ON THE $\mathbb{A}^1$-CONNECTED VARIETIES

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Abstract. We introduce a comb smoothing technique of $\mathbb{A}^1$-connected varieties. For an application, we investigate the Zariskie density of integral points of $\mathbb{A}^1$-connected varieties over function field of curves under appropriate assumptions. As examples, the $\mathbb{A}^1$-connectedness of semisimple groups is studied.

Contents

1. Introduction

1.1. **Some backgrounds.** In this section, we temporarily restrict ourselves to work over the field of complex numbers. We hope this could give the reader a good picture of the story while avoiding unnecessary complexities.

The behavior of rational curves over projective varieties are intensively studied during the past decades. The great progresses along this line naturally leads to the consideration of non-proper cases.

Starting with a quasi-projective variety U, by the log resolution one could embed U into a smooth projective variety X such that $D = X \setminus U$ is a simple normal crossings boundary. Note that when $D = \emptyset$, we get back to the proper case.

The next question is what are the right notion of "rational curves" over a pair (X, D)? A crucial observation of Keel-McKernan [KM99] suggests that the right notion of *log rational curves on log pairs* should consists of the following two types of rational curves:

- A *rational curve* on a log smooth pair (X, D) is a morphism of pairs $f : (\mathbb{P}^1, \emptyset) \to (X, D)$, i.e., the image of $\mathbb{P}^1$ avoids D.

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- An $\mathbb{A}^1$-*curve* on a log smooth pair (X, D) is a morphism of pairs $f : (\mathbb{P}^1, \{\infty\}) \to (X, D)$, i.e., the image of $\mathbb{P}^1$ meets D exactly once at ∞.

For the first type, we have the standard definitions of uniruledness and rational connectedness as below:

- A smooth proper variety X is *uniruled* if there exists a rational curve through a general point on X;
- A smooth proper variety X is *rationally connected* if there exists a rational curve through a general pair of points on X.

These concepts were introduced and studied by Mori [Mor79], Campana [Cam92], and Kollár-Miyaoka-Mori [KMM92]. Rationally connected varieties are generalizations of projective spaces, which include the class of smooth Fano varieties. They admits nice arithmetic properties over non-closed fields, e.g. function fields of curves [GHS03, dJS03, HT06] and large fields [Kol99, KS03].

The study of the second type of rational curves in a pair is of different flavor, and is more involved. Luckily, the recent development of stable log maps [GS13, Che10, AC] provides a solid foundation for working with rational curves of the second type. Mimicking the above definition, we formulate the following concepts:

- A log smooth variety (X, D) is $\mathbb{A}^1$-*uniruled* if there exists an $\mathbb{A}^1$-curve through a general point on X;
- A log smooth variety (X, D) is $\mathbb{A}^1$-*connected* if there exists an $\mathbb{A}^1$-curve through a general pair of points on X.

From the deformation point of view, the $\mathbb{A}^1$-curves are better understood as log maps. We refer to Section 2 for more details.

The idea of $\mathbb{A}^1$-connectedness already has its application for even the study of type one rational curves — it plays a central roll in the proof of separably rationally connectedness of general Fano complete intersections over all characteristics [CZ13]. Many interesting examples of $\mathbb{A}^1$-connected varieties were provided there:

Theorem 1.1 ([CZ13]). *Given a log smooth log Fano variety (X, D) with D irreducible, (X, D) is $\mathbb{A}^1$-connected if (X, D) is $\mathbb{A}^1$-uniruled. Moreover, the latter condition holds when the normal bundle of D is nontrivial and effective, e.g., D is ample.*

One would like to ask for more interesting examples, and even arithmetic applications of $\mathbb{A}^1$-connected varieties. One purpose of this paper is to study $\mathbb{A}^1$-connectedness from those points of view.

1.2. **Organization and main results of the paper.** $\mathbb{A}^1$-connected varieties should be thought as generalizations of affine spaces, where the latter are obviously $\mathbb{A}^1$-connected. Although many results of this paper can be formulated using the usual schematic language, the setting of logarithmic geometry in the sense of Kato-Fontaine-Illusie [Kat89] is more natural and crucial to the proof.

In Section 2.1, we define and explore basic properties of $\mathbb{A}^1$-uniruledness and $\mathbb{A}^1$-connectedness using stable log maps in the sense of [GS13, Che10, AC].

Section 3 contains the main technique of this paper – the comb smoothing under the logarithmic setting. This provides us a constructive method for producing very free $\mathbb{A}^1$-curves for general log smooth varieties with possibly toric singularities, see Theorem 3.1.

The searching of integral points over non-closed field is an old and difficult arithmetic question. A geometric formulation of the Zariski density of integral points over function fields of curves was introduced, and the important case of Fano smooth pairs was studied by Hassett-Tschinkel [HT08]. Section 4 can be viewed as a continuation of their foundational work under the logarithmic settings.

Since $\mathbb{A}^1$-connected varieties are generalizations of affine spaces, we propose the following question similar to [Has10], and hope to further pursue along this direction in our future work:

Question 1.2. Does strong approximation hold for $\mathbb{A}^1$-connected varieties? Or at least at places of good reduction?

In section 5, we investigate the $\mathbb{A}^1$-connectedness of semi-simple groups. These provide many interesting examples with reducible, and even toroidal boundary divisors. Furthermore, we verify that they fulfill the conditions for Zariski density in Theorem 4.1.

1.3. **Notations.** Throughout the rest of this paper, we work over an algebraically closed field $\mathbf{k}$ of characteristic $\operatorname{char}\mathbf{k} \geq 0$. We will specify the conditions on $\operatorname{char}\mathbf{k}$ when it is necessary.

All logarithmic structures are assumed to be fine and saturated. Capital letters like X, Y are reserved for log schemes, and the locus of X and Y with the trivial log structure are denoted by X°, Y° respectively. The underlying structure of a log scheme X is denoted by $\underline{X}$.

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2. DEFINITIONS OF $\mathbb{A}^1$-CONNECTEDNESS

2.1. **Notions of log maps.** We first recall some basic terminology of log maps, and refer to [GS13, Che10, AC] for detailed discussions.

Definition 2.1. Let $X \to B$ be a morphism of log schemes. A *stable log map* over a log scheme S is a commutative diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ S \arrow[r] & B \end{tikzcd} \tag{2.1.1}$$

such that $C \to S$ is a family of log curves over S as defined in [Kat96, Ols07], and the underlying map $\underline{f}$ is a family of usual stable maps to the underlying family of targets X/B.

For the rest of this paper, we will only consider the case when B is a geometric point with the trivial log structure. For simplicity, we may write $f : C/S \to X$ for a family of stable log maps over S. When S is a geometric point with the trivial log structure, we will write $f : C \to X$ for simplicity.

Consider a geometric fiber $f : C/S \to X$. For each marking $\sigma \in C$, we have a canonical map of monoids

$$c_\sigma := \bar{f}^\flat : f^*\overline{\mathcal{M}}_{X,\sigma} \to \mathbb{N}.$$

as in [AC], called the *contact order* at σ. A marking σ is called a *contact marking* if c_σ is non-trivial. Otherwise, we call σ a *usual marking*.

Let X be a log scheme, and $\Gamma = (g, n, \beta, \{c_i\}_{i=1}^n)$ the set of data consisting of the genus $g \in \mathbb{N}$, the number of markings n, the curve class $\beta \in H_2X$, and the set of contact orders $\{c_i\}$. Denote by $\mathfrak{M}_\Gamma(X)$ for the stack of stable log maps with discrete data Γ.

Definition 2.2. A log map $f : C/S \to X$ is called *k-pointed* if it has precisely k usual markings.

