Berkeley 2006

Slides for two lectures on

Distribution of rational points and classification of complex algebraic varieties

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$\underline{X}$ smooth proj $/\mathbb{C}$ $\quad \dim X = n$

$\underline{K}_X$ = canonical line bundle

$m \in \mathbb{N}$ if $H^0(X, mK_X) \neq 0$ get a $\underline{\text{rat'l}}$ map

$$\varphi_m : X \dashrightarrow \mathbb{P}(H^0(X, mK_X)^*)$$

The KODAIRA dimension $\kappa(X)$ of X is

$$\kappa(X) := \begin{cases} -\infty & \text{if } H^0(X, mK_X) = 0 \\ \max_m \{\dim \varphi_m(X)\} & \text{otherwise} \end{cases}$$

(1) $\kappa(X) \in \{-\infty, 0, 1, \ldots, n\}$

if $\kappa(X) = n$ $\quad X$ is called of general type

(2) BIRATIONAL INVARIANCE

(3) $\kappa(X \times Y) = \kappa(X) + \kappa(Y)$

If $\kappa(X) \geq 0$ $\forall m \gg 0$ get a canonical map

$$\varphi = \varphi_{m \gg 0} : X \xrightarrow[\substack{\uparrow \\ \kappa(\text{FIBER}) = 0}]{} \varphi(X) \leftarrow \dim = \kappa$$

examples

(1) X rational $\Rightarrow \kappa(X) = -\infty$

(2) $X = B \times \mathbb{P}^r \quad r \geq 1 \Rightarrow \kappa(X) = -\infty$

(3) X abelian variety $\Rightarrow \kappa(X) = 0$

(4) $\dim X = 1$ $\quad$ X of general type

$$\Updownarrow$$

$$g(X) \geq 2$$

$$\Updownarrow \quad \text{FALTINGS 83}$$

$X(k)$ is finite. $\forall$ number field K.

How to generalize Faltings to higher dimension?

(WL) WEAK LANG CONJ: X of general type $\Rightarrow X(k)$ no[t]
dense in X

BOMBIERI / VOJTA

$\Uparrow$

(SL) STRONG LANG CONJ: X of general type $\Rightarrow$

$\exists$ open $\neq \emptyset$ $U_X \subset X : \forall K \quad X(k) \cap U_X$ is finite

<u>OPEN</u>! Known if $\dim X = 1$ and $X \subsetneq$ ABELIAN va[riety]

Other known cases satisfying WL or SL

Let $f : X \longrightarrow B$ be dominant. Then

(1) If B satisfies WL so does X

- because $f(X(k)) \subset B(k)$

(2) If B satisfies SL and the gen'l fiber of f is a smooth curve of genus $\geq$ X satisfies SL

$W_f := \{b \in B : X_b \text{ smooth of genus} \geq 2\} \subset B$

open in B

Then $U_X = f^{-1}(W_f \cap U_B)$

Example: B a curve of genus ≥ 2

There exist <u>LOTS</u> of such $X \xrightarrow{f} B$

BASIC FACTS about M_g

$M_g :=$ moduli variety of smooth curves of genus g

- Quasiprojective over Spec $\mathbb{Z}$
- Normal, irreducible
- $g \geq 1$ <u>not</u> complete
- $g \geq 2$ $\dim M_g = 3g-3$

$\forall f : X \longrightarrow B$ fibers are smooth curves of genus g

$\exists !$ moduli map $\mu_f : B \longrightarrow M_g$

$$b \longmapsto [X_b]$$

Def: f is ISOTRIVIAL iff $\mu_f(B)$ is a POINT

f is TRULY VARYING iff $\dim B = \dim \mu_f(B)$

- There exist LOTS of coverings of M_g having a GLOBAL FAMILY of curves of genus g

LESSON: Many <u>UNIFORMITY</u> phenomena are <u>PLAUSIBLE</u>

Uniform boundedness of rational points ?

(WU) Weak Uniformity Conj: $\forall g \geq 2 \ \forall K$

$$\exists B(g,K): \ \forall X \in M_g(K)$$

$$\Uparrow \qquad \#X(K) \leq B(g,K)$$

(SU) Strong Uniformity Conj: $\forall g \geq 2$

$\exists N(g)$: $\forall K$ the set of $X \in M_g(K)$ such that $\#X(K) > N(g)$ is finite

OPEN!

Thm (-Harris Mazur 95) (1) W. Lang $\Rightarrow$ W. Uniformity

(2) S. Lang $\Rightarrow$ S. Uniformity

Outline for (1)

$$\boxed{WL} \xRightarrow[\text{Fibered Power Theorem}]{} \boxed{\begin{array}{c}\text{Open Uniformity}\\ \text{for arbitrary families}\end{array}} \xRightarrow[\text{Moduli}]{} \boxed{WU}$$

The ROLE of W/L

<u>PRETEND</u> that a family $X \xrightarrow{f} B$ of curves of general type has X of general type.

