S.G.A III Vol. 2, Expose VIII: Groups Diagonalisables. Lecture Notes in Mathematics 152.

[Hin 88] M. Hindry, *Autour d'une conjecture de Serge Lang.* Inven. Math. 94 (1988) 575-603.

[Kwm 80] Y. Kawamata, *On Bloch's conjecture.* Inv. Math 57, p. 97-100 (1980).

[Luo 87] Z.-H. Luo, *Kodaira dimension of algebraic function fields.* Amer. J. Math. 109 (1987) no. 4, 669-693.

[Luo 88] Z.-H. Luo, *An invariant approach to the theory of logarithmic Kodaira dimension of algebraic varieties.* Bull. A. M. S. (N. S.) 19 (1988) no. 1, 319-323.

[Lng AV] S. Lang, Abelian Varieties. Interscience, New York 1959.

[Lng 60] S. Lang, *Some theorems and conjectures in diophantine equations.* Bul. Amer. Math. Soc. 66 (1960) 240-249.

[Lng 65] S. Lang, *Division points on curves.* Annali di Mat. Pura ed Appl. LXX (1965) p. 229-234.

[Lng 74] S. Lang, *Higher dimensional diophantine problems.* Bull. A.M.S. 80 (1974) p. 779-787.

[Lng 86] S. Lang, *Hyperbolic diophantine analysis.* Bull. A.M.S. 14 (1986) p. 159-205.

[Lng 87] S. Lang, *Diophantine problems in complex hyperbolic analysis* in : *Current trends in arithmetic algebraic geometry.* Contemp. Math. 67, A.M.S. Providence, R.I. (1987) p. 229-249.

[Lng NT] S. Lang, Number theory III: Diophantine Geometry. Encyclopedia of Mathematical Sciences vol. 60. Springer Berlin - New York 1991.

[Lng DG] S. Lang, Fundamentals of Diophantine Geometry. Springer, New York 1983.

[Lau 84] M. Laurent, *Equations Diophantiènnes exponentielles,* Invent. Math. 78 (1984) 299-327.

[Man 63] Yu. L. Manin, *Rational points on an algebraic curve over function fields.* Transl. Am. Math. Soc. II Ser. 50, p. 189-234 (1966). (Russian original: Izv. Acad. Nauk U.S.S.R. 1963.)

[Man 90] Yu. L. Manin, Letter to the editor. Math. USSR Izv. 34, 465-466 (1990).

[McQ 91] M. McQuillan, private circulation.

[Nog 81] J. Noguchi, *Lemma on logarithmic derivatives and holomorphic curves in algebraic varieties.* Nagoya Math. Journal 83 (1981) p. 213-233.

[Nog 85] J. Noguchi, *Hyperbolic fiber spaces and Mordell's conjecture over function fields.* Publ. Res. Inst. Math. Sci. 21 no. 1 (1985) 27-46.

[Oor 73] F. Oort, *The isogeny class of a CM-type abelian variety is defined over a finite extension of the prime field.* J. Pure Appl. Alg. 3 (1973) 399-408.

[Ray 83a] M. Raynaud, *Courbes sur une variété abélienne et point de torsion.* Invent. Math. 71, 201-233 (1983).

[Ray 83b] M. Raynaud, *Around the Mordell conjecture for function fields and a conjecture of Serge Lang.* Algebraic Geometry (Tokyo / Kyoto 1982) 1-19. Lecture Notes in Math. 1016, Springer Berlin - New York, 1983.

[Ray 83c] M. Raynaud, *Sous variétés d'une variété abélienne et points de torsion.* Arithmetic and Geometry vol I 327-352, Progr. Math. 35 Birkhäuser, Boston 1983.

[Sam TL] P. Samuel, *Lectures on old and new results on Algebraic curves.* Bombay: Tata Inst. Fund. Res. 1966.

[Sam 66] P. Samuel, *Compléments à un article de Hans Grauert sur la conjecture de Mordell.* I. H. E. S. Publ. Math. no. 29 (1966) 55-62.

[Serre] J. P. Serre, Exp. 10: *Morphismes universels et variété d'Albanese;* Exp. 11: *Morphisms Universels et différentielles de troisiéme espèce.* Seminair C. Chevalley, 3e année: 1958/59.

[Vol 90] J. F. Voloch, *On the conjectures of Mordell and Lang in positive characteristic.* Inv. Math. 104, 643-646 (1991).

[Voj DA] P. Vojta, *Diophantine approximation and value distribution theory,* Lecture Notes in Math. 1239, Springer New York, 1987.

[Voj 91] P. Vojta, *Integral points on subvarieties of semiabelian varieties*, preprint 1991.

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