BIRATIONAL GEOMETRY FOR NUMBER THEORISTS: COMPANION NOTES

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Contents

Introduction

When thinking about the course "birational geometry for number theorists" I so naïvely agreed to give at the Göttingen summer school, I cannot avoid imagining the spirit of the late Serge Lang, not so quietly beseeching one to do things right, keeping the theorems functorial with respect to ideas, and definitions natural. But most important is the fundamental tenet of diophantine geometry, for which Lang was one of the stongest and loudest advocates, which was so aptly summarized in the introduction of [16]:

GEOMETRY DETERMINES ARITHMETIC.

To make sense of this, largely conjectural, epithet, it is good to have some loose background in birational geometry, which I will try to provide. For the arithmetic motivation I will explain conjectures of Bombieri, Lang and Vojta, and new and exciting versions of those due to Campana. In fact, I imagine Lang would insist (strongly, as only he could) that Campana's conjectures most urgently need further investigation, and indeed in some sense they form the centerpiece of these notes.

Unfortunately, birational geometry is too often rightly subject to another of Lang's beloved epithets:

21 July 2006 version: won't get any better before the lectures.

YOUR NOTATION SUCKS!

which has been a problem in explaining some basic things to number theorists. I'll try to work around this problem, but I can be certain some problems will remain! One line of work which does not fall under this criticism, and is truly a gem, is Mori's "bend and break" method. It will be explained in due course.

These pages are meant to contain a very rough outline of ideas and statements of results which are relevant to the lectures. I do not intend this as a prerequisite to the course, but I suspect it will be of some help to the audience. Some exercises which might also be helpful are included.

IMPORTANT:

- some of the material in the lectures is not (yet) discussed here, and
- only a fraction of the material here will be discussed in the lectures.

Our convention: a variety over k is an *absolutely* reduced and irreducible scheme of finite type over k.

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Lecture 0. Geometry and arithmetic of curves

The arithmetic of algebraic curves is one area where basic relationships between geometry and arithmetic are known, rather than conjectured.

0.1. **Closed curves.** Consider a smooth projective algebraic curve C defined over a number field k. We are interested in a qualitative relationship between its arithmetic and geometric properties.

We have three basic facts:

0.1.1. A curve of genus 0 becomes rational after at most a quadratic extension k' of k, in which case its set of rational points $C(k')$ is infinite (and therefore dense in the Zariski topology).

0.1.2. A curve of genus 1 has a rational point after an extension k' of k (though the degree is not a priori bounded), and has positive Mordell–Weil rank after a further quadratic extension k''/k, in which case again its set of rational points $C(k'')$ is infinite (and therefore dense in the Zariski topology).

We can immediately introduce the following definition:

Definition 0.1.3. Let X be an algebraic variety defined over k. We say that rational points on X are potentially dense, if there is a finite extension k'/k such that the set $X(k')$ is dense in $X_{k'}$ in the Zariski topology.

Thus rational points on a curve of genus 0 or 1 are potentially dense.

Finally we have

Theorem 0.1.4 (Faltings, 1983). *Let C be an algebraic curve of genus > 1 over a number field k. Then $C(k)$ is finite.*

In other words, rational points on a curve C of genus g are potentially dense if and only if $g \leq 1$.

0.1.5. So far there isn't much birational geometry involved, because we have the old theorem:

Theorem 0.1.6. *A smooth algebraic curve is uniquely determined by its function field.*

But this is an opportunity to introduce a tool: on the curve C we have a canonical divisor class K_C, such that $\mathcal{O}_C(K_C) = \Omega^1_C$, the sheaf of differentials, also known by the notation ω_C - the dualizing sheaf. We have:

(1) $\deg K_C = 2g - 2 = -\chi(C_{\mathbb{C}})$, where $\chi(C_{\mathbb{C}})$ is the topological Euler characteristic of the complex Riemann surface $C_{\mathbb{C}}$.
(2) $\dim H^0(C, \mathcal{O}_C(K_C)) = g$.

For future discussion, the first property will be useful. We can now summarize, follwing [16]:

0.1.7.

Degree of K_C	rational points
$2g-2 \leq 0$	potentially dense
$2g-2 > 0$	finite

0.2. **Open curves.**

0.2.1. Consider a smooth quasi-projective algebraic curve C defined over a number field k. It has a unique smooth projective completion $C \subset \overline{C}$, and the complement is a finite set $\Sigma = \overline{C} \smallsetminus C$. Thinking of Σ as a reduced divisor of some degree n, a natural line bundle to consider is $\mathcal{O}_{\overline{C}}(K_{\overline{C}} + \Sigma)$, the sheaf of differentials with logarithmic poles on Σ, whose degree is again $-\chi^{top}(C) = 2g - 2 + n$. The sign of $2g - 2 + n$ again serves as the geometric invariant to consider.

0.2.2. Consider for example the affine line. Rational points on the affine line are not much more interesting than those on $\mathbb{P}^1$. But we can also consider the behavior of *integral* points, where interesting results do arise. However, what does one mean by integral points on $\mathbb{A}^1$? The key is that integral points are an invariant of an "integral model" of $\mathbb{A}^1$ over $\mathbb{Z}$.

0.2.3. Consider the ring of integers $\mathcal{O}_k$ and a finite set $S \subset \operatorname{Spec} \mathcal{O}_k$ of finite primes. One can associate to it the ring $\mathcal{O}_{k,S}$ of S-integers, of elements in K which are in $\mathcal{O}_\wp$ for any prime $\wp \notin S$.

Now consider a *model* of C over $\mathcal{O}_{k,S}$, namely a scheme $\mathcal{C}$ of finite type over $\mathcal{O}_{k,S}$ with an isomorphism of the generic fiber $\mathcal{C}_k \simeq C$. It is often useful to start with a model $\overline{\mathcal{C}}$ of $\overline{C}$, and take $\mathcal{C} = \overline{\mathcal{C}} \smallsetminus \overline{\Sigma}$.

Now it is clear how to define integral points: an S-integral point on $\mathcal{C}$ is simply an element of $\mathcal{C}(\mathcal{O}_{k,S})$, in other words, a section of $\mathcal{C} \to \operatorname{Spec}(\mathcal{O}_{k,S})$. This is related to rational points on a proper curve as follows:

0.2.4. If $\Sigma = \emptyset$, and the model chosen is proper, the notions of integral and rational points agree, because of the valuative criterion for properness.

Exercise 0.2.5. Prove this!

We have the following facts:

0.2.6. If C is rational and $n \leq 2$, then after possibly enlarging k and S, any integral model of C has an infinite collection of integral points.

Exercise 0.2.7. Prove this!

On the other hand, we have:

Theorem 0.2.8 (Siegel's Theorem). *If $n \geq 3$, or if $g > 0$ and $n > 0$, then for any integral model $\mathcal{C}$ of C, the set of integral points $\mathcal{C}(\mathcal{O}_{k,S})$ is finite.*

A good generalization of Definition 0.1.3 is the following:

Definition 0.2.9. Let X be an algebraic variety defined over k with a model $\mathcal{X}$ over $\mathcal{O}_{k,S}$. We say that integral points on X are potentially dense, if there is a finite extension k'/k, and an enlargement S' of the set of places in k' over S, such that the set $\mathcal{X}(\mathcal{O}_{k',S'})$ is dense in $X_{k'}$ in the Zariski topology.

We can thus generalize 0.1.7, as in [16], as follows:

0.2.10.

degree of $K_{\overline{C}} + \Sigma$	integral points
$2g - 2 + n \leq 0$	potentially dense
$2g - 2 + n > 0$	finite

0.2.11. One lesson we must remember from this discussion is that

For **open** varieties we use **integral** points on **integral models**.

0.3. **Faltings implies Siegel.** Siegel's theorem was proven years before Faltings's theorem. Yet it is instructive, especially in the later parts of these notes, to give the following argument showing that Faltings's theorem implies Siegel's.

Theorem 0.3.1 (Hermite-Minkowski, see [16] page 264)**.** *Let k be a number field, $S \subset \operatorname{Spec} \mathcal{O}_{k,S}$ a finite set of finite places, and d a positive integer. Then there are only finitely many extensions k'/k of degree $\leq d$ unramified outside S.*

From which one can deduce

Theorem 0.3.2 (Chevalley-Weil, see [16] page 292)**.** *Let $\pi : \mathcal{X} \to \mathcal{Y}$ be a finite étale morphism of schemes over $\mathcal{O}_{k,S}$. Then there is a finite extension k'/k, with S' lying over S, such that $\pi^{-1}\mathcal{Y}(\mathcal{O}_{k,S}) \subset \mathcal{X}(\mathcal{O}_{k',S'})$.*

On the geometric side we have an old topological result

Theorem 0.3.3. *If C is an open curve with $2g-2+n > 0$ and $n > 0$, defined over k, there is a finite extension k'/k and a finite unramified covering $D \to C$, such that $g(D) > 1$.*

Exercise 0.3.4. Combine these theorems to obtain a proof of Siegel's theorem assuming Faltings's theorem.

0.3.5. Our lesson this time is that

Rational and integral points can be controlled in finite étale covers.

0.4. **Function field case.** There is an old and distinguished tradition of comparing results over number fields with results over function fields. To avoid complications I will concentrate on function fields of characteristic 0, and consider closed curves only.

0.4.1. If K is the function field of a complex variety B, then a variety X/K is the generic fiber of a scheme $\mathcal{X}/B$, and a K-rational point $P \in X(K)$ can be thought of as a rational section of $\mathcal{X} \to B$. If $\dim B = 1$ and $\mathcal{X} \to B$ is proper, then again a K-rational point $P \in X(K)$ is equivalent to a *regular* section $B \to \mathcal{X}$.

Exercise 0.4.2. Prove (i.e. make sense of) this!

0.4.3. The notion of *integral points* is similarly defined using sections. When $\dim B > 1$ there is an intermediate notion of *proper rational points*: a K-rational point p of X is a proper rational point of $\mathcal{X}/B$ if the closure B' of p in $\mathcal{X}$ maps properly to B. Most likely this notion will not play a central role here.

Consider now C/K a curve. Of course it is possible that $\mathcal{C}$ is, or is birationally equivalent to, $C_0 \times B$, in which case we have plenty of constant sections coming from $C_0(\mathbb{C})$, corresponding to constant points $C(K)^{const}$. But that is almost all there is:

Theorem 0.4.4 (Manin [25], Grauert [14])**.** *Assume $g(C) > 1$. Then the set of nonconstant points $C(K) \smallsetminus C(K)^{const}$ is finite.*

Exercise 0.4.5. What does this mean for constant curves $C_0 \times B$?

Working inductively on transcendence degree, and using Faltings's Theorem, we obtain:

Theorem 0.4.6. *Let C be a curve of genus > 1 over a field k finitely generated over $\mathbb{Q}$. Then the set of k-rational points $C(k)$ is finite.*

Exercise 0.4.7. Prove this, using previous results as given!

See [30] for an appropriate statement in positive characteristics.

Lecture 1. Kodaira dimension

1.1. **Iitaka dimension.** Consider now a smooth, projective variety X of dimension d over a field k of characteristic 0. We seek an analogue of the sign on $2g-2$ in this case. The approach is by counting sections of the canonical line bundle $\mathcal{O}_X(K_X) = \wedge^d\Omega^1_X$. Iitaka's book [17] is a good reference.