Let X be a log smooth scheme. A log map $f : C/S \to X$ is said to be *n-contacted* if it has precisely n contact markings. A 1-contacted genus zero stable log maps with smooth domain curve is called an *$\mathbb{A}^1$-curve*. An $\mathbb{A}^1$-curve could also be k-pointed for any non-negative k.

An $\mathbb{A}^1$-curve $f : C/S \to X$ over a geometric point $\underline{S}$ is called *free (resp. very free)* if f^*T_X is semi-positive (resp. ample).

We have the following analogue of [Kol96, II. 3.14]:

Proposition 2.3. *Let X be a log smooth log scheme with $\dim X = r$, and $f : C \to X$ be a free $\mathbb{A}^1$-curve. Assume the splitting type*

$$f^*T_X = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_r),$$

with $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$. Then

(1) If $a_2 \geq 1$, then a general deformation of f is an immersion;
(2) If $a_3 \geq 1$, then a general deformation of f is an embedding.

Proof. The proof is identical to [Kol96, II 3.14] by working with the stack of stable log maps and the log deformation theory, see for example [Che10, Section 2.5]. ♠

Consider a geometric fiber log map: $f : C/S \to X$ with a subset of markings $P = \{\Sigma_i\}_i$. Then f is called *P-free* if $H^1(f^*T_X(-P)) = 0$, and $f^*T_X(-P)$ is globally generated.

2.2. $\mathbb{A}^1$-connectedness.

Definition 2.4. Let X be a proper log smooth scheme. It is called *separably $\mathbb{A}^1$-connected* (resp. *separably $\mathbb{A}^1$-uniruled*) if it admits a very free (resp. free) $\mathbb{A}^1$-curve.

Remark 2.5. The reader should be notified that the definition of $\mathbb{A}^1$-connectedness and $\mathbb{A}^1$-uniruled in [KM99] includes the case of complete rational curves in X°. However, we find it is more convenient to study the two cases separately, as they require different techniques.

Proposition 2.6. *Let X be a proper log smooth scheme. The following are equivalent:*

(1) X is separably $\mathbb{A}^1$-uniruled;
(2) there exists a family of 1-pointed $\mathbb{A}^1$ curves $f : C/S \to X$ such that the one-pointed evaluation is dominant and separable.
(3) there exists a flat family $\pi : U \to S$ of usual schemes with geometric fiber isomorphic to $\mathbb{A}^1$, and a separable and dominant morphism $f : U \to X^\circ$ such that each geometric fiber $f_s : \mathbb{A}^1 \to X^\circ$ is a proper morphism for each $s \in S$,

When $\operatorname{char}\mathbf{k} = 0$*, the separable conditions in (2) and (3) can be removed.*

If further assume $\operatorname{char}\mathbf{k} = 0$ *and* $\mathbf{k}$ *is uncountable, then X is $\mathbb{A}^1$-uniruled if and only if there exists an $\mathbb{A}^1$ curve though general points on X.*

Proof. The equivalence of the three conditions follows from the standard argument of [Kol96, IV 1.3] for the stack of stable log maps.

Notice that the stack of $\mathbb{A}^1$-curves in X has countably many irreducible components. The last statement follows from the same argument as in [Kol96, IV (1.3.5)]. ♠

Proposition 2.7. *Let X be a proper log smooth scheme. The following are equivalent:*

(1) X is separably $\mathbb{A}^1$-connected;
(2) there exists a family of two-pointed log rational curves on X such that the two-pointed evaluation is separable and dominant.
(3) there exists a flat family $\pi : U \to S$ of usual schemes with geometric fiber isomorphic to $\mathbb{A}^1$, and a morphism $f : U \to X^\circ$ such that the fiber $f_s : \mathbb{A}^1 \to X^\circ$ is a proper morphism for each $s \in S$, and the two evaluation map
$$f^{(2)} : U \times_S U \to X^\circ$$
is separable and dominant.

When $\operatorname{char}\mathbf{k} = 0$*, the separable condition in (2) and (3) can be removed.*

If further assume $\operatorname{char}\mathbf{k} = 0$ *and* $\mathbf{k}$ *is uncountable, then X is $\mathbb{A}^1$-connected if and only if there exists an $\mathbb{A}^1$ curve though general pair of points in X°.*

Proof. The statements follows from the standard arguments, see for example [Kol96, IV 3.4, 3.5, 3.7] ♠

Definition 2.8. A smooth usual scheme X° is called *separably $\mathbb{A}^1$-connected* (resp. *separably $\mathbb{A}^1$-uniruled*) if it satisfies Proposition 2.7(3) (resp. Proposition 2.6(3)).

Corollary 2.9. *$g^\circ : X^\circ \to Y^\circ$ be a finite étale morphism of two usual smooth schemes. Then any proper $\mathbb{A}^1$-curve in Y° lifts to a proper $\mathbb{A}^1$-curve in X°. Furthermore, X is separably $\mathbb{A}^1$-connected (resp. $\mathbb{A}^1$-uniruled) if and only if Y is so.*

Proof. The first statement follows from the fundamental group of $\mathbb{A}^1$ is trivial. The second statement follows from the first one and Definition 2.8. ♠

2.3. Contact orders along the boundary.

Notation 2.10. Let X be a log scheme. Following [Ols03], we introduce a canonical stratification $\{X_\lambda\}_{\lambda\in\Lambda}$ of X such that

(1) X_λ is a connected, locally closed subset of X;
(2) X is given by the disjoint union of X_λ;
(3) the sheaf of monoid $\overline{\mathcal{M}}_X|_{X_\lambda}$ is a locally constant sheaf of monoids with $\overline{\mathcal{M}}_{X,x} \cong P_\lambda$ for any $x \in X_\lambda$. Here P_λ is a sharp saturated monoid.

Let $\overline{X_\lambda}$ be the closure of X_λ in X with the reduced structure. A stratum X_λ is called a *center* if $X_\lambda = \overline{X_\lambda}$ in X.

We have a natural partial ordering on the index set Λ by $\lambda' > \lambda$ if $X_{\lambda'} \subset \overline{X_\lambda}$. Note that $\overline{X_\lambda}$ has a natural boundary $\partial\overline{X_\lambda} = \cup_{\lambda'>\lambda} X_{\lambda'}$. In general, the pair $(\overline{X_\lambda}, \partial\overline{X_\lambda})$ can be very singular. But in case X is log smooth, one can check that the pair $(\overline{X_\lambda}, \partial\overline{X_\lambda})$ is also log smooth. Denote by $X_\lambda^\dagger$ the log scheme associated to such pair.

Definition 2.11. Let X be a log smooth scheme with the stratification $\{X_\lambda\}$ as above. We call $X_\lambda \subset X$ a *(very) free support* if there is a (very) free $\mathbb{A}^1$-curve with its unique contact marking mapping to a point in X_λ.

Note that if X_λ is a (very) free support, then through general point of X_λ there is a (very) free $\mathbb{A}^1$ curve.

Let X_λ be a (very) free support. Let σ_λ be the dual monoid of P_λ. Let N_λ be the lattice generated by σ_λ, and $(N_\lambda)_{\mathbf{k}} = N_\lambda \otimes_{\mathbb{Z}} \mathbf{k}$.

Following [ACMW], we view contact orders supported on X_λ as integral points in σ_λ. Denote by $\mathcal{F}(X_\lambda)$ (resp. $\mathcal{V}(X_\lambda)$) the subset of σ_λ consisting of contact orders of free (resp. very free) $\mathbb{A}^1$ curves. Let $\mathcal{A}(X_\lambda) \subset N_\lambda$ be the sub-lattice spanned by the contact orders of any $\mathbb{A}^1$-curve supported on X_λ. We call $\mathcal{A}(X_\lambda)$ the $\mathbb{A}^1$ *lattice* of X_λ.

