Artin fans

AMS special session on Combinatorics and Algebraic Geometry

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October 24, 2014

Heros:

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- Martin Olsson

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- Jonathan Wise
- Qile Chen, Steffen Marcus,
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- Martin Ulirsch

Superabundance

Mikhalkin-Speyer: there is a tropical cubic curve C of genus 1 in TP^3 which does not lift to an algebraic curve (Speyer, *Tropical Geometry*, Berkeley thesis 2005, Figure 5.1).

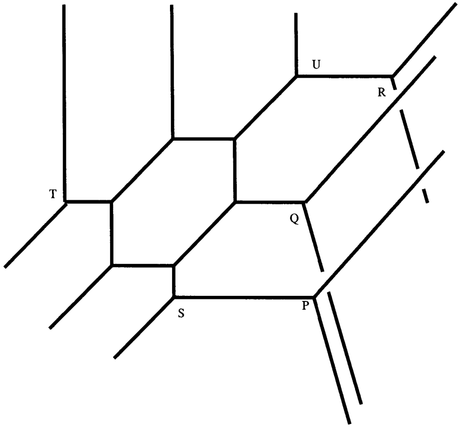

Figure 5.1: A Genus 1 Zero Tension Curve which is not Tropical

Superabundance (continued)

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- But logarithmic stable maps are **obstructed**.

Question

Is there a world in which they are not obstructed?

Logarithmic structures

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Definition

It is a *logarithmic structure* if $\alpha : \alpha^{-1}\mathcal{O}^*_{\underline{X}} \to \mathcal{O}^*_{\underline{X}}$ is an isomorphism.

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$$M_D(U) = \left\{ f \in \mathcal{O}_{\underline{X}}(U) \;\middle|\; f_{U \smallsetminus D} \in \mathcal{O}^\times_{\underline{X}}(U \smallsetminus D) \right\}.$$

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- Let k be a field,

$$\begin{array}{ccc} \mathbb{N} \oplus k^\times & \to & k \\ (n, z) & \mapsto & z \cdot 0^n \end{array}$$

defined by sending $0 \mapsto 1$ and $n \mapsto 0$ otherwise.

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- A nodal curve is logarithmically smooth over a logarithmic point.

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These can be nonzero on a broken cubic curve!

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Proposition (Wise; ℵ, Chen, Marcus)

There is an initial factorization $X \to \mathcal{A}_X \to \text{Log}$ *such that* $\mathcal{A}_X \to \text{Log}$ *is étale, representable, strict.*

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The stack $\mathcal{A}_X$ is small, totally combinatorial.

$\mathbb{P}^3$ and $\mathcal{A}_{\mathbb{P}^3}$

$$\mathbb{P}^3 = (\mathbb{A}^4 \smallsetminus \{0\})/\mathbb{G}_m.$$

So

$$\{C \to \mathbb{P}^3\} \leftrightarrow \{(\mathcal{L}, s_0, \ldots, s_3) | s_i \text{ do not vanish together}\}.$$

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The obstructions are gone!

Sample theorem

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If $Y \to X$ is a toric modification, then

> *Logarithmic Gromov–Witten invariants of X coincide with those of Y.*

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Reason: $\mathfrak{M}(\mathcal{A}_Y) \to \mathfrak{M}(\mathcal{A}_X)$ is birational. So $\overline{\mathcal{M}}(Y) \to \overline{\mathcal{M}}(X)$ is virtually birational.

Tropicalization

Things are connected in Martin Ulirsch's fundamental commutative diagram:

$$\begin{array}{ccccc}
X^{\beth} & \longrightarrow & \mathcal{A}_X^{\beth} & \longrightarrow & \Sigma_X \\
{\scriptstyle \rho_X} \downarrow \downarrow {\scriptstyle r_X} & & {\scriptstyle \rho_{\mathcal{A}}} \downarrow \downarrow {\scriptstyle r_{\mathcal{A}}} & & {\scriptstyle \rho_{\Sigma}} \downarrow \downarrow {\scriptstyle r_{\Sigma}} \\
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- $\mathbb{P}^3$ and $\mathcal{A}_{\mathbb{P}^3}$ share their tropicalization $\overline{TP}^3$.
- $\mathcal{A}_X^\beth \to \Sigma_X$ **is a homeomorphism**.