Theorem 1.1.1. *If L is a line bundle on X. Assume $h^0(X, L^n)$ does not vanish for all positive integers n. Then there is a unique integer $\kappa = \kappa(X, L)$ with $0 \leq \kappa \leq d$ such that*

$$\limsup_{n\to\infty} \frac{h^0(X, L^n)}{n^\kappa}$$

exists and is nonzero.

Definition 1.1.2.
(1) The integer $\kappa(X, L)$ in the theorem is called *the Iitaka dimension of* (X, L).
(2) In the special case $L = \mathcal{O}_X(K_K)$ we write $\kappa(X) := \kappa(X, L)$ and call $\kappa(X)$ *the Kodaira dimension of* X.
(3) It is customary to set $\kappa(X, L) =$ to be either -1 or $-\infty$ if $h^0(X, L^n)$ vanishes for all positive integers n. It is safest to say in this that the Iitaka dimension is *negative*.

We will see an algebraic justification for the -1 convention soon, and a geometric justification for $-\infty$ in a bit.

An algebraically meaningful presentation of the Iitaka dimension is the following:

Proposition 1.1.3. *Consider the algebra of sections*

$$\mathcal{R}(X, L) := \bigoplus_{n\geq 0} H^0(X, L^n).$$

Then, with the -1 *convention,*

$$\operatorname{tr.deg}\mathcal{R}(X,L) = \kappa(X,L) + 1.$$

Definition 1.1.4. We say that a property holds for a sufficiently high and divisible n, if there exists $n_0 > 0$ such that the property holds for every positive multiple of n_0.

A geometric meaning of $\kappa(X,L)$ is given by the following:

Proposition 1.1.5. *Assume* $\kappa(X,L) \geq 0$*. then for sufficiently high and divisible* n*, the dimension of the image of the rational map* $\phi_{L^n} : X \dashrightarrow \mathbb{P}H^0(X,L^n)$ *is precisely* $\kappa(X,L)$*.*

Even more precise is:

Proposition 1.1.6. *There is* $n_0 > 0$ *such that the image* $\phi_{L^n}(X)$ *is birational to* $\phi_{L^{n_0}}(X)$ *for all* $n_0|n$*.*

Definition 1.1.7.
1. The birational equivalence class of $\phi_{L^{n_0}}(X)$ is denoted $I(X,L)$.
2. The rational map $X \to I(X,L)$ is called the *Iitaka fibration* of (X,L).
3. In case L is the canonical bundle, this is called the Iitaka fibration of X, written $X \to I(X)$

The following notion is important:

Definition 1.1.8. The variety X is said to be *of general type* of $\kappa(X) = \dim X$.

Remark 1.1.9. The name definitely leaves something to be desired. It comes from the observation that surfaces not of general type can be nicely classified, whereas there is a whole zoo of surfaces of general type.

Exercise 1.1.10. Prove Proposition 1.1.6:

1. Show that if $n,d > 0$ and $H^0(X,L^n) \neq 0$ then there is a dominant $\phi_{L^{nd}}(X) \dashrightarrow \phi_{L^n}(X)$ such that the following diagram is commutative:

$$\begin{array}{ccc} X & \overset{\phi_{L^{nd}}}{\dashrightarrow} & \phi_{L^{nd}}(X) \\ & \underset{\phi_{L^n}}{\searrow} & \downarrow \\ & & \phi_{L^n}(X). \end{array}$$

2. Conclude that $\dim \phi_{L^n}(X)$ is a constant κ for large and divisible n.
3. Suppose $n > 0$ satisfies $\kappa := \dim \phi_{L^n}(X)$. Show that for any $d > 0$, the function field of $\phi_{L^{nd}}(X)$ is algebraic over the function field $\phi_{L^n}(X)$.
4. Recall that for any variety X, and subfield L of $K(X)$ containing k is finitely generated. Apply this to the algebraic closure of $\phi_{L^n}(X)$ to complete the proof of the proposition.

Exercise 1.1.11. Use proposition 1.1.6 to prove Theorem 1.1.1.

1.2. Properties and examples of the Kodaira dimension.

Exercise 1.2.1. Show that $\kappa(\mathbb{P}^n) = -\infty$ and $\kappa(A) = 0$ for an abelian variety A.

1.2.2. *Curves:*

Exercise. Let C be a smooth projective curve and L a line bundle. Prove that

$$\kappa(C, L) = \begin{cases} 1 & \text{if } \deg_C L > 0, \\ 0 & \text{if } L \text{ is torsion, and} \\ < 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\kappa(C) = \begin{cases} 1 & \text{if } g > 1, \\ 0 & \text{if } g = 1, \text{ and} \\ < 0 & \text{if } g = 0. \end{cases}$$

1.2.3. *Birational invariance:*

Exercise. Let $X' \dashrightarrow X$ be a birational map of smooth projective varieties. Show that the spaces $H^0(X, \mathcal{O}_X(mK_X))$ and $H^0(X', \mathcal{O}_{X'}(mK_{X'}))$ are canonically isomorphic.

Deduce that $\kappa(X) = \kappa(X')$.

1.2.4. *Generically finite maps.*

Exercise. Let $f : X' \to X$ be a generically finite map of smooth projective varieties.

Show that $\kappa(X') \geq \kappa(X)$.

1.2.5. *Finite étale maps.*

Exercise 1.2.6. Let $f : X' \to X$ be a finite étale map of smooth projective varieties.

Show that $\kappa(X') = \kappa(X)$.

1.2.7. *Field extensions:*

Exercise. Let k'/k be a field extension, X a variety over k with line bundle L, and $X_{k'}, L_{k'}$ the result of base change.

Show that $\kappa(X, L) = \kappa(X_{k'}, L_{k'})$. In particular $\kappa(X) = \kappa(X_{k'})$.

1.2.8. *Products.*

Exercise. Show that, with the $-\infty$ convention,

$$\kappa(X_1 \times X_2, L_1 \boxtimes L_2) = \kappa(X_1, L_1) + \kappa(X_2, L_2).$$

Deduce that $\kappa(X_1 \times X_2) = \kappa(X_1) + \kappa(X_2)$.

This "easy additivity" is the main reason for the $-\infty$ convention. We'll see more about fibrations below.

1.2.9. *Fibrations.* The following is subtle and difficult:

Theorem (Siu's theorem on deformation invariance of plurigenera)**.** *Let $X \to B$ be a smooth projective morphism with connected geometric fibers. Then $h^0(X_b, \mathcal{O}(K_{X_b}))$ is independent of $b \in B$. In particular $\kappa(X_b)$ is independent of $b \in B$.*

Exercise 1.2.10. Let $X \to B$ be a morphism of smooth projective varieties with connected fibers. Let $b \in B$ be such that $X \to B$ is smooth over b, and let $\eta_B \in B$ be the generic point.

Use "cohomology and base change" and Siu's theorem to deduce that

$$\kappa(X_b) = \kappa(X_{\eta_B}).$$

Definition 1.2.11. The Kodaira defect of X is $\delta(X) = \dim(X) - \kappa(X)$.

Exercise 1.2.12. Let $X \to B$ be a morphism of smooth projective varieties with connected fibers. Show that $\delta(X) \geq \delta(X_{\eta_B})$. Equivalently $\kappa(X) \leq \dim(B) + \kappa(X_{\eta_B})$.

Exercise 1.2.13. Let $Y \to B$ be a morphism of smooth projective varieties with connected fibers, and $Y \to X$ a generically finite map. Show that $\delta(X) \geq \delta(Y_{\eta_B})$.

This "easy subadditivity" has many useful consequences.

Definition 1.2.14. We say that X is uniruled if there is a variety B of dimension $\dim X - 1$ and a dominant rational map $B \times \mathbb{P}^1 \dashrightarrow X$.

Exercise 1.2.15. If X is uniruled, show that $\kappa(X) = -\infty$.

The converse is an important conjecture, sometimes known as the $-\infty$-Conjecture. It is a consequence of the "good minimal model" conjecture:

Conjecture 1.2.16. *Assume X is not uniruled. Then $\kappa(X) \geq 0$.*

Exercise 1.2.17. If X is covered by a family of elliptic curves, show that $\kappa(X) \leq \dim X - 1$.

1.2.18. *Surfaces.* Surfaces of Kodaira dimension < 2 are "completely classified". Some of these you can place in the table using what you have learned so far. In the following description we give a representative of the birational class of each type:

κ	description
$-\infty$	$\mathbb{P}^2$ or $\mathbb{P}^1 \times C$
0	a. abelian surfaces b. bielliptic surfaces k. K3 surfaces e. Enriques surfaces
1	many elliptic surfaces

1.2.19. *Iitaka's program.* Here is a central conjecture of birational geometry:

Conjecture (Iitaka)**.** *Let $X \to B$ be a surjective morphism of smooth projective varieties. Then*

$$\kappa(X) \geq \kappa(B) + \kappa(X_{\eta_B}).$$

1.2.20. Major progress on this conjecture was made through the years by several geometers, including Fujita, Kawamata, Viehweg and Kollár. The key, which makes this conjecture plausible, is the semipositivity properties of the relative dualizing sheaf $\omega_{X/B}$, which originate from work of Arakelov and rely on deep Hodge theoretic arguments.

Two results will be important for these lectures.

Theorem 1.2.21 (Kawamata)**.** *Iitaka's conjecture follows from the Minimal Model Program: if X_{η_B} has a good minimal model then $\kappa(X) \geq \kappa(B) + \kappa(X_{\eta_B})$.*

Theorem 1.2.22 (Viehweg)**.** *Iitaka's conjecture holds in case B is of general type, namely:*

Let $X \to B$ be a surjective morphism of smooth projective varieties, and assume $\kappa(B) = \dim B$. Then $\kappa(X) = \dim(B) + \kappa(X_{\eta_B})$.

Note that equality here is forced by the easy subadditivity inequality: $\kappa(X) \leq \dim(B) + \kappa(X_{\eta_B})$ always holds.

Exercise 1.2.23. Let X, B_1, B_2 be smooth projective varieties. Suppose $X \to B_1 \times B_2$ is generically finite to its image, and assume both $X \to B_i$ surjective.

(1) Assume B_1, B_2 are of general type. Use Viehweg's theorem and the Kodaira defect inequality to conclude that X is of general type.
(2) Assume $\kappa(B_1), \kappa(B_2) \geq 0$. Show that if Iitaka's conjecture holds true, then $\kappa(X) \geq 0$.

Exercise 1.2.24. Let X be a smooth projective variety. Show that there is a dominant rational map

$$L_X : X \dashrightarrow L(X)$$

such that

(1) $L(X)$ is of general type, and

(2) the map is universal: if $g : X \dashrightarrow Z$ is a dominant rational map with Z of general type, there is a unique rational map $L(g) : L(X) \dashrightarrow Z$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \overset{L_X}{\dashrightarrow} & L(X) \\ & \underset{g}{\searrow} & \downarrow L(g) \\ & & Z. \end{array}$$

The map L_X is called *the Lang map of* X, and $L(X)$ the *Lang variety of* X.

1.3. **Uniruled varieties and rationally connected fibrations.**

1.3.1. *Uniruled varieties.* For simplicity let us assume here that k is algebraically closed.

