NOTES ON GLUING FORMULAS FOR LOG GW INVARIANTS

Contents

1. The tropical point of view and splittings

For a scheme X or Deligne-Mumford stack $\mathcal{X}$, we denote by $|X|$ or $|\mathcal{X}|$ the topological space of geometric points of X or $\mathcal{X}$. Given a geometric point x of $\mathcal{X}$, by an étale neighbourhood of x we mean the data of an étale morphism $U \to \mathcal{X}$ with U a scheme and $x_U \in |U|$ a geometric point mapping to x. We write this as (U, x_U).

Definition 1.1. Let $\mathcal{X}$ be a Deligne-Mumford stack. We define the *category of points* of $\mathcal{X}$ to be the category $\mathrm{Points}_{\mathcal{X}}$ whose objects are geometric points of $\mathcal{X}$. If x_1, x_2 are two geometric points of $\mathcal{X}$, then an element α of $\mathrm{Hom}(x_1, x_2)$ consists of an assignment

$$(U, x_{2,U}) \mapsto \alpha(U, x_{2,U}) \in |U|$$

which associates to every étale neighbourhood $(U, x_{2,U})$ of x_2 a point $x_{1,U} = \alpha(U, x_{2,U}) \in |U|$ mapping to x_1 and with $x_{2,U} \in \mathrm{cl}\{x_{1,U}\}$. We require this correspondence to satisfy that given $(U', x_{2,U'}) \to (U, x_{2,U}) \to \mathcal{X}$, the image of $x_{1,U'}$ in U should be $x_{1,U}$.

Composition of morphisms is defined, for $\alpha_1 \in \mathrm{Hom}(x_1, x_2)$, $\alpha_2 \in \mathrm{Hom}(x_2, x_3)$, by $(\alpha_2 \circ \alpha_1)(U, x_{3,U}) = \alpha_1(U, \alpha_2(U, x_{3,U}))$.

If $f : \mathcal{X} \to \mathcal{Y}$ is a representable morphism, there is a functor

$$\mathrm{Points}_f : \mathrm{Points}_{\mathcal{X}} \to \mathrm{Points}_{\mathcal{Y}}.$$

This map is the obvious map on points for objects, and, if $y_2 = f(x_2)$, $\alpha \in \mathrm{Hom}(x_1, x_2)$,

$$\mathrm{Points}_f(\alpha)(U, y_{2,U}) = \text{the composition } \alpha(\mathcal{X} \times_{\mathcal{Y}} U, x_{2,U}) \to \mathcal{X} \times_{\mathcal{Y}} U \to U,$$

where $x_{2,U}$ is the geometric point of the fibred product induced by $x_2 \in |\mathcal{X}|$ and $y_{2,U} \in |U|$.

Example 1.2. If X is a nodal cubic, let x_2 be the nodal point and x_1 the geometric generic point. Then $\mathrm{Hom}(x_1, x_2)$ consists of two morphisms, corresponding to the two branches of X passing through x_2. The whole point of this definition is to take care of different branches.

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Now suppose that $\mathcal{X}$ has a fs log structure, hence for each $x \in \mathrm{Ob}(\mathrm{Points}_{\mathcal{X}})$ a stalk $\overline{\mathcal{M}}_{\mathcal{X},x}$, and for each $x_1 \to x_2$ a generization map $\overline{\mathcal{M}}_{\mathcal{X},x_2} \to \overline{\mathcal{M}}_{\mathcal{X},x_1}$. This gives a covariant functor

$$\mathrm{Trop}_{\mathcal{X}} : \mathrm{Points}_{\mathcal{X}} \to \mathrm{Cones},$$

where Cones is the category of strictly convex rational polyhedral cones, with morphisms being integral maps of cones.[1] Here

$$\mathrm{Trop}_{\mathcal{X}}(x) = \mathrm{Hom}(\overline{\mathcal{M}}_{\mathcal{X},x}, \mathbb{R}_{\geq 0})$$

and $\mathrm{Trop}_{\mathcal{X}}(x_1 \to x_2)$ is induced by the generization map. Note such maps are always inclusions of faces.

Note this construction is functorial: given a representable morphism of Deligne-Mumford fs log stacks $f : \mathcal{X} \to \mathcal{Y}$, we obtain a natural transformation of functors

$$\mathrm{Trop}_f : \mathrm{Trop}_{\mathcal{X}} \to \mathrm{Trop}_{\mathcal{Y}} \circ \mathrm{Points}_f .$$

In relatively nice cases, instead of using the functor $\mathrm{Trop}_{\mathcal{X}}$ it is nicer to use the topological space $\lim \mathrm{Trop}_{\mathcal{X}}$, which is a complex of polyhedral cones. For example, if $C \to (\mathrm{Spec}\, \Bbbk, Q)$ is a log smooth curve, then we obtain a continuous map

$$\lim \mathrm{Trop}_C \to \lim \mathrm{Trop}_{(\mathrm{Spec}\, \Bbbk, Q)} = \mathrm{Hom}(Q, \mathbb{R}_{\geq 0})$$

which is a family of tropical curves.

Recall that given a monoid P we have an algebraic stack

$$\mathfrak{T}_P := [\mathrm{Spec}\, \Bbbk[P] / \mathrm{Spec}\, \Bbbk[P^{\mathrm{gp}}]],$$

and given a surjection $P_1 \to P_2$ one obtains an open embedding

$$\mathfrak{T}_{P_2} \to \mathfrak{T}_{P_1}.$$

Given $\mathcal{X}$, we can also define

$$\mathfrak{T}_{\mathcal{X}} := \lim_{x \in |\mathcal{X}|} \mathfrak{T}_{\overline{\mathcal{M}}_{\mathcal{X},x}}.$$

This is a direct limit over the category $\mathrm{Points}_{\mathcal{X}}$. There is a natural map $\mathcal{X} \to \mathfrak{T}_{\mathcal{X}}$ through which the map $\mathcal{X} \to \mathcal{L}og_{\Bbbk}$ factors.