Then by W/L $\exists$ $X \supsetneq \underset{\text{closed}}{Z} \supset X(k)$

The restriction $f_{/Z} : Z \longrightarrow B$ has finite degree d $\Rightarrow$ (Open Uniformity) $\exists$ $\underset{\text{open}}{U} \subset B$ $\forall u \in U$ $\# X_u(k) \leq d$

<u>BUT</u> there are LOTS of $f : X \longrightarrow B$ with <u>X NOT of general type</u>

(1) $\forall B : \kappa(B) < \dim B$ $\qquad$ $B \times C$ not of g.t.

(2) Pencils of plane curves of degree ≥ 4

$$X \longrightarrow \mathbb{P}^1$$

$X =$ Blow up of $\mathbb{P}^2$ at a finite set

$\Rightarrow X \sim \mathbb{P}^2 \Rightarrow \kappa(X) = -\infty$

GENERAL GEOMETRIC PROBLEM

Q: How close to be of general type is a variety X that carries a fibration $X \xrightarrow{f} B$ with general fiber of general type?

Fibered Power Theorem: For $m \gg 0$ the m-th fibered power X^m_B of X over B dominates a variety of gen'l type

i.e. $\exists$ dominant map

$$X^m_B \longrightarrow W \qquad W \text{ of general type}$$

If f is a family of curves $\Rightarrow \dim W = m + \dim \mu_f(B$

- if f is truly varying X^m_B is of gen'l type

(-HARRIS MAZUR) / HASSETT / ABRAMOVICH
curves / surfaces / general case

How many rational points can a curve have ?

W. UNIFORMITY ?

$B(2, \mathbb{Q}) \geqslant 588$ KULESZ ~98

$$y^2 = 278271081\, x^{2}(x^2-9)^2 - 229833600\,(x^2-1)^{\cancel{4}}$$

$\# \mathrm{Aut} \geq 12$

$B(3, \mathbb{Q}) \geqslant 112$ KELLER, KULESZ

$$y^2 = 4839795 0000\,(x^2+1)^4 - 939127350499\,\cancel{(x^3-x)}^{2}$$

$\# \mathrm{Aut} \geqslant 16$

S. UNIFORMITY ?

To prove that $\underline{L} \leq N(g)$ have to find infinitely many curves of genus g over a n.fi[eld] K having at least $\underline{L}$ ratl points over K

g	2	3	4	5	6	9	45	g
$N(g)$	128	100	126	132	146	180	$\underline{\underline{781}}$	$16(g+1)$

ELKIES, HARRIS

BRUMER/MEST[RE]

RATIONAL POINTS OVER FUNCTION FIELDS

B smooth proj $/\mathbb{C}$ $\qquad$ $L = \mathbb{C}(B)$ a function field

<u>Theorem</u> (MANIN 63 / GRAUERT): Let $\mathcal{X}$ be a <u>non isotrivial</u> smooth curve of genus $g \geq 2$ over L.

$\Rightarrow \mathcal{X}(L)$ is FINITE

$\mathcal{X}$ smooth curve $/L \rightsquigarrow f : X \longrightarrow B$ a "model" (unique up to $\simeq$)

$W_f := \{ b \in B : X_b \text{ is a smooth curve} \}$ is OPEN $\neq \emptyset$ in B

- $\mathcal{X}$ is non isotrivial iff f is
- $\mathcal{X}$ is truly varying iff f is

$$\mathcal{X}(L) \longleftrightarrow \text{Rational sections of } f$$

Rk: If $X = C \times B$ for some $C \in M_g$ theorem fails

- there are infinitely many CONSTANT sections of $f : C \times B \longrightarrow B$

RESTRICTED UNIFORMITY over FUNCTION FIELDS

PROP 1

$\forall g \geq 2 \quad \forall \mathcal{L} = \mathbb{C}(B) \quad \exists B^{(2)}(g, \mathcal{L})$:

$\forall \mathcal{X}$ non isotrivial sm. curve of genus g over $\mathcal{L}$

having "good reduction in codimension 1"

(i.e. $\exists f : \operatorname{codim}(B - W_f) \geq 2$)

$$\# \mathcal{X}(\mathcal{L}) \leq B^{(2)}(g, \mathcal{L})$$

PROP 2

$\forall g \geq 24 \quad \forall \mathcal{L} : \operatorname{trdeg}_{\mathbb{C}} \mathcal{L} = 3g - 3$

$\exists B^{(*)}(g, \mathcal{L}) : \forall$ truly varying smooth curve

$\mathcal{X}$ of genus g over $\mathcal{L}$

$$\# \mathcal{X}(\mathcal{L}) \leq B^{(*)}(g, \mathcal{L})$$

Rk: $g \geq 24$ $\overline{M}_g$ is of general type

DICHOTOMY IN DIMENSION 1

$\dim X = 1$, then:

$X(K)$ is dense (for some K)

$\kappa(X) = -\infty, 0 \iff X(K)$ is dense (for some K)

$g_X = 0, 1 \iff d_X \equiv 0$ $\quad d_X :=$ Kobayashi pseudo metric

$\iff \exists f : \mathbb{C} \longrightarrow X$ holomorphic NOT constant

Is there an analogous picture in higher dimension?