As indicated above, a variety X is said to be *uniruled* if there is a $d-1$-dimensional variety B and a dominant rational map $B \times \mathbb{P}^1 \dashrightarrow X$. Instead of $B \times \mathbb{P}^1$ one can take any variety $Y \to B$ whose generic fiber has genus 0. As discussed above, if X is uniruled then $\kappa(X) = -\infty$. The converse is the important $-\infty$-Conjecture 1.2.16

A natural question is, can one "take all these rational curves out of the picture?" The answer is yes, in the best possible sense.

Definition 1.3.2. A smooth projective variety P is said to be *rationally connected* if through any two points $x, y \in P$ there is a morphism from a rational curve $C \to P$ having x and y in its image.

There are various equivalent ways to characterize rationally connected varieties.

Theorem 1.3.3 (Campana, Kollár-Miyaoka-Mori)**.** *Let P be a smooth projective variety. The following are equivalent:*

(1) P is rationally connected.
(2) Any two points are connected by a chain of rational curves.
(3) For any finite set of points $S \subset P$, there is a morphism from a rational curve $C \to P$ having S in its image.
(4) There is a "very free" rational curve on P - if $\dim P > 2$ this means there is a rational curve $C \subset P$ such that the normal bundle $N_{C \subset P}$ is ample.

Key properties:

Theorem 1.3.4. *Let X and X' be smooth projective varieties, with X rationally connected.*

(1) If $X \dashrightarrow X'$ is a dominant rational map (in particular when X and X' are birationally equivalent) then X' is rationally connected.

(2) If X' is deformation-equivalent to X then X' is rationally connected.
(3) If $X' = X_{k'}$ where k'/k is an algebraically closed field extension, then X' is rationally connected if and only if X is.

Exercise 1.3.5. A variety is unirational if it is a dominant image of $\mathbb{P}^n$. Show that every unirational variety is rationally connected.

More generally: a smooth projective variety X is Fano if its anti-canonical divisor is ample.

Theorem 1.3.6 (Kollár-Miyaoka-Mori, Campana)**.** *A Fano variety is rationally connected.*

On the other hand:

Conjecture 1.3.7 (Kollár)**.** *There is a rationally connected threefold which is not unirational. There should also exist some hypersurface of degree n in $\mathbb{P}^n$, $n \geq 4$ which is not unirational.*

Conjecture 1.3.8 (Kollár-Miyaoka-Mori, Campana)**.** *(1) A variety X is rationally connected if and only if*

$$H^0(X, (\Omega^1_X)^{\otimes n}) = 0$$

for every positive integer n.

(2) A variety X is rationally connected if and only if every positive dimensional dominant image $X \dashrightarrow Z$ has $\kappa(Z) = -\infty$.

This follows from the minimal model program.

Now we can break any X up:

Theorem 1.3.9 (Campana, Kollár-Miyaoka-Mori, Graber-Harris-Starr)**.** *Let X be a smooth projective variety. There is a birational morphism $X' \to X$, a variety $Z(X)$, and a dominant morphism $X' \to Z(X)$ with connected fibers, such that*

(1) The general fiber of $X' \to Z(X)$ is rationally connected, and
(2) $Z(X)$ is not uniruled.

Moreover, $X' \to X$ is an isomorphism in a neighborhood of the general fiber of $X' \to Z(X)$.

1.3.10. The rational map $r_X : X \dashrightarrow Z(X)$ is called the *maximally rationally connected fibration of* X (or MRC fibration of X) and $Z(X)$, which is well defined up to birational equivalence, is called the *MRC quotient* of X.

1.3.11. The MRC fibration has the universal property of being "final" for dominant rational maps $X \to B$ with rationally connected fibers.

One can construct similar fibrations with similar universal property for maps with fibers having $H^0(X_b, (\Omega^1_{X_b})^{\otimes n}) = 0$, or for fibers having no dominant morphism to positive dimensional varieties of nonnegative Kodaira dimension. Conjecturally these agree with r_X. Also conjecturally, assuming Iitaka's conjecture, there exists $X \dashrightarrow Z'$ which is initial for maps to varieties of non-negative Kodaira dimension. This conjecturally will also agree with r_X. All these conjecture would follow from the "good minimal model" conjecture.

1.3.12. *Arithmetic, finally.* The set of rational points on a rational curve is Zariski dense. The following is a natural extension:

Conjecture 1.3.13 (Campana)**.** *Let P be a rationally connected variety over a number field k. Then rational points on P are potentially dense.*

This conjecture and its sister below 1.4.2 was implicit in works of many, including Bogomolov, Colliot-Thélène, Harris, Hassett, Tschinkel.

1.4. **Geometry and arithmetic of the Iitaka fibration.** We now want to understand the geometry and arithetic of varieties such as $Z(X)$, i.e. non-uniruled varieties. Conj:k¡0-uniruled

So let X satisfy $\kappa(X) \geq 0$, and consider the Iitaka fibration $X \dashrightarrow I(X)$.

Proposition 1.4.1. *Let F be a general fiber of $X \to I(X)$. Then $\kappa(F) = 0$*

Conjecture 1.4.2 (Campana)**.** *Let F be a variety over a number field k satisfying $\kappa(F) = 0$. Then rational points on F are potentially dense.*

Exercise 1.4.3. Recall the Lang map in 1.2.24. Assuming Conjecture 1.2.16, show that $L(X)$ is the result of applying MRC fibrations and Iitaka fibrations untill the result stabilizes.

1.5. **Lang's conjecture.** A highly inspiring conjecture in diophantine geometry is the following:

Conjecture (Lang's conjecture, weak form)**.** *Let X be a smooth projective variety of general type over a number field, or any finitely generated field, k. Then $X(k)$ is not Zariski-dense in X.*

In fact, motivated by analogy with conjectures on the Kobayashi pseudometric of a variety of general type, Lang even proposed the following:

Conjecture (Lang's geometric conjecture)**.** *Let X be a smooth projective variety of general type. There is a Zariski closed proper subset $S(X) \subset X$, whose irreducible components are not of general type, and such that every subset $T \subset X$ not of general type is contained in $S(X)$.*

The two combine to give:

Conjecture (Lang's conjecture, srtong form)**.** *Let X be a smooth projective variety of general type over a number field, or any finitely generated field, k. Then for any finite extension k'/k, the set $(X \smallsetminus S(X))(k')$ is finite.*

Here is a simple consequence:

Proposition 1.5.1. *Assume Lang's conjecture holds true. Let X be a smooth projective variety over a number field k. Assume there is a dominant rational map $X \to Z$, such that Z is a positive dimensional variety of general type (i.e.,* $\dim L(X) > 0$*). Then $X(k)$ is not Zariski-dense in X.*

1.6. **Uniformity of rational points.** Lang's conjecture can be investigated whenever one has a variety of general type around. By considering certain subvarieties of the moduli space $\mathcal{M}_{g,n}$ of curves of genus g with n distinct points on them, rather surprizing and inspiring implications on the arithmetic of curves arise. This is the subject of the work [9] of L. Caporaso, J. Harris and B. Mazur. Here are their key results:

Theorem 1.6.1. *Assume that the weak Lang's conjecture holds true. Let k be a number field, or any finitely generated field, and let $g > 1$ be an integer. Then there exists an integer $N(k, g)$ such that for every algebraic curve C over k we have*

$$\#C(k) \leq N(k, g).$$

Theorem 1.6.2. *Assume that the strong Lang's conjecture holds true. Let $g > 1$ be an integer. Then there exists an integer $N(g)$ such that for every finitely generated field k there are, up to isomorphisms, only finitely many algebraic curves C over k with $\#C(k) > N(g)$.*

Further results along these lines, involving higher dimensional varieties and involving more stronger results on curves can be found in [15], [1], [28], [4], [2]. For instance, P. Pacelli's result in [28] says that the number $N(k, g)$ can be replaced for number field by $N(d, g)$, where $d = [k : \mathbb{Q}]$.

The reader may decide whether this shows the great power of the conjectures or their unlikelihood. I prefer to be agnostic and rely on the conjectures for inspiration.

1.7. **The search for an arithmetic dichotomy.** As demonstrated in table 0.1.7, potential density of rational points on curves is dictated by geometry. Lang's conjecture carves out a class of higher dimensional varieties for which rational points are, conjecturally, not potentially dense. Can this be extended to a dichotomy as we have for curves?

One can naturally wonder - is the Kodaira dimension itself enough for determining potential density of points? Or else, maybe just the inexistence of a map to a positive dimensional variety of general type?

1.7.1. *Rational points on surfaces.* The following table, which I copied from a lecture of L. Caporaso, describes what is known about surfaces.

Caporaso's table: rational points on surfaces

Kodaira dimension	$X(k)$ potentially dense	$X(k)$ never dense
$\kappa = -\infty$	$\mathbb{P}^2$	$\mathbb{P}^1 \times C$ $(g(C) \geq 2)$
$\kappa = 0$	$E \times E$, many others	**none known**
$\kappa = 1$	many examples	$E \times C$ $(g(C) \geq 2)$
$\kappa = 2$	**none known**	many examples

The bottom row is the subject of Lang's conjecture, and the $\kappa = 0$ row is the subject of Conjecture 1.4.2.

1.7.2. *Failure of the dichotomy using $\kappa(X)$.* The first clear thing we learn from this is, as Caporaso aptly put it in her lecture, that diophantine geometry is not governed by the Kodaira dimension. On the top row we see that clearly: on a ruled surface over a curve of genus ≥ 2, rational points can never be dense by Faltings's theorem. So it behaves very differently from a rational surface.

Even if one insists on working with varieties of non-negative Kodaira dimension, the $\kappa = 1$ row gives us trouble.

Exercise. Take a Lefschetz pencil of cubic curves in $\mathbb{P}^2$, parametrized by t, and assume that it has two sections s_1, s_2 whose difference is not torsion on the generic fiber. We use s_1 as the origin.

(1) Show that the dualizing sheaf of the total space S is $\mathcal{O}_S(-2[F])$, where F is a fiber.
(2) Show that the relative dualizing sheaf is $\mathcal{O}_S([F])$. Take the base change $t = s^3$. We still have two sections, still denoted s_1, s_2, such that the difference is not torsion. We view s_1 as origin.

    Show that the relative dualizing sheaf of the new surface X is $\mathcal{O}_X(3[F])$ and its dualizing sheaf is $\mathcal{O}_X([F])$. Conclude that the resulting surface X has Kodaira dimension 1.
(3) For any rational point p on $\mathbb{P}^1$ such where the section s_2 of $X \to \mathbb{P}^1$ is not torsion, the fiber and dense set of rational points.
(4) Conclude that X has a dense set of rational points.

1.7.3. *Failure of the dichotomy using the Lang map.* The examples given above still allow for a possible dichotomy based on the existence of a nontrivial map to a variety of general type. But the following examples, which fits on the right column on row $\kappa = 1$, shows this doesn't work either. The example is due to Colliot-Thélène, Skorobogatov and Swinnerton-Dyer [10].

Example. Let C be a curve with an involution $\phi : C \to C$, such that the quotient is rational. Consider an elliptic curve E with a 2-torsion point a, and consider the fixed-point free action of $\mathbb{Z}/2\mathbb{Z}$ on $Y = E \times C$ given by:

$$(x, y) \mapsto (x + a, \phi(y)).$$

Let the quotient of Y by the involution be X. Then $L(X)$ is trivial, though rational points on X are not potentially dense by Chevalley-Weil and Faltings.