The following conception turns out to be crucial to the $\mathbb{A}^1$-connectedness:

Definition 2.12. A (very) free support X_λ is called *fully* if $\mathcal{F}(X_\lambda)$ (resp. $\mathcal{V}(X_\lambda)$) spans the vector space $(N_\lambda)_{\mathbf{k}}$. It is called *primitive* if $\mathcal{F}(X_\lambda)$ (resp. $\mathcal{V}(X_\lambda)$) spans the lattice N_λ.

The following shows how contact orders control the topology of log schemes:

Proposition 2.13. *Let X be a proper log smooth, separably $\mathbb{A}^1$-connected variety. For any codimension one boundary stratum $X_\lambda \subset X$, assume that $\mathcal{A}(X_\lambda)$ spans the lattice N_λ. Then any Kummer étale cover $Y \to X$ is trivial. Equivalently, this means that any exact log étale morphism $Y \to X$ is an isomorphism of log schemes. In particular, the log fundamental group as in* [Ill02, Section 4] *of X is trivial.*

Proof. Note that X is separably rationally connected, hence has trivial fundamental group. Thus any non-trivial Kummer étale cover $g : Y \to X$ must ramify along the boundary. Let $X_\lambda \subset X$ be a codimension one boundary stratum, and let $Y_\lambda \subset Y$ be the stratum over X_λ. Assume that g ramifies along X_λ of index r. Then there is an inclusion of monoids $\tilde{g} : \sigma_{Y_\lambda} \to \sigma_{X_\lambda}$ with finite cokernel of index r.

Consider an $\mathbb{A}^1$-curve $f : C \to X$ with the contact order c generating the monoid σ_{X_λ}. Note that f can be lift to an $\mathbb{A}^1$-curve $f' : C \to Y$ with contact order c' such that $\tilde{g}(c') = c$. The condition $\mathcal{A}(X_\lambda) = N_\lambda$ implies that $\tilde{g}$ is surjective, and hence $r = 1$. ♠

3. A CRITERION FOR $\mathbb{A}^1$-CONNECTEDNESS

Our next goal is to prove the following criterion for the existence of very free $\mathbb{A}^1$-curves.

Theorem 3.1. *Let X be an irreducible log smooth scheme. Assume there is a separably rationally connected center $X_\lambda \subset X$, or a separably $\mathbb{A}^1$-connected stratum closure $X_\lambda^\dagger$, such that X_λ is a fully free support. Then*

1. *X is separably $\mathbb{A}^1$-connected with a very free support X_λ;*
2. *If X_λ is a center, then through every point of X_λ there is an $\mathbb{A}^1$-curve.*

The proof of this theorem will be given in the end of this section after various preparations.

3.1. **Delign-Faltings log structures.**

We introduce the log structure associated to cartier divisors to slightly generalizes the case of simple normal crossings boundaries. Readers may consult [Che10, Appendix A] for more details.

Let $D \subset \underline{X}$ be a connected cartier divisor. Such D corresponds to a global section $s \in H^0(\mathcal{O}_{\underline{X}}(D))$, hence a rank one Deligne-Faltings log structure $\mathcal{M}_X$ over $\underline{X}$, see [Che10, A.2]. We have the fiber

$$\overline{\mathcal{M}}_{X,x} = \begin{cases} \mathbb{N}, & \text{if } b \in D; \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.1}$$

where $\overline{\mathcal{M}}_X := \mathcal{M}_X/\mathcal{O}^*$ is called the *characteristic sheaf of* $\mathcal{M}_X$. Denote by $X = (\underline{X}, \mathcal{M}_X)$ the log scheme.

One may consider the a finite set of connected Cartier divisors

$$D_1, D_2, \cdots, D_m$$

over $\underline{X}$. For each D_i, we obtain a rank one Deligne-Faltings log structure $\mathcal{M}_i$ as above, hence a log scheme X_i associated to the pair $(\underline{X}, D_i)$. Consider the Cartesian product of log schemes:

$$X = X_1 \times_{\underline{X}} \cdots \times_{\underline{X}} X_m \tag{3.1.2}$$

where we view $\underline{X}$ as the log scheme with the trivial log structure. The log structure $\mathcal{M}_X$ of X is called a *rank* m *Deligne-Faltings log structure*. We have the fiber

$$\overline{\mathcal{M}}_{X,x} \cong \mathbb{N}^d \tag{3.1.3}$$

if x is contained in the intersection of precisely d divisors $\cap_{j=1}^{d} D_{i_j}$.

3.2. **Construct combs with handles on the boundary.**

Let X be our target log scheme.

Notation 3.2. We first consider the case of rank one Deligne-Faltings log structure $\mathcal{M}_X$ associated to a connected cartier divisor $D \subset \underline{X}$. We fix the following:

1. $\underline{f}_0 : \underline{C}_0 \to D$ is a usual stable map with $\underline{C}_0 \cong \mathbb{P}^1$ and marked points $p_1, \cdots, p_k \in \underline{C}_0$;
2. for each $i = 1, \cdots, k$, let $\underline{f}_i : \underline{C}_i \to \underline{X}$ be a usual stable map with at least one marking $q_i \in \underline{C}_i$ such that $\underline{f}_i(q_i) = p_i$, $\underline{f}_i(\underline{C}_i) \not\subset D$, and the map $\underline{f}_i$ tangent to D at q_i with contact order $c_i \in \mathbb{Z}_{>0}$;
3. $\sum_i c_i + (\underline{f}_0)_*[\underline{C}_0] \cap D \geq 0$.

By gluing q_i with p_i, we obtain a usual stable map $\underline{f} : \underline{C} \to \underline{X}$.

Proposition 3.3. *With the notations above, there is a stable log map* $f : C/S \to X$ *over the underlying map* $\underline{f}$ *such that*

1. *if* $\sum_i c_i + (\underline{f}_0)_*[\underline{C}_0] \cap D > 0$*, we can choose the stable log map* f *having exactly one contact marking on the component* $\underline{C}_0$ *at any smooth point.*
2. *if* $\sum_i c_i + (\underline{f}_0)_*[\underline{C}_0] \cap D = 0$*, we can choose the stable log map* f *having no marking on* $\underline{C}_0$*.*

Proof. Denote by $\mathbb{P} := \mathbb{P}_D(\mathcal{O}(D)|_D \oplus \mathcal{O}_D)$ with two Cartier divisors D_0 and D_∞ isomorphic to D defined by

$$\mathcal{O}_{\mathbb{P}}(D_0)|_{D_0} \cong \mathcal{O}_{\underline{X}}(-D)|_D \quad \text{and} \quad \mathcal{O}_{\mathbb{P}}(D_\infty)|_{D_\infty} \cong \mathcal{O}_{\underline{X}}(D)|_D$$

Gluing D_0 with D, we obtain a scheme $\underline{Y} = \underline{X} \cup_{D \cong D_0} \mathbb{P}$. Since $D \subset \underline{X}$ is Cartier, $\underline{Y}$ can be obtained as the pull-back of the universal expansion of the smooth pairs [ACFW11]. Thus, there is a natural morphism of log schemes:

$$Y \to p^\dagger \tag{3.2.1}$$

together with a log map

$$Y \to X \tag{3.2.2}$$

where $p^\dagger$ is is the standard log points, i.e. $\overline{\mathcal{M}}_{p^\dagger} \cong \mathbb{N}$. By the same argument of [CZ13, Lemma 3.5], we may choose a underlying map

$$\underline{C}_0 \to \mathbb{P}$$

tangent to D_∞ at a single point of contact order $\sum_i c_i + (\underline{f}_0)_*[\underline{C}_0] \cap D$ at any point away from the nodes, and tangent to D_0 at $p_1, \cdots, p_k \in D \cong D_0$ with contact order c_i respectively. The construction in [Kim10, 5.2.3] yields a log map $f : C/S \to Y/p^\dagger$, whose composition with $Y \to X$ is the log map needed. ♠

For later use, we slightly generalize our construction to the case of rank m Deligne-Faltings log structures.