General idea: $\boxed{\text{GEOMETRIC HYPERBOLICITY}} \iff$ FEW RATIONAL POINTS

$\kappa(X) = -\infty$

Rational $\Rightarrow$ Rationally Connected $\Rightarrow$ Uniruled $\Rightarrow$ $\kappa = -\infty$

$\Leftarrow$ Conjecture!

R.C. = 2 general pts of X are joined by a rational curve of X

MRC filtration: $\exists!\ r : X \longrightarrow r(X)$

KOLLÁR, MIYAOKA, MORI 92 CAMPANA

(the Maximally Rationally Connected f.b. ...)

(a) the gen'l fiber is RC

GRABER HARRIS STARR 01 (b) $r(X)$ is NOT uniruled

$\kappa(X) \geq 0$

CANONICAL fib: $\exists!\ \varphi : X \longrightarrow \varphi(X) \subset \mathbb{P}(H^0(mK_X ...$

$m >> ...$

(a) $\kappa(\text{gen'l fiber}) = 0$

(b) $\dim \varphi(X) = \kappa(X)$

X is RC $\Leftrightarrow r(X)$ a point | $\kappa(X) = 0 \Longleftrightarrow \varphi(X)$ a point

RATIONAL POINTS on SURFACES

dim $X = 2$

	$X(K)$ dense $\exists K$	$X(K)$ NOT dense $\forall K$
$\kappa = -\infty$	$\mathbb{P}^2$	$\mathbb{P}^1 \times C$ $\quad g_C \geq 2$
$\kappa = 0$	$E \times E$ many	NONE
$\kappa = 1$	many	$E \times C$ $\quad g_C \geq 2$
$\kappa = 2$	NONE	many
	"special"	"general"

BOGOMOLOV
HARRIS
HASSETT
SILVERMAN
TSCHINKEL
....

Conj: $\kappa = 0 \Rightarrow X(K)$ dense

Lesson: Diophantine geometry is NOT governed by Kodaira dimension

Classification from diophantine perspective ?

WL: X of general type $\Longrightarrow$ $X(k)$ not dense

$C \times \mathbb{P}^1 \longrightarrow C$ $\quad \cancel{\Longleftarrow}$

$g_C \geq 2$

$WL^{(1)}$: X dominates general type $\Longrightarrow$ $X(k)$ not dense

$X \xrightarrow{c} \mathbb{P}^1$ $\quad \cancel{\Longleftarrow}$ COLLIOT-THÉLÈNE, SKOROBOGATOV, SWINNERTON-DYER 97

$2g+2$ double fibers
general fiber elliptic curve

$WL^{(2)}$: $\boxed{\text{Some étale covering of } X \text{ dominates general type}}$ $\Longrightarrow$ $X(k)$ not dense

$\Longleftarrow$ OPEN $\quad$ CT/ABRAMOVICH

Def X NOT Weakly special.

MAIN EXAMPLE

H hyperelliptic $g \geqslant 2$ $\quad \iota : H \circlearrowleft$ hyp. involuti

E elliptic $\quad \tau : E \circlearrowleft$ fixed pt free involution

$\tilde{X} = H \times E \qquad \alpha = (\iota, \tau) : \tilde{X} \circlearrowleft$

$$\begin{tikzcd}
\tilde{X} = H \times E \arrow[r, "p_H"] \arrow[d, "u"', "\text{étale} \to"] & H \arrow[d, "v"] \\
\tilde{X}/\langle \alpha \rangle = X \arrow[r, "c"] & \mathbb{P}^1 = H/\langle \iota \rangle
\end{tikzcd}$$

elliptic fibers
$2g+2$ double

$\tilde{X}(k)$ NOT dense

$\Downarrow$

$X(k)$ NOT dense

Thm Chevalley Weil : $\tilde{X} \xrightarrow{u} X$ étale =
$\forall k \; \exists k' \supset k : X(k) \subset u(\tilde{X}(k')$