In the next lecture we address a conjectural approach to a dichotomy - due to F. Campana - which has a chance to work .

1.8. **Logarithmic Kodaira dimension and the Lang-Vojta conjectures.** We now briefly turn our attention to open varieties, following the lesson in section 0.2.11.

Let $\overline{X}$ be a smooth projective variety, D a reduced normal crossings divisor. We can consider the quasiprojective variety $X = \overline{X} \smallsetminus D$.

The logarithmic Kodaira dimension of X is defined to be the Iitaka dimension $\kappa(X) := \kappa(\overline{X}, K_{\overline{X}} + D)$. We say that X is of *logarithmic general type* if $\kappa(X) = \dim X$.

It can be easily shown that $\kappa(X)$ is independent of the completion $X \subset \overline{X}$, as long as $\overline{X}$ is smooth and D is a normal crossings divisor. More invariance propertiers can be discussed, but will take us too far afield.

Now to arithmetic: suppose $\mathcal{X}$ is a model of X over $\mathcal{O}_{k,S}$. We can consider integral points $\mathcal{X}(\mathcal{O}_{L,S_L})$ for any finite extension L/k and enlargement S_L of the set of places over S.

The Lang-Vojta conjecture is the following:

Conjecture 1.8.1. *If X is of logarithmic general type, then integral points are not potentially dense on X, i.e. $\mathcal{X}(\mathcal{O}_{L,S_L})$ is not Zariski dense for any L, S_L.*

1.8.2. In case $X = \overline{X}$ is already projective, the Lang-Vojta conejcture reduces to Lang's conjecture: X is simply a variety of general type, integral points on X are the same as rational points, and Lang's conjecture asserts that $X(k)$ is not Zariski-dense in X.

1.8.3. The Lang-Vojta conjecture turns out to be a praticular case of a more precise and more refined conjecture of Vojta, which will be discussed in a later lecture.

Lecture 2. Campana's program

For this section one important road sign is

THIS SITE IS UNDER CONSTRUCTION DANGER! HEAVY EQUIPMENT CROSSING

A quick search on the web shows close to the top a number of web sites deriding the idea of "site under construction". Evidently these people have never engaged in research!

2.0.4. Campana's program is a new method of breaking algebraic varieties into "pieces" which builds upon Iitaka's program, but, by using a particular structure on varieties which I will call "Campana constellations" enables one to get closer to a classification which is compatible with arithmetic properties. There is in fact an underlying more refined structure which I call "firmament" for the Campana constellation, which might be the more fundamental structure to study. It truly does say something about rational points.

I am not entirely satisfied with my definition of firmaments, as the definition is quite technical and the geometry behind the structure is not easy to describe. I will speculate about a stack theoretic approach, following ideas due to martin Olsson, at the end of this lecture.

2.0.5. The term "constellation" is inspired by Aluffi's celestial [5], which is in turn inspired by Hironaka.

Campana used the term "orbifold", in analogy to orbifolds used in geometry, but the analogy breaks very early on. A suggested replacement "orbifold pair" still does not make me too happy. Also, "Campana pair" is a term which Campana himself is not comfortable using, nor could he shorten it to just "pair", which is insufficient. I was told by Campana that he would be happy to use "constellations" if the term catches.

2.1. **One dimensional Campana constellations.**

2.1.1. *The two key examples: elliptic surfaces.* Let us inspect again Caporaso's table of surfaces, and concentrate on $\kappa = 1$. We have in 1.7.2 and 1.7.3 two examples, say $S_1 \to \mathbb{P}^1$ and $S_2 \to \mathbb{P}^1$ of elliptic surfaces of Kodaira dimension 1 fibered over $\mathbb{P}^1$. But their arithmetic behavior is very different.

Campana asked the question: is there an underlying structure on the base $\mathbb{P}^1$ from which we can deduce this difference of behavior?

The key point is that the example in 1.7.3 has $2g+2$ double fibers lying over a divisor $D \subset \mathbb{P}^1$. This means that the elliptic surface $S_2 \to \mathbb{P}^1$ can be lifted to $S_2 \to \mathcal{P}$, where $\mathcal{P}$ is the orbifold structure $\sqrt{(\mathbb{P}^1, D)}$ on $\mathbb{P}^1$ obtained by taking the square root of D. Following the ideas of Darmon and Granville in [12], one should consider the canonical divisor class $K_{\mathcal{P}}$ of $\mathcal{P}$, viewed as a divisor with rational coefficients on $\mathbb{P}^1$, namely $K_{\mathbb{P}^1} + (1-1/2)D$. In general, when one has an m-fold fiber over a divisor D, one wants to take D with coefficient $(1-1/m)$.

Darmon and Granville prove, using Chevalley-Weil and Faltings, that such an orbifold $\mathcal{P}$ has potentially dense set of integral points if and only if the Kodaira dimension $\kappa(\mathcal{P}) = \kappa(\mathcal{P}, K_{\mathcal{P}}) < 1$. And the image of a rational point on S_2 is an integral point on $\mathcal{P}$. This fully explains our example: since integral points on $\mathcal{P} = \sqrt{(\mathbb{P}^1, D)}$ are not Zariski dense, and since rational points on S_2 map to integral points on $\mathcal{P}$, rational points on S_2 are not dense.

2.1.2. *The multiplicity divisor.* What should we declare the structure to be when we have a fiber that looks like $x^2y^3 = 0$, i.e. has two components of multiplicities 2 and 3? Here Campana departs from the classical orbifold picture: the highest classical orbifold to which the fibration lifts has no new structure under such a fiber, because $\gcd(2,3) = 1$. Campana makes the key observation that a rich and interesting classification theory arises if one instead considers $\min(2,3) = 2$ as the basis of the structure.

Definition 2.1.3 (Campana)**.** Consider a dominant morphism $f : X \to Y$ with X, Y smooth and $\dim Y = 1$. Define a divisor with rational coefficients $\Delta_f = \sum \delta_p p$ on Y as follows: assume the divisor f^*p on X decomposes as $f^*p = \sum m_i C_i$, where C_i are the distinct irreducible components of the fiber taken with reduced structure. Then set

$$\delta_p \ = \ 1 - \frac{1}{m_p}, \qquad \text{where} \qquad m_p \ = \ \min_i m_i.$$

Definition 2.1.4 (Campana)**.** A Campana constellation curve (Y/Δ) is a pair consisting of a curve Y along with a divisor $\Delta = \sum \delta_p p$ with rational coefficients, where each δ_p is of the form $\delta_p = 1 - 1/m_p$ for some integer m_p.

The Campana constellation base of $f : X \to Y$ is the structure pair consisting of Y with the divisor Δ_f defined above, denoted (Y/Δ_f).

The word used by Campana is *orbifold*, but as I have argued, the analogy with orbifolds is shattered in this very definition. At the end of the lecture I speculate about a stack-theoretic approach, but that involves Artin stacks, which again take a departure from orbifolds.

The new terminology "constellation" will become better justified and much more laden with meaning when we consider Y of higher dimension.

Campana's definition deliberately does not distinguish between the structure coming from a fiber of type $x^2 = 0$ and one of type $x^2y^3 = 0$. We will see later a way to resurrect the difference to some extent using the notion of *firmament*, by which Campana's constellations hang.

Definition 2.1.5 (Campana)**.** The Kodaira dimension of a Campana constellation curve (Y/Δ) is defined as the following Iitaka dimension:

$$\kappa\left(\, (Y/\Delta)\,\right) = \kappa(Y, K_Y + \Delta).$$

We say that (Y/Δ) is of general type if it has Kodaira dimension 1. We say that it is *special* if it is not of general type.

Exercise 2.1.6. Classify special Campana constellation curves over $\mathbb{C}$. See [8]

2.1.7. *Models and integral points.* Now to arithmetic. As we learned in Lesson 0.2.11, when dealing with a variety with a structure given by a divisor, we need to speak about *integral points* on an *integral model* of the structure. Thus let $\mathcal{Y}$ be an integral model of Y, proper over $\mathcal{O}_{k,S}$, and denote by $\widetilde{\Delta}$ the closure of Δ. It turns out that there is more than one natural notion to consider - soft and firm. The firm notion will be introduced when higher dimensions are considered.

Definition. A k rational point x on Y, considered as an integral point $\tilde{x}$ of $\mathcal{Y}$, is said to be a *soft S-integral points on* $(\mathcal{Y}/\widetilde{\Delta})$ if for any nonzero prime $\wp \subset \mathcal{O}_{k,S}$ where $\bar{x}$ reduces to some $\bar{z}_\wp \in \tilde{\Delta}_\wp$, we have

$$\mathrm{mult}_\wp(\bar{x} \cap \bar{p}) \geq m_p.$$

A key property of this definition is:

Proposition 2.1.8. *Assume $f : X \to Y$ extends to a good model $\tilde{f} : \mathcal{X} \to \mathcal{Y}$. Then The image of a rational point on X is a soft S-integral point on $(\mathcal{Y}/\widetilde{\Delta}_f)$.*

So rational points on X can be investigated using integral points on Y. This makes the following very much relevant:

Conjecture 2.1.9 (Campana)**.** *If the Campana constellation curve (Y/Δ) is of general type then the set of soft S-integral point on any model $\mathcal{Y}$ is not Zariski dense.*

This conjecture does not seem to follow readily from Faltings's theorem. As we'll see it does follow from the *abc* conjecture, in particular we have the following theorem.

Theorem 2.1.10 (Campana)**.** *If (Y/Δ) is a Campana constellation curve of general type defined over over the function field K of a curve B then for any finite set $S \subset B$, the set of non-constant soft S-integral point on any model $\mathcal{Y} \to B$ is not Zariski dense.*

2.2. **Higher dimensional Campana constellations.** We turn now to the analogous situation of $f : X \to Y$ with higher dimensional Y. Unfortunately, points on Y are no longer divisors. And divisors on Y are not quite sufficient to describe codimension > 1 behavior. Campana resolves this by considering all birational models of Y separately. I prefer to put all this data together using the notion of a b-divisor, inrtoduced by Shokurov [29], based on ideas by Zariski [33]. See also [5].

Definition 2.2.1. A *rank 1 discrete valuation* on the function field $\mathcal{K} = \mathcal{K}(Y)$ is a surjective group homomorphism $\nu : \mathcal{K}^\times \to \mathbb{Z}$ satisfying

$$\nu(x + y) \geq \min(\nu(x), \nu(y))$$

with equality unless $\nu(x) = \nu(y)$. We define $\nu(0) = \infty$.

The *valuation ring of* ν is defined as

$$R_\nu \;=\; \left\{x \in \mathcal{K} \;\middle|\; \nu(x) \geq 0\right\}.$$

Denote by $Y_\nu = \operatorname{Spec} R_\nu$, and its unique close point s_ν.

A rank 1 discrete valuation ν is *divisorial* if there is a birational model Y' of Y and an irreducible divisor $D' \subset Y'$ such that for all $x \in \mathcal{K}(X) = \mathcal{K}(X')$ we have

$$\nu(x) \;=\; \operatorname{mult}_{D'} x.$$

In this case we say ν has divisorial center D' in Y'.