Notation 3.4. Let $\mathcal{M}_X$ be the log structure given by the following Cartier divisors $D_1, \cdots, D_m$. Assume that

1. $\underline{f}_0 : \underline{C}_0 \to \cap_j D_j$ is a usual stable map with $\underline{C}_0 \cong \mathbb{P}^1$ with marked points $p_1, \cdots, p_k \in \underline{C}_0$;
2. For each $i = 1, \cdots, k$, let $\underline{f}_i : \underline{C}_i \to \underline{X}$ be a usual stable map with at least one marking $q_i \in \underline{C}_i$ such that $\underline{f}_i(q_i) = \underline{f}_0(p_i)$, $\underline{f}_i(\underline{C}_i) \not\subset D_j$ for all j, and the map $\underline{f}_i$ tangent to D_j at q_i with contact order $c_{ij} \in \mathbb{Z}_{>0}$;
3. $\sum_i c_{ij} + (\underline{f}_0)_*[\underline{C}_0] \cap D_j \geq 0$ for all j.

Gluing p_i and q_i for all i, we obtain a usual stable map $\underline{f} : \underline{C} \to \underline{X}$.

Proposition 3.5. *With the notations above, there is a stable log map $f : C/S \to X$ over the underlying map $\underline{f}$ such that*

(1) If $\sum_i c_i + (\underline{f}_0)_*[\underline{C}_0] \cap D_j > 0$ for some j, we can choose the stable log map f having exactly one contact marking on the component $\underline{C}_0$ at any smooth point of $\underline{C}$.

(2) If $\sum_i c_i + (\underline{f}_0)_*[\underline{C}_0] \cap D_j = 0$ for all j, we can choose the stable log map f having no marking on $\underline{C}_0$.

Proof. By Proposition 3.3, we may construct log maps $f_j : C_j/S \to X_j$ with $X_j = (\underline{X}, \mathcal{M}_j)$ where $\mathcal{M}_j$ is the rank one Deligne-Faltings log structure associated to D_j for each j. We may assume that the marked point on the component $\underline{C}_0$ is given by the same point for all j. Using (3.1.2), the product structures of the stacks of stable log maps in [AC] yields the log map as in the statement. ♠

3.3. **The case of simple normal crossings boundaries.** We first verify Theorem 3.1 for the case of simple normal crossings boundaries. Let X be a log smooth scheme with simple normal crossings boundary D given by smooth irreducible components $D_1, \cdots, D_k$. For any $\lambda \subset [k] := \{1, \cdots, k\}$, denote by $D_\lambda = \cap_{i \in \lambda} D_i$. Following Notation 2.10, let $X_\lambda^\dagger$ be the log scheme associated to the pair $(D_\lambda, \cup_{i\in I(\lambda)}(D_\lambda \cap D_i))$, where $I(\lambda)$ is the collection of index i such that $D_i \cap D_\lambda$ is a divisor in D_λ. Let Y be the log scheme associated to the pair $(\underline{X}, D = \cup_{i\notin\lambda} D_i)$.

We learned the following lemma from Yi Hu.

Lemma 3.6. *There is a natural exact sequence*

$$0 \to \Omega_{X^\dagger_\lambda} \to \Omega_X|_{D_\lambda} \to \mathcal{O}_{D_\lambda}^{\oplus|\lambda|} \to 0.$$

Proof. There is a strict log map

$$D_\lambda \to Y$$

hence an exact sequence

$$0 \to N^\vee_{D_\lambda/\underline{X}} \to \Omega_Y|_{D_\lambda} \to \Omega_{X^\dagger_\lambda} \to 0 \tag{3.3.1}$$

On the other hand, there is an exact sequence

$$0 \to \Omega_Y \to \Omega_X \to \sum_{i\in\lambda} \mathcal{O}_{D_i} \to 0 \tag{3.3.2}$$

Tensoring with $\mathcal{O}_{D_\lambda}$, we obtain the exact sequence:

(3.3.3)

$$0 \to \sum_{i\in\lambda} Tor_1^{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{D_i}) \to \Omega_Y|_{D_\lambda} \to \Omega_X|_{D_\lambda} \to \sum_{i\in\lambda} \mathcal{O}_{D_i} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_\lambda} \to 0$$

For the term on the right, we have

$$\sum_i \mathcal{O}_{D_\sigma} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_\lambda} = \sum_{i\in\lambda} \mathcal{O}_{D_\lambda}.$$

To calculate the $Tor_1^{\mathcal{O}_X}(-,-)$, we take the resolution

$$0 \to \mathcal{O}_X(-D_i) \to \mathcal{O}_X \to \mathcal{O}_{D_i} \to 0$$

Tensoring with $\mathcal{O}_{D_\lambda}$ we have the exact sequence:

$$0 \to Tor_1^{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{D_i}) \to \mathcal{O}_{D_\lambda}(-D_i) \to \mathcal{O}_{D_\lambda} \to \mathcal{O}_{D_i} \otimes \mathcal{O}_{D_\lambda} \to 0 \tag{3.3.4}$$

Note that if $i \notin \lambda$, then we have

$$Tor_1^{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{D_i}) = 0$$

since they intersect transversally.

If $i \in \lambda$, we have

$$\mathcal{O}_{D_i} \otimes \mathcal{O}_{D_\lambda} = \mathcal{O}_{D_\lambda}$$

Thus the arrow

$$\mathcal{O}_{D_\lambda} \to \mathcal{O}_{D_i} \otimes \mathcal{O}_{D_\lambda}$$

in (3.3.4) is an isomorphism. This yields

$$Tor_1^{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{E_i}) = \mathcal{O}_{D_\lambda}(-D_i) = N^\vee_{D_i/\underline{X}}|_{D_\lambda}$$

hence

$$\sum_i Tor_1^{\mathcal{O}_X}(\mathcal{O}_{D_\lambda}, \mathcal{O}_{D_i}) \cong N^\vee_{D_\lambda/\underline{X}}|_{D_\lambda}.$$

We deduce

$$0 \to N^{\vee}_{D_\lambda/\underline{X}}|_{D_\lambda} \to \Omega_Y|_{D_\lambda} \to \Omega_X|_{D_\lambda} \to \sum_{i\in\lambda} \mathcal{O}_{D_\lambda} \to 0 \tag{3.3.5}$$

Putting (3.3.5) and (3.3.1) together, we have the following commutative diagram with the exact row and column:

(3.3.6)

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & & N^{\vee}_{D_\lambda/\underline{X}} \arrow[dl, "\cong"'] \arrow[d] & & & \\
0 \arrow[r] & N^{\vee}_{D_\lambda/\underline{X}} \arrow[r] & \Omega_Y|_{D_\lambda} \arrow[r] \arrow[d] & \Omega_X|_{D_\lambda} \arrow[r] & \sum_{i\in\lambda} \mathcal{O}_{D_\lambda} \arrow[r] & 0 \\
 & & \Omega_{X^{\dagger}_{\lambda}} \arrow[ur, dashed] \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

Now the statement follows. ♠

We next study the comb-smoothing technique under the logarithmic setting. For later use, we consider slightly more general situations:

Notation 3.7. Let $f : C/S \to X$ be a genus g stable log map over the geometric point $\underline{S}$ with one contact marking σ such that

1. $\underline{C} = \cup_{i=0}^{m} \underline{C}_i$ with smooth irreducible component $\underline{C}_i \cong \mathbb{P}^1$ for $i \neq 0$, and a smooth genus g component $\underline{C}_0$.
2. For each $i \neq 0$, we have a unique node $r_i \in C$ joining $\underline{C}_i$ and $\underline{C}_0$. All nodes of $\underline{C}$ are of this form. Let p_i and q_i denote the special point on $\underline{C}_i$ and $\underline{C}_0$ corresponding to r_i.
3. $f(C_i \setminus \{p_i\}) \subset X^{\circ}$ and $f(p_i) \in X_\lambda$ for $i \neq 0$.
4. $f(C_0 \setminus \{\sigma\}) \subset X_\lambda$.