$\kappa(X) = 1, \; q(X) = 1 \implies X$ does not domin
general type

Rk: Existence of $\tilde{X} \longrightarrow X$ étale with $\tilde{X}(k$ not dense can be deduced from the geometry of the fibration $c: X \longrightarrow \mathbb{P}$

$$\begin{tikzcd}
 & H \times E \arrow[r, "p_H"] \arrow[dl, "p_E"'] \arrow[d, "u"] & H \arrow[d, "v"] \\
E \arrow[d, "\text{étale}"'] & X \arrow[r, "c"] \arrow[dl, "f"] & \mathbb{P}^1 \\
= E' & &
\end{tikzcd}$$

ramified

isotrivial (H)
<u>no</u> singular fibers

Program: (1) Develop a KODAIRA-type theory of fibrat-

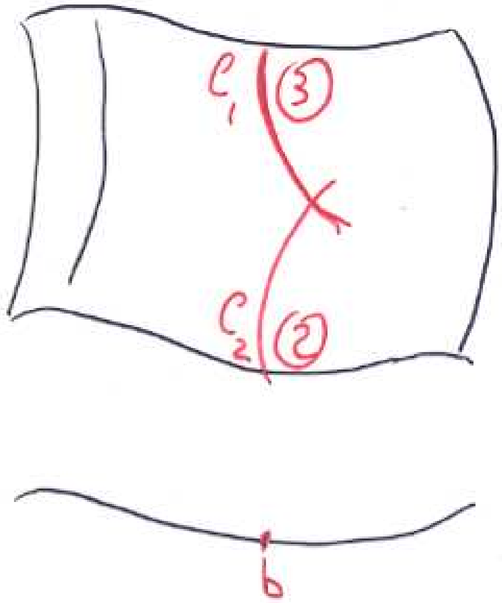
(2) Find the (if $\exists$) distinguished one

(3) Use to understand arithmetic and other geometric aspects.

FIBRATIONS of GENERAL TYPE (CAMPANA/OL

$f : X \longrightarrow B$

$\boxed{\dim X = 2 \qquad \dim B = 1}$ ⟸ to simplify



$b \in B$

$X_b = \sum \mathrm{mult}(C_b)\, C_b$

$X_b = 3C_1 + 2C_2$

$m_f(b) := \min_{C_b} \{\mathrm{mult}(C_b)\}$

$m_f(b) = 2$

$\Delta_f := \sum_b \left(1 - \frac{1}{m_f(b)}\right) b \in \mathrm{Div}\, B \otimes \mathbb{Q}$

Def: The fibration f is of general type if

$$\deg(K_B + \Delta_f) > 0$$

- If f has NO multiple fibers, then f of general type iff B is

The pair (B, Δ_f) is called an <u>ORBIFOLD</u>

Similar construction in the arithmetic setting:
M-curves DARMON, GRANVILLE 97 $\rightsquigarrow$ "Faltings + E

- Δ_f is such that if $v: B' \longrightarrow B$ is a covering KILLING the multiple fibers of f, B' is of general type if f is

Example: $c: X \longrightarrow \mathbb{P}^1$ of CT-S-SD is of general t
(neither the base nor the fiber are!)

<u>Def</u>: X is <u>SPECIAL</u> if it does <u>NOT</u> admit a <u>fibration</u> of <u>general type</u> (to a variety o dim > 0

CAMPANA 01

Examples (1) X of general type $\Rightarrow$ X NOT special

(2) $\dim X = 1$ $\quad$ X special $\Longleftrightarrow$ $g_X \leq 1$

Theorems

CORE fibration (1) $\exists !\ c: X \longrightarrow c(X)$
(the CORE fib. of X) such that

CAMPANA 01

(a) the general fiber is SPECIAL
(b) c is a fibration of general type
(or $c(X)$ is a point)

... loosely ... (2) c admits a natural factorizat[ion]
through canonical and MRC f[ibrations]

$$c = (\varphi \circ r)^{\ell}$$

X special $\Leftrightarrow$ $c(X) =$ point

$$\left.\begin{array}{l} X \text{ R.C} \\ \\ k(X) = 0 \end{array}\right\} \Longrightarrow X \text{ is SPECIAL}$$

(Recall: X is RC $\Leftrightarrow$ $r(X) =$ point
$k(X) = 0 \Leftrightarrow \varphi(X) =$ point)

Unified Conjectural picture

dim X = arbitrary

$X(k)$ dense $\iff$ X SPECIAL $\iff$ $d_X \equiv 0$

for some k

↑ Kobayashi

Strong Lang Conj. revisited

X not special $\implies$ $\exists$ open $U_X \subset \mathcal{C}(X)$: $\forall k$

$c(X(k)) \cap U_X$ is finite

Back to $WL^{(2)}$

$\mathcal{C}$

$X(k)$ dense $\iff$ X SPECIAL

$\Downarrow$ $\cancel{\Uparrow}$ BOGOMOLOV or TSCHINKEL dim ≥ 3

X WEAKLY special

($\nexists$ étale $\tilde{X} \to X$ dominating genl ty

CT

A

? CAMPANA PAUN OS: BT examples behave "hyperbolically"