Definition 2.2.2. A b-divisor $\mathbf{\Delta}$ on Y is an expression of the form

$$\mathbf{\Delta} \;=\; \sum_\nu c_\nu \cdot \nu,$$

a possibly infinite sum over divisorial valuations of $\mathcal{K}(Y)$ with rational coefficients, which satisfies the following finiteness condition:

- for each birational model Y' there are only finitely many ν with divisorial center on Y' having $c_\nu \neq 0$.

A b-divisor is of *orbifold type* if for each ν there is a positive integer m_ν such that $c_\nu = 1 - 1/m_\nu$.

Definition 2.2.3. Let Y be a variety, X a reduced scheme, and let $f : X \to Y$ be a morphism, surjective on each irreducible component of X. For each divisorial valuation ν on $\mathcal{K}(Y)$ consider $f' : X'_\nu \to Y_\nu$, where X' is a desingularization of the (main component of the) pullback $X \times_Y Y_\nu$. Write $f^* s_\nu = \sum m_i C_i$. Define

$$\delta_\nu \;=\; 1 - \frac{1}{m_\nu} \quad \text{with} \quad m_\nu = \min_i m_i.$$

The Campana b-divisor on Y associated to a dominant map $f : X \to Y$ is defined to be the b-divisor

$$\mathbf{\Delta}_f = \sum \delta_\nu \nu.$$

Exercise 2.2.4. The definition is independent of the choice of desingularization X'_ν.

This makes the b-divisor $\mathbf{\Delta}_f$ a proper birational invariant of f. In particular we can apply it to a dominant rational map f.

Definition 2.2.5. A *Campana constellation* $(Y/\mathbf{\Delta})$ consists of a variety Y with a b-divisor $\mathbf{\Delta}$ such that, locally in the étale topology on Y, there is $f : X \to Y$ with $\mathbf{\Delta} = \mathbf{\Delta}_f$.

The trivial constellation on Y is given by the zero b-divisor.

For each birational model Y', define the Y'-divisorial part of $\mathbf{\Delta}$:

$$\Delta_{Y'} = \sum_{\nu \text{ with divisorial support on } Y'} \delta_\nu \nu.$$

This feels rather unsatisfactory because it relies, at least locally, on the choice of f. But using the notion of firmament below we will make this structure more combinatorial and less dependent on f.

2.2.6. Here's why I like the word "constellation": think of a valuation ν as a sort of "generalized point" on Y. Putting $\delta_\nu > 0$ suggests viewing a "star" at that point. Replacing Y by higher and higher models Y' is analogous to using stronger and stronger telescopes to view farther stars deeper into space. The picture I have in my mind is somewhat reminiscent of the astrological meaning of "constellation", not as just one group of stars, but rather as the arrangement of the entire heavens at the time the "baby" $X \to Y$ is born. But hopefully it is better grounded in reality.

We now consider morphisms. For constellations we consider only dominant morphisms.

Definition 2.2.7. (1) Let $(X/\mathbf{\Delta}_X)$ be a Campana constellation, and $f : X \to Y$ a dominant morphism. The constellation base $(Y, \mathbf{\Delta}_{f,\mathbf{\Delta}_X})$ is defined as follows: for each divisorial valuation ν of Y and each divisorial valuation μ of X with center D dominating the center E of ν, let

$$m_{\mu/\nu} = m_\mu \cdot \mathrm{mult}_D(f^*E).$$

Define

$$m_\nu = \min_{\mu/\nu} m_{\mu/\nu} \quad \text{and} \quad \delta_\nu = 1 - \frac{1}{m_\nu}.$$

Then set as before

$$\mathbf{\Delta}_{f,\mathbf{\Delta}_X} = \sum_\nu \delta_\nu \nu.$$

(2) Let $(X/\mathbf{\Delta}_X)$ and $(Y/\mathbf{\Delta}_Y)$ be Campana constellations and $f : X \to Y$ a dominant morphism. Then f is said to be a *constellation morphism* if $\mathbf{\Delta}_Y \le \mathbf{\Delta}_{f,\mathbf{\Delta}_X}$, in other words, if for every divisorial valuation ν on Y and any μ/ν we have $m_\nu \le m_{\mu/\nu}$.

Definition 2.2.8. A rational m-canonical differential ω on Y is said to be regular on $(Y/\mathbf{\Delta})$ if for every divisorial valuation ν on $\mathcal{K}(Y)$, the polar multiplicity satisfies

$$(\omega)_{\infty,\nu} \le m\delta_\nu.$$

In other words, it is a section of $\mathcal{O}_{Y'}(m(K_{Y'} + \Delta_{Y'}))$ on every birational model Y'.

The Kodaira dimension $\kappa(\,(Y/\mathbf{\Delta})\,)$ is defined using regular m-canonical differentials on $(Y/\mathbf{\Delta})$.

Exercise 2.2.9. This is a birational invariant.

Theorem 2.2.10 (Campana)**.** *There is a birational model Y' such that*

$$\kappa(\,(Y/\mathbf{\Delta})\,) = \kappa(Y', K'_Y + \Delta_{Y'}).$$

This is proven using Bogomolov sheaves, an important notion which is a bit far afield for the present discussion.

Definition 2.2.11. A Campana constellation $(Y/\mathbf{\Delta})$ is said to be *of general type* if $\kappa(\,(Y/\mathbf{\Delta})\,) = \dim Y$.

A Campana constellation $(X/\mathbf{\Delta})$ is said to be *special* if there is no dominant morphism $(X/\mathbf{\Delta}) \to (Y/\mathbf{\Delta}')$ where $(Y/\mathbf{\Delta}')$ is of general type.

Definition 2.2.12.
(1) A morphism $f : (X/\mathbf{\Delta}_X) \to (Y/\mathbf{\Delta}_Y)$ of Campana constellation is *special*, if its generic fiber is special.
(2) Given a Campana constellation $(X/\mathbf{\Delta}_X)$, a morphism $f : X \to Y$ is said to have *general type base* if $(Y/\mathbf{\Delta}_{f,\mathbf{\Delta}_X})$ is of general type.
(2') In particular, considering X with trivial constellation, a morphism $f : X \to Y$ is said to have *general type base* if $(Y/\mathbf{\Delta}_f)$ is of general type.

Here is the main classification theorem of Campana:

Theorem 2.2.13 (Campana)**.** *Let $(X/\mathbf{\Delta}_X)$ be a Campana constellation. There exists a dominant rational map $c : X \dashrightarrow C(X)$, unique up to birational equivalence, such that*

(1) it has special general fibers, and
(2) it has Campana constellation base of general type.

This map is final for (1) and initial for (2).

This is the Campana core map of $(X/\mathbf{\Delta}_X)$, the constellation $(C(X)/\mathbf{\Delta}_{c,\mathbf{\Delta}_X})$ being the core of $(X/\mathbf{\Delta}_X)$. The key case is when X has the trivial constellation, and then $c : X \dashrightarrow (C(X)/\mathbf{\Delta}_c)$ is the Campana core map of X and $(C(X)/\mathbf{\Delta}_c)$ the core of X.

2.2.14. *Examples of constellation bases.*

Exercise 2.2.15. Describe the constellation:

(1) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^2$
(2) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^2y$
(3) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^2y^2$
(4) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^2y^3$
(5) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^3y^4$
(6) $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $s = x^2; t = y$
(7) $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $s = x^2; t = y^2$
(8) $f : \mathbb{A}^2 \sqcup \mathbb{A}^2 \to \mathbb{A}^2$ given by $s = x_1^2; t = y_1$ and $s = x_2; t = y_2^2$
(9) $f : X \to \mathbb{A}^2$ given by $\operatorname{Spec} \mathbb{C}[s, t, \sqrt{st}]$.
(10) $f : \mathbb{A}^3 \to \mathbb{A}^2$ given by $s = x^2y^3; t = z$

(more details)

2.2.16. *Rational points and the question of integral points.* Campana made the following bold conjecture:

Conjecture 2.2.17 (Campana)**.** *Let X/k be a variety over a number field. Then rational points are potentially dense on X if and only if X is special, i.e. if and only if the core of X is a point.*

It is natural seek a good definition of integral points on a Campana constellation and translate the non-special case of the conjecture above to a conjecture on integral points on Campana constellations of general type. This may be possible, but I believe a more natural framework is that of firmaments, where the definition of integral points is natural.

2.3. **Firmaments supporting constellations and integral points.** It seems that Campana constellations are wonderfully suited for purposes of birational classification. Still they seem to lack some subtle information necessary for good definitions of structures such as non-dominant morphisms and integral points - at least I have not been successful in doing this directly on constellations. For these purposes I propose the notion of firmaments. It is very much possible that at the end a simpler formalism will be discovered, and the whole notion of firmaments will be redundant.

The right foundation to use for my proposed firmaments is that of logarithmic structures. However the book [6] on logarithmic structure has not been written. Therefore I will use toroidal embeddings instead. The examples above, which are all toric, show that the toric cases are easy to figure out, and the idea is to reduce all cases to toric situations. The speculations in the end of the lecture involve Olsson's toric stacks, so toric geometry seems to be a useful formalism.

Definition 2.3.1 ([19], [18], [3])**.**

(1) A toroidal embedding $U \subset X$ is the data of a variety X and a dense open set U with complement a Weil divisor $D = X \smallsetminus U$, such that locally in the étale, or analytic, topology, or formally, near every point, $U \subset X$ admits an isomorphism with (a neighborhood of a point in) $T \subset V$, with T a torus and V a toric variety. (It is sometimes convenient to refer to the toroidal structure using the divisor: (X, D).)

(2) Let $U_X \subset X$ and $U_Y \subset Y$ be toroidal embeddings, then a dominant morphism $f : X \to Y$ is said to be toroidal if étale locally near every point of X there is a toric chart for X near x and for Y near $f(x)$ which is a torus-equivariant morphism of toric varieties.

2.3.2. *The cone complex.* Recall that, to a toroidal embedding $U \subset X$ we can attach an integral polyhedral cone complex Σ_X, consisting of strictly convex cones, attached to each other along faces, and in each cone σ a finitely generated, unit free integral saturated monoid $N_\sigma \subset \sigma$ generating σ as a real cone. In [19], [18] the monoid M_σ dual to N_σ is used. While the use of M_σ is natural from the point of view of logarithmic structures, all the action with firmaments happens on N_σ, so I use it instead.

2.3.3. *Valuation rings and the cone complex.* The complex Σ_X can be pieced together using the toric charts, where the picture is well known: for a toric variety V, cones correspont to toric affine opens V_σ, and the lattice N_σ is the monoid of one-parameter subgroups having a limit point in V_σ; it is dual to the lattice of effective toric Cartier divisors M_σ, which is the quotient of the lattice of regular monomials $\tilde{M}_\sigma$ by the unit monomials.

For our purposes it is convenient to recall the characterization of toric cones using valuations given in [19]: let R be a discrete valuation ring with valuation ν, special point s_R and generic point η_R; let $\phi : \operatorname{Spec} R \to X$ be a morphism such that $\phi(\eta_R) \subset U$ and $\phi(s_R)$ lying in a stratum having chart $V = \operatorname{Spec} k[\tilde{M}_\sigma]$. One associates to ϕ the point n_ϕ in N_σ given by the rule:

$$n(m) = \nu(\phi^* m) \quad \forall m \in M.$$

In case $R = R_\nu$ is a valuation ring of Y, I'll call this point n_ν. One can indeed give a coherent picture including the case $\phi(\eta_R) \not\subset U$, but I won't discuss this here. (It is however important for a complete picture of the category and of the arithmetic structure.).