It follows from the assumption that the restriction $f|_{\underline{C}_i}$ defines an $\mathbb{A}^1$-curve $f_i : C'_i \to X$ for $i \neq 0$ of contact order c_i, and the one-contacted genus g log map $f_0 : C'_0 \to X^{\dagger}_{\lambda}$.

We further assume that $c_i \in N_k$ is a non-trivial integral vector, and f is a local immersion away from the nodes and the contact marking.

Consider the following sequences:

$$\mathrm{d}f : f^*\Omega_X \to \Omega_{C/S}.$$

and

$$\mathrm{d}f_i : f^*\Omega_X \to \Omega_{C'_i}$$

By [Ols05], the above sequences governs the deformation of f and f_i respectively. The assumption on c_i and the local immersion of f implies that $\mathrm{d}f$ and $\mathrm{d}f_i$ are surjective with locally free kernel $N_f^\vee$ and $N_{f_i}^\vee$ respectively. We now consider the following commutative diagram:

(3.3.7)
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & N_{f_0}^\vee \arrow[r] \arrow[d] & (N_f^\vee)_{\underline{C}_0} \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & f_0^*\Omega_{X_\lambda^\dagger} \arrow[r] \arrow[d] & f^*\Omega_X|_{C_0} \arrow[r] \arrow[d] & \mathcal{O}^{\oplus|\lambda|} \arrow[r] \arrow[d, "\phi"] & 0 \\
0 \arrow[r] & \Omega_{C_0'} \arrow[r] \arrow[d] & \Omega_C|_{C_0} \arrow[r] \arrow[d] & \sum_i \mathbf{k}_{q_i} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where each rows and columns are exact. A local calculation at q_i shows that

Lemma 3.8. *The restriction $\phi|_{q_i}$ is given by the $(1 \times |\lambda|)$-matrix $(-c_i)$, where c_i is the contact order at q_i.*

We then have the following:

Proposition 3.9. *Theorem 3.1 is true when X has simple normal crossings boundary.*

Proof. We only consider the case that X_λ is a center. The proof for the other case is completely identical.

Since X_λ is separably rationally connected, we may choose a very free curve $\underline{f}_0 : \underline{C}_0 \cong \mathbb{P}^1 \to X_\lambda$ through general points of X_λ. We may further assume that $\underline{f}_0$ is an immersion by [Kol96, II. 1.8]. This can be achieved by for example replacing X by $X \times \mathbb{P}^1$. Note that if we could construct the desired very free $\mathbb{A}^1$-curve in $X \times \mathbb{P}^2$, then taking the composition with the projection $X \times \mathbb{P}^2 \to X$ will yield the desired very free $\mathbb{A}^1$-curve in X. For any fixed positive integer n, further require that $N_{\underline{f}_0}(-n)$ to be ample, where $N_{\underline{f}_0} = (N_{\underline{f}_0}^\vee)^\vee$.

By the assumption, we may also choose free $\mathbb{A}^1$-curves $f_i : C_i \to X$ for $i = 1, 2, \cdots, k$, such that the contact marking send to a general point of the image of $\underline{f}_0$ with contact order c_i. We may further assume that f_i is an immersion for all i by replacing X with $X \times \mathbb{P}^1$, and applying Proposition 2.3.

Let k be sufficiently larget. By Proposition 3.5, we can glue $\underline{f}_0, f_1, \cdots, f_k$ together to form a log map $f : C \to X$ with a single contact marking $\sigma \in \underline{C}_0$. We then arrived at the situation of (3.3.7).

Fix any positive integer n. Since X_λ is a fully free support, we may choosing k sufficiently large, and c_i sufficiently general for all i, by Lemma 3.8 we could require $V^\vee(-n)$ to be ample. In particular, $(N_f)_{C_0}(-n)$ is ample. We then deform f fixing the contact marking σ. By upper semi-continuity, A general deformation f_t of f provides the very free $\mathbb{A}^1$ curve in X as in the statement. ♠

We also proved that

Corollary 3.10. *Assume that* $\dim X \geq 2$. *For any fixed positive integer* n*, we may construct* f *as in the proof of Proposition 3.9 whose general deformation* f_t *is an immersion of* $\mathbb{A}^1$*-curve away from the contact marking, and the conormal* $N_{f_t}(-n)$ *is ample.*

The following result helps us to generalize the above result to arbitrary log smooth schemes.

Proposition 3.11. *Let* X *be a log smooth scheme with not necessarily simple normal crossings boundary, and let* X_λ *be a (very) free support of* X*. Then the set of integral points* $\overline{\mathcal{F}}(X_\lambda) = \mathcal{F}(X_\lambda)$ *(resp.* $\overline{\mathcal{V}}(X_\lambda) = \mathcal{V}(X_\lambda)$*) is closed under addition in* σ_λ*.*

Proof. We only consider the case $\mathcal{F}(X_\lambda)$. The other case is identical.

For any two points $c_1, c_2 \in \mathcal{F}(X_\lambda)$, we choose two free log maps $f_1 : C_1 \to X$ and $f_2 : C_2 \to X$ with the corresponding contact orders. Since the deformations of f_1 and f_2 swept out a general points of X_λ, we may assume that $p = f_1(\sigma_1) = f_2(\sigma_2)$, where σ_i is the contact marking of f_i.

Let $\underline{C}$ be a rational nodal curve given by three irreducible components $\underline{C}_0 \cup \underline{C}_1 \cup \underline{C}_2$ such that $q_i = \underline{C}_i \cap \underline{C}_0$ for $i = 1, 2$. Choose another smooth point $\sigma \in \underline{C}_0$ to be the marking. Let $\underline{f} : \underline{C} \to \underline{X}$ be the stable map defined by gluing $\underline{f}_1$ and $\underline{f}_2$ along the contracted component $\underline{C}_0$. We next lift $\underline{f}$ to a log map f with a unique marking σ with tangency c_σ. It is then clear that f deforms to a free $\mathbb{A}^1$-curve with tangency $c_\sigma = c_1 + c_2$, which proves the statement.