2.3.4. *Functoriality.* Given toridal embeddings $U_X \subset X$ and $U_Y \subset Y$ and a morphism $f : X \to Y$ carrying U_X into U_Y (but not necessarily toroidal) the description above functorially associates a polyhedral morphism $f_\Sigma : \Sigma_X \to \Sigma_Y$ which is integral, that is, $f_\Sigma(N_\sigma) \subset N_\tau$ whenever $f_\Sigma(\sigma) \subset \tau$.

2.3.5. *Toroidalizing a morphism.* While most morphisms are not toroidal, we have the following:

Theorem (Abramovich-Karu)**.** *Let $f : X \to Y$ be a dominant morphism of varieties. Then there exist modifications $X' \to X$ and $Y' \to Y$ and toroidal structures $U_{X'} \subset X'$, $U_{Y'} \subset Y'$ such that the resulting rational map $f' : X' \to Y'$ is a toroidal morphism:*

$$\begin{array}{ccccc} U_{X'} & \hookrightarrow & X' & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ U_{Y'} & \hookrightarrow & Y' & \longrightarrow & Y \end{array}$$

Furthermore, f' can be chosen flat.

We now define firmaments:

Definition 2.3.6. A toroidal *firmament* on a toroidal embedding $U \subset X$ with complex Σ is a finite collection $\mathbf{\Gamma} = \{\Gamma^i_\sigma \subset N_\sigma\}$, where

- each $\Gamma^i_\sigma \subset N_\sigma$ is a finitely generate submonoid, not-necessarily saturated.

- each Γ^i_σ generates the corresponding σ as a cone,
- the collection is closed under restrictions to faces $\tau \prec \sigma$, i.e. $\Gamma^i_\sigma \cap \tau = \Gamma^j_\tau$ for some j, and
- it is irredundant, in the sense that $\Gamma^i_\sigma \not\subset \Gamma^j_\sigma$ for different i, j.

A morphism from a torodial firmament $\mathbf{\Gamma}_X$ on a toroidal embedding $U_X \subset X$ to $\mathbf{\Gamma}_Y$ on $U_Y \subset Y$ is a morphism $f : X \to Y$ with $f(U_X) \subset U_Y$ such that for each σ and i, we have $f_\Sigma(\Gamma^i_\sigma) \subset \Gamma^j_\tau$ for some j.

We say that the firmament $\mathbf{\Gamma}_X$ is *induced* by $f : X \to Y$ from $\mathbf{\Gamma}_Y$ if for each $\sigma \in \Sigma_X$ such that $f_\Sigma(\sigma) \subset \tau$, we have $\Gamma^i_\sigma = f_\Sigma^{-1}\Gamma^i_\tau \cap N_\sigma$.

Given a proper birational equivalence $\phi : X_1 \dashrightarrow X_2$, then two toroidal firmaments $\mathbf{\Gamma}_{X_1}$ and $\mathbf{\Gamma}_{X_2}$ are said to be *equivalent* if there is a toroidal X_3, and a commutative diagram

$$\begin{array}{ccc} & X_3 & \\ {\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_2} \\ X_1 & \overset{\phi}{\dashrightarrow} & X_2, \end{array}$$

where f_i are modifications, such that the two firmaments on X_3 induced by f_i from $\mathbf{\Gamma}_{X_i}$ are identical.

A firmament on an arbitrary X is an equivalence class represented by a modification $X' \to X$ with a toroidal embedding $U' \subset X'$ and a toroidal firmament $\mathbf{\Gamma}$ on $\Sigma_{X'}$.

The trivial firmament is defined by $\Gamma_\sigma = N_\sigma$ for all σ in Σ.

For the discussion below one can in fact replace $\mathbf{\Gamma}$ by the union of the Γ^i_σ, but I am not convinced that makes things better.

Definition 2.3.7. (1) Let $f : X \to Y$ be a flat toroidal morphism of toroidal embeddings. The *base firmament* $\mathbf{\Gamma}_f$ associated to $X \to Y$ is defined by the images $\Gamma^\tau_\sigma = f_\Sigma(N_\tau)$ for each cone $\tau \in \Sigma_X$ over $\sigma \in \Sigma_Y$.

(2) Let $f : X \to Y$ be a dominant morphism of varieties. The base firmament of f is represented by any $\mathbf{\Gamma}_{f'}$, where $f' : X' \to Y'$ is a flat toroidal birational model of f.

(3) If X is reducible, decomposed as $X = \cup X_i$, but $f : X_i \to Y$ is dominant for all i, we define the base firmament by the (maximal elements of) the union of all the firmaments associated to $X_i \to Y$.

Definition 2.3.8. Let $\mathbf{\Gamma}$ be a firmament on Y. Define the Campana constellation $(Y/\mathbf{\Delta})$ hanging from $\mathbf{\Gamma}$ (or supported by $\mathbf{\Gamma}$) as follows: say $\mathbf{\Gamma}$ is a toroidal firmament on some birational model Y'. Let ν be a divisorial valuation. We have associated to it a point $n_\nu \in \sigma$ for the cone σ associated to the stratum in which s_ν lies. Define

$$m_\nu \quad = \quad \min\{k \mid k \cdot n_\nu \ \in \ \Gamma^i_\sigma \ \text{ for some } \ i\}.$$

2.3.9. Note that, according to the definition above, every firmament supports a unique constellation, though a constellation can be supported by more than one firmament. Depending on one's background, this might agree or disagree with the primitive cosmology of one's culture. Think of it this way: as we said before, the word "constellation" refers to the entire "heavens", visible through stronger and stronger telescopes Y'. The word "firmament" refers to an overarching solid structure supporting the heavens, but solid as it may be, it is entirely imaginary and certainly not unique.

An absolutely important result is:

Proposition 2.3.10. *The formation of constellation hangign by* $\mathbf{\Gamma}$ *is independent of the choice of representative in the equivalence class* $\mathbf{\Gamma}$*, and is a constellation, i.e. always induced, locally in the étale topology, from a morphism* $X \to Y$*.*

Also, the Campana constellation supported by the base firmament of a dominant morphism $X \to Y$ *is the same as the base constellation associated to* $X \to Y$*.*

2.3.11. *Examples.*

(1) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^2$: $\tau = \mathbb{R}_{\geq 0}; N_\tau = \mathbb{N}; \mathbf{\Gamma} = \{2\mathbb{N}\}$. Supported constellation: $\Delta = D_0/2$
(2) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^2 y$: $\mathbf{\Gamma} = \{\mathbb{N}\}$, the trivial structure. Supported constellation: $\Delta = 0$
(3) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^2 y^2$: $\mathbf{\Gamma} = \{2\mathbb{N}\}$. Supported constellation: $\Delta = D_0/2$
(4) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^2 y^3$: $\mathbf{\Gamma} = \{2\mathbb{N} + 3\mathbb{N}\}$. Supported constellation: $\Delta = D_0/2$. Note: same constellation hanging by different firmaments.
(5) $f : \mathbb{A}^2 \to \mathbb{A}^1$ given by $t = x^3 y^4$: $\mathbf{\Gamma} = \{3\mathbb{N} + 4\mathbb{N}\}$. Note: not saturated in its associated group. Supported constellation: $\Delta = 2D_0/3$.
(6) $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $s = x^2; t = y$: $\mathbf{\Gamma} = \{2\mathbb{N} \times \mathbb{N}\}$. For constellation: coefficient of y axis in Δ_Y is $1/2$. In the blowup of Y at origin, the coefficient of exceptional is again $1/2$, but blowing up the intersection one gets a coefficient of 0 on the second exceptional.
(7) $f : \mathbb{A}^2 \to \mathbb{A}^2$ given by $s = x^2; t = y^2$: $\mathbf{\Gamma} = \{2\mathbb{N} \times 2\mathbb{N}\}$. For constellation: $\Delta_Y = 1/2(D_x + D_y)$, the coefficient of exceptional on blowup is again $1/2$.
(8) $f : \mathbb{A}^2 \sqcup \mathbb{A}^2 \to \mathbb{A}^2$ given by $s = x_1^2; t = y_1$ and $s = x_2; t = y_2^2$: $\mathbf{\Gamma} = \{2\mathbb{N} \times \mathbb{N}, \mathbb{N} \times 2\mathbb{N}\}$. Note: more than one semigroup. $\Delta_Y = 0$, but on blowup the exceptional gets $1/2$.
(9) $f : X \to \mathbb{A}^2$ given by $\operatorname{Spec} \mathbb{C}[s, t, \sqrt{st}]$: $\mathbf{\Gamma} = \{\langle (2,0), (1,1), (0,2) \rangle\}$. Now $\Delta_Y = 1/2(D_x + D_y)$, but the exceptional on the blowup gets 0.
(10) $f : \mathbb{A}^3 \to \mathbb{A}^2$ given by $s = x^2 y^3; t = z$: $\mathbf{\Gamma} = \{(2\mathbb{N} + 3\mathbb{N}) \times \mathbb{N}\}$. The constellation is pretty interesting!

2.3.12. *Arithmetic.* We have learned our lesson - for arithmetic we need to talk about integral points on integral models. I'll restrict to the toroidal case, since that's what I understand.

Definition. An S-integral model of a toroidal firmament $\mathbf{\Gamma}$ on Y consists of an integral toroidal model $\mathcal{Y}'$ of Y'.

Definition 2.3.13. Consider a toroidal firmament $\mathbf{\Gamma}$ on Y/k, and a rational point y such that the firmament is trivial in a neighborhood of y. Let $\mathcal{Y}$ be a toroidal S-integral model.

Then y is a *firm integral point of* $\mathcal{Y}$ with respect to $\mathbf{\Gamma}$ if the section $\operatorname{Spec}\mathcal{O}_{k,S} \to \mathcal{Y}$ is a morphism of firmaments, when $\operatorname{Spec}\mathcal{O}_{k,S}$ is endowed with the trivial firmament.

Explicitly, at each prime $\wp \in \operatorname{Spec}\mathcal{O}_{k,S}$ where y reduces to a stratum with cone σ, consider the associated point $n_{y_\wp} \in N_\sigma$. Then y is firmly S-integral if for every $\wp$ we have $n_{y_\wp} \in \Gamma_\sigma^i$ for some i.

Theorem 2.3.14. *Let $f : X \to Y$ be a proper dominant morphism of varieties over k. There exists a toroidal birational model $X' \to Y'$ and an integral model $\mathcal{Y}'$ such that image of a rational point on X' is a firm S-integral point on $\mathcal{Y}'$ with respect to $\mathbf{\Gamma}_f$.*

In fact, at least after throwing a few small primes into the trash-bin S, a point is S integral on $\mathcal{Y}'$ with respect to $\mathbf{\Gamma}_f$ if and only if locally in the étale topology on $\mathcal{Y}'$ it lifts to a rational point on X. This is the motivation of the definition.