Note that the image of $\underline{C}_0$ under $\underline{f}$ is the point p. We may focus on a neighborhood of p, and assume a chart $\beta : \overline{\mathcal{M}}_p \to \mathcal{M}_X$. Since $\overline{\mathcal{M}}_p$ is fine, saturated, and sharp, we have a coequalizer diagram of monoids

$$\begin{tikzcd} P_2 \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & P_1 \arrow[r, "g"] & \overline{\mathcal{M}}_p \end{tikzcd} \tag{3.3.8}$$

where both P_1 and P_2 are finitely generated free monoids, see [Ogu06, Theorem 2.1.9 (7)]. As in [AC, (3.12.3)], the above (3.3.8) lifts to a coequalizer of log structures around p

$$\begin{tikzcd} \mathcal{M}_2 \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & \mathcal{M}_1 \arrow[r, "g"] & \mathcal{M}_X \end{tikzcd} \tag{3.3.9}$$

with charts $\beta_1 : P_1 \to \mathcal{M}_1$ and $\beta_2 : P_2 \to \mathcal{M}_2$ induced by β. Let $\{\delta_i\}_i$ be a set of generators of P_1, and $\{e_j\}_j$ be a set of generators of P_2. Now for each δ_i, consider the pull-back of $\beta_1(\delta_i)$ over $\underline{C}_1$ and $\underline{C}_2$, and denote by d_{1i} and d_{2i} the corresponding degree. We thus fix a rational function s_i over $\underline{C}_0$ with d_{1i}-th order poles at q_1, d_{2i}-th order poles at q_2, and $(d_{1i}+d_{2i})$-th zero at σ. Thus same as the proof of Proposition 3.3, a choice of the rational functions for each i:

$$u_i \cdot s_i \tag{3.3.10}$$

where $u_i \in \mathbf{k}^*$ defines a log map $f' : C'/S \to X'$ where $X' = (\underline{X}, \mathcal{M}_1)$. By the universality of minimality as in [AC, Proposition 3.18], we may assume f' is minimal for the convenience of calculation. In fact, a direct calculation using [AC, 4.1.2] shows that $\overline{\mathcal{M}}_S \cong \mathbb{N}$.

To show that f' descends to a log map f to X, it remains to show that there exists a choice of $\{u_i\}_i$, such that the two compositions

$$\begin{tikzcd} \underline{f}^*\mathcal{M}_2 \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & \underline{f}^*\mathcal{M}_1 \arrow[r, "(f')^\flat"] & \mathcal{M}_{C'} \end{tikzcd} \tag{3.3.11}$$

are equal. Note that the two composition $r_1 \circ f'$ and $r_2 \circ f'$ induces two log maps to $(\underline{X}, \mathcal{M}_s)$.

In fact, since the tangency at nodes and the marked point is governed by the contact orders c_1 and c_2, it suffices to check this over a general point of $x \in \underline{C}_0$. We first verify that the two compositions are equal on the level of characteristics. Consider

$$\begin{tikzcd} (\underline{f}^*\overline{\mathcal{M}}_2)_x \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & (\underline{f}^*\overline{\mathcal{M}}_1)_x \arrow[r, "(f')^\flat"] & \overline{\mathcal{M}}_{C',x} = \overline{\mathcal{M}}_S \end{tikzcd} \tag{3.3.12}$$

For any element $e \in (\underline{f}^*\overline{\mathcal{M}}_2)_x$, since $r_1(e)$ and $r_2(e)$ correspond to the same element in $\overline{\mathcal{M}}_p$, we have

$$c_1(g(r_1(e))) = c_2(g(r_2(e)))$$

Since f' is minimal, the two composition are equal by a direct calculation using [AC, 4.1.2].

Since groupification of monoids has a right adjoint, we obtain a coequalizer

$$\begin{tikzcd} P_2^{gp} \arrow[r, shift left, "r_1^{gp}"] \arrow[r, shift right, "r_2^{gp}"'] & P_1^{gp} \arrow[r, "g^{gp}"] & \overline{\mathcal{M}}_p^{gp} \end{tikzcd} \tag{3.3.13}$$

Let $K = \ker g^{gp}$. Then $K \subset P_1^{gp}$ is the subgroup given by the image of the following

$$\gamma : P_2^{gp} \to P_1^{gp} \quad e \mapsto r_1^{gp}(e) - r_2^{gp}(e). \tag{3.3.14}$$

Restricting to $x \in \underline{C}_0$, the choice of $\{u_i\}$ as in (3.3.10) is equivalent to an element in

$$[s] = (u_i \cdot s_{i,x}) \in \mathrm{Hom}(P_1^{gp}, \mathbf{k}^*).$$

We have a commutative diagram

(3.3.15)
$$\begin{tikzcd}
& \mathrm{Hom}(P_1^{gp}, \mathbf{k}^*) \arrow[ld] \arrow[rd, "\gamma^\vee"] & \\
\mathrm{Hom}(K, \mathbf{k}^*) \arrow[rr] & & \mathrm{Hom}(P_2^{gp}, \mathbf{k}^*)
\end{tikzcd}$$

To finish the proof, it suffices to show the existence of $\{u_i\}$ such that $\gamma^\vee([s])$ is the trivial element. This follows from the surjectivity of the left skew arrow in the above diagram.

♠

Remark 3.12. The proof of Proposition 3.11 can be used to construct log comb for general log smooth target as in Proposition 3.5. But we will not carry out the details here, since it is not needed in the following discussions.

Proof of Theorem 3.1. By the log étale modification of [Niz06] and [ACMW], there is a birational log étale modification $\pi : Y \to X$ such that Y has simple normal crossings boundary. Consider the strata of Y which is birational to X_λ under π. Clearly if X_λ is a center, then all strata of Y birational X_λ are again centers. By Proposition 3.11 and [ACMW, Proposition 3.2], at least one stratum Y_λ over X_λ is a fully free support. We obtain the desired very free $\mathbb{A}^1$-curve in X by first producing a very free $\mathbb{A}^1$-curve with the corresponding properties using Proposition 3.9 for Y and Y_λ, and then taking the composition with π. ♠

The comb smoothing construction also provides us the following:

Corollary 3.13. *Let X be a proper log smooth variety with a free $\mathbb{A}^1$-support X_λ such that $X_\lambda^\dagger$ is separably $\mathbb{A}^1$-connected. Then for any positive number n, there exists an $\mathbb{A}^1$-curve passing through m general points on X.*

Proof. By the resolution argument, we may assume X has simple normal crossings boundary. Now the statement follows from attaching m free $\mathbb{A}^1$-curves to a very free $\mathbb{A}^1$-curve in $X_\lambda^\dagger$, and smoothing the comb with m usual markings on the m-teeth fixed. ♠

4. Integral points over function field of curves.

Let $\mathbb{F}$ be the function field of a proper, smooth, and irreducible algebraic curve B of genus g. Here we view B as the log scheme with the trivial log structure. Inspired by the work of Hassett and Tschinkel [HT08], we study the Zariski density of integral points of log smooth schemes over F. This can be also viewed as the logarithmic version of Graber-Harris-Starr [GHS03], and de Jong-Starr [dJS03].

Theorem 4.1. *Let $\pi : X \to B$ be a proper, flat morphism of log smooth schemes. Assume there is a stratum $X_\lambda \subset X$ such that*

(1) *The restriction* $\pi|_{X_\lambda} : X_\lambda \to B$ *is flat with general fiber smooth and separably rationally connected;*
(2) *General fibers of* $\pi|_{X_\lambda}$ *are fully very free centers of the corresponding fibers of* π.

Let $S \subset B$ *be any non-empty finite subset of points containing the images of the singularities of* π *and* $\pi|_{X_\lambda}$*, and the boundary of* $X_\lambda^\dagger$*. Then there exists a log map* $s : B^\dagger \to X$ *with* $\underline{B}^\dagger = B$ *and* $s(B \setminus S) \subset X^\circ$*, where* X° *is the interior of* X*. Furthermore, general deformations of* s *fixing* S *swept out an open dense subset of* X*.*

Proof. The proof of the above theorem in several steps, which is parallel to that of Theorem 3.1.

First consider a log étale modification $g : Y \to X$ such that Y has simple normal crossings boundary. Same as in the proof of Theorem 3.1, we observe that there is a stratum Y_λ birational to X_λ which satisfies assumptions (1) and (2) as in the statement for X_λ. Furthermore, the projection $Y_\lambda \to B$ is smooth away from the fiber of S. By replacing X with a resolution, we conclude that

Lemma 4.2. *To prove Theorem 4.1, it suffices to consider the case* X *with simple normal crossings boundary.*

For what follows, we assume X having simple normal crossings boundary.