Conjecture 2.3.15 (Campana)**.** *Let $(Y/\mathbf{\Delta})$ be a smooth projective Campana constellation supported by firmament $\mathbf{\Gamma}$. Then points on Y integral with respect to Γ are potentially dense if and only if $(Y/\mathbf{\Delta})$ is special.*

Corollary 2.3.16 (Campana)**.** *Assume the conjecture holds true. Let X be a smooth projective variety. Then rational points are potentially dense if and only if X is special.*

2.4. **Speculations about a toric stack approach.** Recall that a constellation, and a firmament supporting it, is a structure on a variety Y induced at least locally from a dominant morphism $X \to Y$. On the level of firmaments, the structure is such that maps to $(Y, \mathbf{\Gamma})$ are maps to Y which étale locally admit a lifting to such X. In case $X \to Y$ has toroidal structure, there is essentially such a structure defined by Olsson [27], at least under some assumption. Say for simplicity X and Y are toric, with tori T_X and T_Y. The map induces a homomorphism $T_X \to T_Y$, with kernel T_f. Consider the Artin stack $[X/T_f]$. It admits a morphism to Y, which restricts to an isomorphism over T_Y, but along the boundary we get a non-separated Artin stack, which gives an exotic structure over $Y \smallsetminus T_Y$.

There is no question about the meaning of maps and points and such on $[X/T_f]$. The subtle business is to glue such things together when $X \to Y$

is only toroidal, and to set up the correct framework under which things go through. Further, there is a question of when two such things should be considered equivalent for our purposes, which is yet to be understood.

Lecture 3. The minimal model program

For the "quick and easy" introduction see [13]. For a more detailed treatment starting from surfaces see [26]. For a full treatment up to 1999 see [21]

3.1. **Cone of curves.**

3.1.1. *Groups of divisors and curves modulo numerical equivalence.* Let X be a smooth complex projective variety.

We denote by $N^1(X)$ the image of $\mathbf{Pic}(X) \to H^2(X,\mathbb{Z})/\text{torsion} \subset H^2(X,\mathbb{Q})$. This is the group of Cartier divisors modulo numerical equivalence.

We denote by $N_1(X)$ the subgroup of $H_2(X,\mathbb{Q})$ generated by the fundemental classes of curves. This is the group of algebraic 1-cycles modulo numerical equivalence.

The intersection pairing restricts to $N^1(X) \times N_1(X) \to \mathbb{Z}$, which over $\mathbb{Q}$ is a perfect pairing.

3.1.2. *Cones of divisors and of curves.* Denote by $Amp(X) \subset N^1(X)_{\mathbb{Q}}$ the cone generated by classes of ample divisors. We denote by $NEF(X)$ the closure of $Amp(X) \subset N^1(X)_{\mathbb{R}}$, called the nef cone of X.

Denote by $NE(X) \subset N_1(X)_{\mathbb{Q}}$ the cone generated by classes of curves. We denote its closure by $\overline{NE}(X)$.

Theorem 3.1.3 (Kleiman)**.** *The class $[D]$ of a Cartier divisor is in the closure $NEF(X)$ if and only if $[D]\cdot[C] \geq 0$ for every algebraic curve $C \subset X$.*

In other words, the cones $\overline{NE}(X)$ and $NEF(X)$ are dual to each other.

3.2. **Bend and break.**

For any divisor D on X which is not numerically equivalent to 0, the subset

$$(D \leq 0) := \{v \in NE(X) | v \cdot D \leq 0\}$$

is a half-space. The minimal model program starts with the observation that this set is especially important when $D = K_X$. In fact, in the case of surfaces, $(K_X \leq 0) \cap NE(X)$ is a subcone generated by (-1)-curves, which suggests that it must say something in higher dimensions. Indeed, as it turns out, it is in general a nice cone generated by so called "extremal rays", represented by rational curves $[C]$ which can be contracted in something like a (-1) contraction.

Suppose again X is a smooth, projective variety with K_X not nef. Our first goal is to show that there is *some* rational curve C with $K_X \cdot C < 0$.

The idea is to take an arbitrary curve on X, and to show, using deformation theory, that it has to "move around alot" - it has so many deformations that eventually it has to break, unless it is already the rational curve we were looking for.

3.2.1. *Breaking curves.* The key to showing that a curve breaks is the following:

Lemma 3.2.2. *Suppose C is a projective curve of genus> 0 with a point $p \in C$, suppose B is a one dimensional affine curve, $f : C \times B \to X$ a nonconstant morphism such that $\{p\} \times B \to X$ is constant. Then, in the closure of $f(C \times B) \subset X$, there is a rational curve passing through $f(p)$.*

In genus 0 a little more will be needed:

Lemma 3.2.3. *Suppose C is a projective curve of genus 0 with points $p_1, p_2 \in C$, suppose B is a one dimensional affine curve, $f : C \times B \to X$ a morphism such that $\{p_i\} \times B \to X$ is constant, $i = 1, 2$, and the image is two-dimensional. Then $[f(C)]$ is "reducible": there are effective curves C_1, C_2 passing through p_1, p_2 respectively, such that $[C_1] + [C_2] = [C]$.*

3.2.4. *Some deformation theory.* We need to understand deformations of a map $f : C \to X$ fixing a point or two. The key is that the tangent space of the moduli space of such maps - the deformation space - can be computed cohomologically, and the number of equations of the deformation space is also bounded cohomologically.

Lemma 3.2.5. *The tangent space of the deformation space of $f : C \to X$ fixing points $p_1, \ldots, p_n$ is*

$$H^0\left(C, f^*T_X(-\sum p_i)\right).$$

The obstructions lie in the next cohomology group:

$$H^1\left(C, f^*T_X(-\sum p_i)\right).$$

The dimension of the deformation space is bounded below:

$$\begin{aligned} \dim Def(f : C \to X, p_1, \ldots, p_n) &\geq \chi\left(C, f^*T_X(-\sum p_i)\right) \\ &= -(K_X \cdot C) \ + \ (1 - g(C) - n) \dim X \end{aligned}$$

3.2.6. *Rational curves.* Let us consider the case where C is rational. Suppose we have such a rational curve inside X with $-(K_X \cdot C) \geq \dim X + 2$, and we consider deformations fixing $n = 2$ of its points. Then $-(K_X \cdot C) \ + \ (1 - g(C) - 2) \dim X = -(K_X \cdot C) - \dim X \geq 2$. Since C is inside X, the only ways $f : C \to X$ can deform is either by the 1-parameter group of automorphisms, or, beyond one-parameter, go outside the image of C, and we get an image of dimension at least 2. So the rational curve must break, and one of the resulting components C_1 is a curve with $-(K_X \cdot C_1) \leq -(K_X \cdot C)$.

Suppose for a moment $-K_X$ is ample. In this case the process can only stop once we have a curve C_∞ with

$$-(K_X \cdot C_\infty) \quad \leq \quad \dim X + 1.$$

Note that this is optimal - the canonical line bundle on $\mathbb{P}^r$ has degree $r+1$ on any line.

3.2.7. *Higher genus.* If X is any projective variety with K_X not nef, then there is some curve C with $K_X \cdot C < 0$. To be able to break C we need $-(K_X \cdot C) - g(C) \dim X \geq 1$.

There is apparently a problem: the genus term may offset the positivity of $-(K_X \cdot C)$. One might think of replacing C by a curve covering C, but there is a problem: the genus increses in coverings roughly by a factor of the degree of the cover, and this offsets the increase in $-(K_X \cdot C)$. There is one case when this does not happen, that is in characteristic p we can take the iterated Frobenius morphism $C^{[m]} \to C$, and the genus of $C^{[m]}$ is $g(C)$. We can apply our bound and deduce that there is a rational curve C' on X. If K_X is ample we also have $0 < -(K_X \cdot C) \quad \leq \quad \dim X + 1$.

But our variety X was a complex projective variety. What do we do now? We can find a smooth model $\mathcal{X}$ of X over some ring R finitely generated over $\mathbb{Z}$, and for each maximal ideal $\wp \subset R$ the fiber $\mathcal{X}_\wp$ has a rational curve on it.

How do we deduce that there is a rational curve on the original X? if $-K_X$ is ample, the same is true for $-K_{\mathcal{X}}$, and we deduce that there is a rational curve $C_\wp$ on each $\mathcal{X}_\wp$ such that $0 < -(K_{\mathcal{X}_\wp} \cdot C_\wp) \quad \leq \quad \dim X + 1$. These are parametrized by a Hilbert scheme of finite type over R, and therefore this Hilbert scheme has a point over $\mathbb{C}$, namely there is a rational curve C on X with $0 < -(K_X \cdot C) \quad \leq \quad \dim X + 1$.

In case $-K_X$ is not ample, a more delicate argument is necessary. One fixes an ample line bundle H on $\mathcal{X}$, and given a curve C on X with $-K_X \cdot C < 0$ one shows that there is a rational curve C' on each $\mathcal{X}_\wp$ with

$$H \cdot C' \leq 2 \dim X \frac{H \cdot C}{-K_X \cdot C}.$$

Then one continues with a similar Hilbert scheme argument.

3.3. **Cone theorem.**

Using some additional delicate arguments one proves:

Theorem 3.3.1 (Cone theorem)**.** *Let X be a smooth projective variety. There is a countable collection C_i of rational curves on X with*

$$0 < -K_X \cdot C_i \leq \dim X + 1,$$

whose classes $[C_i]$ are discrete in the half space $N_1(X)_{K_X<0}$, such that

$$\overline{NE}(X) \quad = \overline{NE}(X)_{K_X \geq 0} + \sum_i \mathbb{R}_{\geq 0} \cdot [C_i].$$

The rays $\mathbb{R}_{\geq 0} \cdot [C_i]$ are called extremal rays (or, more precisely, extremal K_X-negative rays) of X.

These extremal rays have a crucial property:

Theorem 3.3.2 (Contraction theorem)**.** *Let X be a smooth complex projective variety and let $R = \mathbb{R}_{\geq 0} \cdot [C]$ be an extremal K_X-negative ray. Then there is a normal projective variety Z and a surjective morphism $c_R : X \to Z$ with connected fibers, unique up to unique isomorphism, such that for an irreducible curve $D \subset X$ we have $c_R(D)$ is a point if and only if $[D] \in R$.*

This map c_R is defined using a base-point-free linear system on X made out of a combination of an ample sheaf H and K_X.

3.4. **The minimal model program.** If X has an extremal ray which gives a contraction to a lower dimensional variety Z, then the fibers of c_R are rationally connected and we did learn something important about the structure of X: it is uniruled.

Otherwise $c_R : X \to Z$ is birational, but at least we have gotten rid of one extremal ray - one piece of obstruction for K_X to be nef.

One is tempted to apply the contraction theorem repeatedly, replacing X by Z, until we get to a variety with K_X nef. There is a problem: the variety Z is often singular, and the theorems apply to smooth varieties. All we can say about Z is that it has somewhat mild singularities: in general it has rational singularites; if the exceptional locus has codimension 1 - the case of a so called *divisorial* contraction - the variety Z has so called *terminal* singularites. For surfaces, terminal singularities are in fact smooth, and in fact contractions of extremal rays are just (-1)-contraction, and we eventually are led to a minimal model. But in higher dimensions singularities do occur.

The good news is that the theorems can be extended, in roughly the same form, to varieties with terminal singularities. (The methods are very different from what we have seen and I would rather not go into them.) So as long as we only ned to deal with divisorial contractions, we can continue as in the surface case.

for non-divisorial contractions - so called small contractions - we have the following:

Conjecture 3.4.1 (Flip Conjecture)**.**

(1) *(Existence) Supposed $c_R : X \to Z$ is a small extremal contraction on a variety X with terminal singularities. Then there exists another small contraction $c_R^+ : X^+ \to Z$ such that X^+ has terminal singularities and $K_{X^+} \cdot C > 0$ for any curve C contracted by c_R^+.*

   The transformation $X \dashrightarrow X^+$ is known as a flip.

(2) *(termination) Any sequence of flips is finite.*

This implies the following:

Conjecture 3.4.2 (Minimal model conjecture)**.** *Let X be a smooth projective variety. Then either X is uniruled, or there is a birational modification $X \dashrightarrow X'$ such that X' has only terminal singularities and $K_{X'}$ is nef*

Often one combines this with the following:

Conjecture 3.4.3 (Abundance)**.** *Let X be a projective variety with terminal singularities and K_X nef. Then for some integer $m > 0$, we have $H^0(X, \mathcal{O}_X(mK_X))$ is base-point-free.*

The two together are sometines named "the good minimal model conjecture".

Lecture 4. Vojta, Campana and *abc*

In [31], Paul Vojta started a speculative investigation in diophantine geometry motivated by analogy with value distribution theory. His conjectures go in the same direction as Lang's - they are concerned with bounding the set of points on a variety rather than constructing many of them. Many of the actual proofs in the subject, such as an alternative proof of Faltings's theorem, use razor-sharp tools such as Arakelov geometry. But to describe the relevant conjectures it will suffice to discuss heights from the classical "naïve" point of view. The reader is encouraged to consult Hindry–Silverman [16] for a user–friendly, Arakelov–Free treatment of the theory of heights (including a proof of Faltings's theorem, following Bombieri).

A crucial feature of Vojta's conjectures is that they are not concerned with rational points, but with algebraic points of bounded degree. To account for varying fields of definition, Vojta's conjecture always has the discriminant of the field of definition of a point P accounted for.

Vojta's conjectures are thus much farther-reaching than Lang's. You might say, much more outrageous. On the other hand, working with all extensions of a bounded degree allows for enormous flexibility in using geometric constructions in the investigation of algebraic points. So, even if one is worried about the validity of the conjectures, they serve as a wonderful testing ground for our arithmetic intuition.

4.1. **Heights and related invariants.**

Consider a point in projective space $P = (x_0 : \ldots : x_r) \in \mathbb{P}^r$, defined over some number field k, with set of places $\mathbb{M}_k$. Define the naïve height of P to be

$$H(P) = \prod_{v \in \mathbb{M}_k} \max(\|x_0\|_v, \ldots, \|x_r\|_v).$$

Here $\|x\|_v = |x|$ for a real v, $\|x\|_v = |x|^2$ for a complex v, and $\|x\|_v$ is normalized so that $\|p\| = p^{-[k_v:\mathbb{Q}_p]}$ otherwise. (If the coordinates can be chosen relatively prime algebraic integers, then the product is of course a finite product over the archimedean places where everything is as easy as can be expected.)

This height is independent of the homogeneous coordinates chosen, by the product formula.

To keep things independent of a chosen field of definition, and to replace products by sums, one defines the normalized logarithmic height

$$h(P) = \frac{1}{k : \mathbb{Q}} \log H(P).$$

Now if X is a variety over k with a very ample line bundle L, one can consider the embedding of X in a suitable $\mathbb{P}^r$ via the complete linear system of $H^0(X, L)$. We define the height $h_L(P)$ to be the height of the image point in $\mathbb{P}^r$.

This definition of $h_L(P)$ is not valid for embeddings by incomplete linear systems, and is not additive in L. But it does satisfy these desired properties "almost": $h_L(P) = h(P)+O(1)$ if we embed by an incomplete linear system, and $h_{L\otimes L'}(P) = h_L(P) + h_{L'}(P)$ for very ample L, L'. This allows us to define

$$h_L(P) = h_A(P) - h_B(P)$$

with A and B are very ample and $L \otimes B = A$. The function $h_L(P)$ is now only well defined as a function on $X(\bar{k})$ up to $O(1)$.

Consider a finite set of places S containing all archimedean places.

Let now $\mathcal{X}$ be a scheme proper over $\mathcal{O}_{k,S}$, and D a Cartier divisor.

The counting function of $\mathcal{X}, D$ reative to k, S is a function on points of $X(\bar{k})$ not lying on D. Suppose $P \in X(E)$, which we view again as an S-integral point of $\mathcal{X}$. Consider a place w of E not lying over S, with residue field $\kappa(w)$. Then the restriction of D to $P \simeq \operatorname{Spec} \mathcal{O}_{E,S}$ is a fractional ideal with some multiplicity n_w at w. We define the **counting function** as follows:

$$N_{k,S}(D, P) = \frac{1}{[E : k]} \sum_{\substack{w \in \mathbb{M}_E \\ w \nmid S}} n_w \log |\kappa(w)|.$$

A variant of this is the **truncated counting function**

$$N^{(1)}_{k,S}(D, P) = \frac{1}{[E : k]} \sum_{\substack{w \in \mathbb{M}_E \\ w \nmid S}} \min(1, n_w) \log |\kappa(w)|.$$

Counting functions and truncated counting functions depend on the choice of S and a model $\mathcal{X}$, but only up to $O(1)$. We'll thus suppress the subscript S.

One defines the **relative logarithmic discriminant** of E/k as follows: suppose the discriminant of a number field k is denoted D_k. Then define

$$d_k(E) = \frac{1}{[E:k]} \log |D_E| \quad - \quad \log |D_k|.$$

4.2. **Vojta's conjectures.**

Conjecture 4.2.1. *Let X be a smooth proper variety over a number field k, D a normal crossings divisor on X, and A an ample line bundle on X. Let r be a positive integer and $\epsilon > 0$. Then there is a proper Zariski-closed subset $Z \subset X$ containing D such that*

$$N_k(D,P) + d_k(k(P)) \geq h_{K_X(D)}(P) - \epsilon h_A(P) - O(1)$$

for all $P \in X(\bar{k}) \smallsetminus Z$ with $[k(P):k] \leq r$.

In the original conjectuer in [31], the discriminant term came with a factor $\dim X$. By the time of [32] Vojta came to the conclusion that the factor was not well justified. A seemingly stronger version is

Conjecture 4.2.2. *Let X be a smooth proper variety over a number field k, D a normal crossings divisor on X, and A an ample line bundle on X. Let r be a positive integer and $\epsilon > 0$. Then there is a proper Zariski-closed subset $Z \subset X$ containing D such that*

$$N_k^{(1)}(D,P) + d_k(k(P)) \geq h_{K_X(D)}(P) - \epsilon h_A(P) - O(1).$$

but in [32], Vojta shows that the two conjectures are equivalent.

4.3. **Vojta and *abc*.**

The following discussion is taken from [32], section 2.

The Masser-Oesterlé *abc* conjecture is the following:

Conjecture 4.3.1. *For any $\epsilon > 0$ there is $C > 0$ such that for all $a, b, c \in \mathbb{Z}$, with $a + b + c = 0$ and $\gcd(a,b,c) = 1$ we have*

$$\max(|a|,|b|,|c|) \quad \leq \quad C \cdot \prod_{p|abc} p^{1+\epsilon}.$$

Consider the point $P = (a:b:c) \in \mathbb{P}^2$. Its height is $\log \max(|a|,|b|,|c|)$. Of course the point lies on the line X defined by $x + y + z = 0$. If we denote by D the divisor of $xyz = 0$, that is the intersection of X with the coordinate axes, and if we set $S = \{\infty\}$, then

$$N^{(1)}_{\mathbb{Q},S}(D,P) = \sum_{p|abc} \log p.$$

So the *abc* conjecture says

$$h(P) \leq (1+\epsilon) N^{(1)}_{\mathbb{Q},S}(D,P) + O(1),$$

which, writing $1 - \epsilon' = (1+\epsilon)^{-1}$, is the same as

$$(1-\epsilon')h(P) \leq N^{(1)}_{\mathbb{Q},S}(D,P) + O(1).$$

This is applied only to rational points on X, so $d_{\mathbb{Q}}(\mathbb{Q}) = 0$. We have $K_X(D) = \mathcal{O}_X(1)$, and setting $A = \mathcal{O}_X(1)$ as well we get that *abc* is equivalent to

$$N^{(1)}_{\mathbb{Q},S}(D,P) \geq h_{K_X(D)}(P) - \epsilon' h_A(P) - O(1),$$

which is exactly what Vojta's conjecture predicts in this case.

Note that the same argument gives the *abc* conjecture over any fixed number field.

4.4. ***abc* and Campana.** Material in this section follows Campana's [**?**].

Let us go back to Campana constellation curves. Recall Conjecture , in particular a campana constellation curve of general type over a number field is conjectured to have a finite number of soft S-integral points.

Simple inequalities, along with Faltings's theorem, allow Campana to reduce to a finite number of cases, all on $\mathbb{P}^1$. The multiplicities m_i that occur in these "minimal" divisors Δ on $\mathbb{P}^1$ are:

$$(2,3,7), (2,4,5), (3,3,4), (2,2,2,3) \quad \text{and} \quad (2,2,2,2,2).$$

Now one claims that Campana conjecture in these cases follows from the *abc* conjecture for the number field k. This follows from a simple application of Elkies's [**?**]. It is easiest to verify in case $k = \mathbb{Q}$ when Δ is supported precisely at 3 points, with more points one needs to use Belyi maps (in the function field case one uses a proven generalization of *abc* instead).

We may assume Δ is supported at $0, 1$ and ∞. An integral point on $(\mathbb{P}^1/\Delta)$ in this case is a rational point a/c such that a, c are integers, satisfying the following:

- whenever $p|a$, in fact $p^{n_0}|a$;
- whenever $p|b$, in fact $p^{n_1}|b$; and
- whenever $p|c$, in fact $p^{n_\infty}|c$,

where $b = c - a$.

Now if $M = \max(|a|, |b|, |c|)$ then

$$M^{1/n_0 + 1/n_1 + 1/n_\infty} \geq |a|^{1/n_0}|b|^{1/n_1}|c|^{1/n_\infty},$$

and by assumption $a^{1/n_0} \geq \prod_{p|a} p$, and similarly for b, c. In other words

$$M^{1/n_0 + 1/n_1 + 1/n_\infty} \geq \prod_{p|abc} p.$$

Since, by assumption, $1/n_0 + 1/n_1 + 1/n_\infty < 1$ we can take any $0 < \epsilon < 1 - 1/n_0 + 1/n_1 + 1/n_\infty$, for which the *abc* conjecture gives $M^{1-\epsilon} < C \prod_{p|abc} p$, for some C. So $M^{1-1/n_0+1/n_1+1/n_\infty - epsilon} < C$ and M is bounded, so there are only finitely many such points.

4.5. **Vojta and Campana.** I speculate: Vojta's higher dimensional conjecture implies the non-special part of Campana's conjecture. Hopefully this will be proven before the lectures.

The problem is precisely in understanding what happens when a point reduces to the singular locus of D.

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