Since the general fiber of $\overline{X_\lambda} \to B$ is separably rationally connected, by [dJS03] there is a section $\underline{f}_0 : B \to \overline{X_\lambda}$. This section $\underline{f}_0$ naturally defines a log map $f_0 : C_0 \to X_\lambda^\dagger$ with $\underline{C}_0 \cong B$, since no components fallen into the boundary of $X_\lambda^\dagger$. Since any S-integral section of $X \times \mathbb{P}^2 \to B$ will induce an integral section of $X \to B$. Replacing X by $X \times \mathbb{P}^2$, we may assume that $\dim \pi|_{X_\lambda} \geq 0$. We may play the gluing-smoothing argument as in [HT06, Proposition 24], and assume that

(1) f_0 is an immersion;
(2) the morphism $\mathrm{d}f_0 : \Omega_{X_\lambda^\dagger} \to \Omega_{C_0}$ is surjective, with locally free kernel N_{f_0};
(3) $H^1(N_{f_0}^\vee(-S)) = 0$, and $N_{f_0}^\vee(-S)$ is globally generated.

Note that the proof of the above argument is identical to the case of usual stable maps, since we do not glue components on the boundary of $X_\lambda^\dagger$.

Parallel to Proposition 3.5, we introduce a gluing-smoothing technique for the higher genus case.

Notation 4.3. let $\underline{f} : \underline{C} \to \underline{X}$ be a usual stable map over a geometric point such that

(1) $\underline{C}$ consists of irreducible components $\underline{C}_0 \cup \underline{C}_1 \cup \cdots \cup \underline{C}_m$, where $\underline{C}_0 \cong B$ is of genus g, and all other irreducible components are rational.

(2) For each $i \neq 0$, the irreducible component $\underline{C}_i$ is attached to $\underline{C}_0$ at a general point $p_i \in \underline{C}_0$, and there is no node on $\underline{C}$ other than p_i for $i = 1, \cdots, m$.
(3) $\underline{f}(\underline{C}_0) \subset \overline{X}_\lambda$, and $q_i = \underline{f}(p_i)$ is in general position of X_λ for all i.
(4) $\underline{f}(\underline{C}_i \setminus \{p_i\}) \in X^\circ$ for any $i \neq 0$.

Lemma 4.4. *Let $D \subset X$ be an irreducible smooth boundary divisor containing X_λ. Denote by $c_i(D) = \deg_{\underline{C}_i} f^*(D)$ for all i, and $c = \sum_i c_i(D)$. Fix any point $\sigma \in \underline{C}_0$. Assume that $c \geq 0$, and there is an isomorphism*

$$N_{D/X}|_{\underline{C}_0} \cong \mathcal{O}_{\underline{C}_0}(c \cdot \sigma - \sum_{i \geq 1} c_i \cdot p_i).$$

Then there is a log map $f_D : C/S_D \to X_D$ with a unique contact marking σ of contact order c, where X_D is the log scheme associated to the pair $(\underline{X}, D)$.

Proof. By assumption, we may choose a surjection

$$\mathcal{O}_{\underline{C}_0} \oplus N_{D/X}|_{\underline{C}_0} \to \mathcal{O}_{\underline{C}_0}(\sum_{i \geq 1} c_i \cdot p_i)$$

where the restriction to the first factor is given by the divisor $\sum_{i \geq 1} c_i \cdot p_i$, and to the second factor is given by $c \cdot \sigma$. This induces a morphism $\underline{C}_0 \to \mathbb{P}(\mathcal{O}_{\underline{C}_0} \oplus N_{D/X}|_{\underline{C}_0})$ tangent to D_∞ at σ of order c, and tangent to D_0 at p_i of order c_i. By the same argument as in Proposition 3.3, the section s induces a map to the expansion, hence a log map as we wanted. ♠

Let $D_1, \cdots, D_k$ be the smooth irreducible boundary divisors of X containing X_λ. Write $c_{ij} = \deg_{\underline{C}_i} f^*(D_j)$ for all i, j. We now form the matrix

$$A = (c_{ij})_{1 \leq i \leq m, 1 \leq j \leq k}.$$

Let J be the Jacobian of $\underline{C}_0$. We have a morphism

$$\phi_A : \underline{C}_0^m \to J^k \tag{4.0.16}$$

given by

$$(x_1, \cdots, x_m) \mapsto (\sum_{i \geq 1} c_{i1} \cdot (x_i - \sigma), \cdots, \sum_{i \geq 1} c_{ik} \cdot (x_i - \sigma)).$$

We first need the following result

Lemma 4.5. *Notations as above, fix k-uple of line bundles $\vec{L} = (L_1, \cdots, L_k)$ corresponding to a point $[\vec{L}] \in J^k$. Assume that m is sufficiently large, and the matrix A is general of rank k. Then the fiber $\phi_A^{-1}([\vec{L}])$ contains a general m-uple of points in $\underline{C}_0$.*

Proof. The case $g = 0$ is trivial. So we assume that $g \geq 1$. Consider the map

$$\phi : B \to J$$

defined by $x \mapsto (x - \sigma)$. Denote by $\mathrm{d}\phi : T_B \to \phi^* T_J$ the morphism of tangent bundles. Then we have $\mathrm{d}\phi_A = A \otimes \mathrm{d}\phi$. Since A is of rank k, we observe that $\mathrm{d}\phi_A$, hence the morphism ϕ_A is surjective.

By choosing sufficiently large m, and general A, we may assume the restriction

$$\mathrm{d}\pi_i : \ker \mathrm{d}\phi_A \to \pi_i^* T_B$$

is surjective for all $i \geq 1$, where $\pi_i : B^m \to B$ is the projection to the i-th factor. Thus, we deduce that the restriction

$$\pi_i : \phi_A^{-1}([\vec{L}]) \to B$$

is surjective for any i. This proves the statement. ♠

Applying Lemma 4.5 to

$$[\vec{L}] = ((N^\vee_{D_1/X})|_{\underline{C}_0} \otimes \mathcal{O}_{c_{01}\cdot\sigma}, \cdots, (N^\vee_{D_k/X})|_{\underline{C}_0} \otimes \mathcal{O}_{c_{01}\cdot\sigma}),$$

and combining Lemma 4.4 and the product argument in Proposition 3.5, we obtain

Lemma 4.6. *Consider the situation of Notation 4.3. Assume that the restriction* $\underline{f}|_{\underline{C}_i} : \underline{C}_i \to \underline{X}$ *is obtained from an* $\mathbb{A}^1$*-curve with sufficiently general contact orders, and* m *is sufficiently large. Then there is a log map* $f : C/S \to X$ *lifts* $\underline{f}$*.*

By the assumption (2) of Theorem 4.1, we may choose $\underline{f}|_{\underline{C}_i} : \underline{C}_i \to \underline{X}$ to be very free $\mathbb{A}^1$-curves of the fiber of π to fulfill the conditions in the above lemma. Furthermore, we may assume that $\underline{f}$ is a local immersion away from the special points. We are now in the situation of (3.3.7). By attaching sufficiently many very free $\mathbb{A}^1$-curves, we may assume that $H^1(N_f^\vee(-S)) = 0$ and $N_f^\vee(-S)$ is globally generated. Theorem 4.1 follows from taking a general deformation of f. ♠

5. Wonderful compactifications of semisimple groups

5.1. **The adjoint case.** In this subsection, we prove that the wonderful compactification of a semisimple algebraic group of adjoint type is $\mathbb{A}^1$-connected.

Notation 5.1. We follow basic notations from [BK05, Chapter 6]. Let G be a simply connected semisimple algebraic $\mathbf{k}$-group. Let X be the wondeful compactification of G_{ad}, which is smooth with simple normal crossing boundary. Let $X^\dagger$ be the log variety with underlying variety X and divisorial log structure on the boundary divisor. Let T be a maximal torus of G and B be a Borel subgroup containing T. Let r be the rank of G, i.e., the rank of the character lattice $\mathbb{X}^*(T)$.

(1) Let $X_1, \cdots, X_r$ be the irreducible components of the simple normal crossing boundary divisor $X \backslash G_{ad}$. Let Y be the common intersection of X_i's, which is isomorphic to $G/B \times G/B$. Write $\iota : Y \to X$ the embedding.

(2) Let D_i be the closure of a codimension one Bruhat cell $\overline{Bs_iB^-}$ for $i = 1, \cdots, r$.

Theorem 5.2. *With the notations as in 5.1 and assuming that* $\operatorname{char} \mathbf{k}$ *is zero, the wonderful compactification* $X^\dagger$ *of a semisimple linear algebraic group* G_{ad} *of adjoint type is* $\mathbb{A}^1$*-connected. Furthermore, the unique closed* $G \times G$*-orbit is a fully very free center.*

We recall two basic propositions for wonderful compactifications of adjoint groups.

Proposition 5.3 ([BK05], Prop. 6.1.11)**.** *(1) The restriction map* $\iota^* : \operatorname{Pic} X \to \operatorname{Pic} Y$ *is injective and the image consists of the classes* $\mathcal{L}_Y(\lambda) = \mathcal{L}_Y(-w_0\lambda) \boxtimes \mathcal{L}_Y(\lambda)$*, where* $\lambda \in \mathbb{X}^*(T)$*. In particular, we use* $\mathcal{L}_X(\lambda)$ *to denote the inverse image of* $\mathcal{L}_Y(\lambda)$*.*

(2) $\mathcal{O}_X(D_i) = \mathcal{L}_X(\chi_i)$ *and* $\mathcal{O}_X(X_i) = \mathcal{L}_X(\alpha_i)$*, for* $i = 1, \cdots, r$*, where* χ_i*'s are the fundamental weights and* α_i*'s are the positive simple roots.*

(3) $\mathcal{L}_X(\lambda)$ *is ample if and only if* λ *is regular dominant, i.e., a positive linear combination of fundamental weights. The nef cone is a smooth cone in* $X^*(T)$*.*

(4) $\mathcal{L}_X(\lambda)$ *is effective if and only if* λ *lies in the cone generated by* α_i*'s. The effective cone is a simplicial cone in* $\mathbb{X}^*(T)$*.* ♠

Proposition 5.4 ([Bri], Thm. 2)**.** *There exist* B*-invariant rational curves* $B_1, \cdots, B_r$ *on* Y *such that the matrix given by*

$$(D_i.\iota_*(B_j)) = (\iota^*\mathcal{L}_X(\chi_i).B_j) = (\mathcal{L}_Y(\chi_i).B_j) = \delta_{ij}. \quad ♠$$

Proposition 5.5. *There exists rational curves* $C_1, \cdots, C_r$ *in* Y *such that*

$$(X_i.\iota_*(C_j)) = d\delta_{ij},$$

where d *is the determinant of the Cartan matrix associated to* G*. In particular,* d *is a positive integer.*

Proof. Let $C = (c_{ij})$ be the Cartan matrix associated to the root system of G. We know that the simple roots and the fundamental weights are related by the Cartan matrix. In particular,

$$X_i = \sum_j D_j c_{ji}.$$

Let $C^* = (c^*_{kl})$ be the adjoint matrix of the Cartan matrix. Let β_k be the curve class $\sum_l a^*_{kl}[B_l]$ in Y. By [LT92], C^* is a positive integral matrix and $\det C$ is a positive integer. Therefore we can find a rational curve C_k with the curve class β_k. Then we have

$$X_i.\iota_*(C_k) = \sum_j D_j c_{ji}. \sum_l a^*_{kl}\iota_*(B_k) = \sum_{k,l} a_{ji}a^*_{kl}\delta_{jl} = \sum_k a_{ji}a^*_{kj} = \det(C)\delta_{ik}.$$

Proof of Theorem 5.2. Since Y is a projective homogeneous space, by Theorem 3.1, it suffices to prove that Y is a fully free support. By taking positive integer combinations of C_i's, we may find rational curves $\tilde{C}_1, \cdots, \tilde{C}_r$ on Y such that $c_{ij} = X_j.\iota_*(\tilde{C}_i)$ is positive for all i's and j's. By Proposition 3.5, for each $\tilde{C}_i$, we could associate a log structure, and lift it to an $\mathbb{A}^1$-curve, still denoted by $\tilde{C}_i \to X^\dagger$. Since the wonderful compactification of an adjoint group is log homogeneous, i.e., the log tangent bundle is globally generated [Bri12, Thm. 5.1], $\tilde{C}_i$ is unobstructed and hence can be deformed to a free $\mathbb{A}^1$-curve whose contact order is $(c_{i1}, \cdots, c_{ir})$. In characteristic zero, we can easily choose $\tilde{C}_i$'s such that the matrix of contact orders (c_{ij}) is invertible. ♠

5.2. **The general case.** We next study the $\mathbb{A}^1$-connectedness for wonderful compactifications of semisimple groups.

Let G be a connected semisimple linear algebraic $\mathbf{k}$-group, and $G_{ad} := G/Z$ be the associated adjoint group, where Z is the center of G. Fix a Boral subgroup $B \subset G_{ad}$ with the maximal torus $T \subset B$. Let $\overline{G}_{ad}$ be the wonderful compactification of G_{ad}, and $\overline{G}$ be a toroidal embedding of G [BK05, 6.2.2] with the equivalent finite morphism of usual schemes

$$\overline{G} \to \overline{G}_{ad}. \tag{5.2.1}$$

Denote by $\partial\overline{G} = \overline{G} \setminus G$ and $\partial\overline{G}_{ad} = \overline{G}_{ad} \setminus G_{ad}$. Since the pairs $(\overline{G}, \partial\overline{G})$ locally having only toroidal singularities, and $(\overline{G}, \partial\overline{G}_{ad})$ is a simple normal crossings pair, we may view the pairs as log smooth schemes, and still denote them by $\overline{G}$ and $\overline{G}_{ad}$ respectively.

Lemma 5.6 ([BK05], Prop. 6.2.3)**.** *The morphism as log schemes*

$$\overline{G} \to \overline{G}_{ad}$$

is Kummer étale in characteristic zero. ♠

Theorem 5.7. *Toroidal G-embeddings are separably $\mathbb{A}^1$-connected in characteristic zero, with a fully very free center.*

Proof. Note that $\overline{G}$ has a unique center Y' which is mapped isomorphically to the center Y' of $\overline{G}_{ad}$. By Theorem 5.2, we could find a set of very free $\mathbb{A}^1$-curves on $\overline{G}_{ad}$, whose contact orders forms a basis of $(N_{Y'})_{\mathbf{k}}$, where N_Y is the lattice associated to Y as in Section 2.3. By Lemma 5.6, the set of $\mathbb{A}^1$-curves on $\overline{G}$ can be lift to $\mathbb{A}^1$-curves on $\overline{G}_{ad}$. Their contact orders again form a basis of $(N_Y)_{\mathbf{k}}$, since the morphism $N_{Y'} \to N_Y$ is a refinement of lattices. ♠

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