[NOTE: I would like to see a proof the direct limit is really an algebraic stack; I'm still not so comfortable with stacky arguments. Also, I should note this is in general slightly different then Qile's version; I'm not sure if this makes a difference.]

The general situation we wish to consider is where we have a Deligne-Mumford fs log stack $\mathcal{X}$ over the standard log point $\mathrm{Spec}\, \Bbbk^{\dagger} = (\mathrm{Spec}\, \Bbbk, \mathbb{N})$. If $\mathcal{X}$ were in fact log smooth over $\mathrm{Spec}\, \Bbbk^{\dagger}$ and relatively well-behaved, then we would expect the irreducible components

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[1] If M and M' are lattices, $\sigma \subseteq M_{\mathbb{R}}$, $\sigma' \subseteq M'_{\mathbb{R}}$ strictly convex rational polyhedral cones, then an integral linear map of cones $\sigma \to \sigma'$ is a map induced by a linear map $M \to M'$.

of $\mathcal{X}$ to be in one-to-one correspondence with rays of $\lim \mathrm{Trop}_{\mathcal{X}}$ mapping surjectively to $\lim \mathrm{Trop}_{\mathrm{Spec}\,\Bbbk^\dagger} = \mathbb{R}_{\geq 0}$. In what follows, we make this precise in a virtual sense.

Definition 1.3. Suppose given a log morphism $\pi : \mathcal{X} \to \mathrm{Spec}\,\Bbbk^\dagger$ with $\mathcal{X}$ a Deligne-Mumford fs log stack. Define an equivalence relation $\sim$ on the set

$$\left\{ \rho \subseteq \mathrm{Trop}_{\mathcal{X}}(x) \;\middle|\; \begin{array}{c} x \in |\mathcal{X}|,\ \rho \text{ a one-dimensional face of } \mathrm{Trop}_{\mathcal{X}}(x), \\ \mathrm{Trop}_{\pi}(x) : \rho \to \mathrm{Trop}_{(\mathrm{Spec}\,\Bbbk,\mathbb{N})} = \mathbb{R}_{\geq 0} \text{ surjective} \end{array} \right\}$$

which is generated by the relation $\rho_1 \sim \rho_2$ if $\rho_i \subseteq \mathrm{Trop}_{\mathcal{X}}(x_i)$ and $\rho_2 = \mathrm{Trop}_{\mathcal{X}}(\alpha)(\rho_1)$ for $\alpha \in \mathrm{Hom}(x_1, x_2)$.

Given an equivalence class Ξ, its *degree* d_Ξ is the positive integer such that the map $\rho \to \mathbb{R}_{\geq 0}$ induced by $\mathrm{Trop}(\pi)$, for $\rho \in \Xi$, takes the value d_Ξ on a primitive generator of ρ. Note this is independent of the choice of $\rho \in \Xi$.

A *splitting* of $\mathcal{X}$ is a union of equivalence classes $\Xi = \bigcup_i \Xi_i$ such that $d_\Xi := d_{\Xi_i}$ is independent of i. We say a splitting is *primitive* if it just consists of a single equivalence class.

We will now define a category of points associated to a choice of splitting:

Definition 1.4. Let Ξ be a splitting of $\mathcal{X}$. We define $\mathrm{Points}_{\mathcal{X},\Xi}$ to be the category whose objects are pairs (x, ρ) with $x \in |\mathcal{X}|$ and $\rho \subseteq \mathrm{Trop}_{\mathcal{X}}(x)$ with $\rho \in \Xi$. An element α of $\mathrm{Hom}((x_1, \rho_1), (x_2, \rho_2))$ is an element $\alpha \in \mathrm{Hom}(x_1, x_2)$ with the property that $\mathrm{Trop}_{\mathcal{X}}(\alpha)(\rho_1) = \rho_2$.

Now given a monoid P and a one-dimensional face $\rho \subseteq \mathrm{Hom}(P, \mathbb{R}_{\geq 0})$, let $q_\rho \in \rho$ be a primitive generator. We define a monoid ideal

$$I_d := \{p \in P \mid \langle q_\rho, p \rangle \geq d\} \subseteq P,$$

and define

$$\mathfrak{T}^d_{P,\rho} := [(\mathrm{Spec}\,\Bbbk[P]/I_d)/\,\mathrm{Spec}\,\Bbbk[P^{\mathrm{gp}}]].$$

Then for a splitting Ξ define

$$\mathfrak{T}_{\mathcal{X},\Xi} := \lim_{(x,\rho)\in \mathrm{Ob}(\mathrm{Points}_{\mathcal{X},\Xi})} \mathfrak{T}^{d_\Xi}_{\overline{\mathcal{M}}_{\mathcal{X},x},\rho}$$

$$\mathfrak{Q}_{\mathcal{X},\Xi} := \lim_{(x,\rho)\in \mathrm{Ob}(\mathrm{Points}_{\mathcal{X},\Xi})} \mathfrak{T}^{1}_{\overline{\mathcal{M}}_{\mathcal{X},x},\rho}$$

Here the limit is over the category $\mathrm{Points}_{\mathcal{X},\Xi}$. There are natural maps

$$\mathfrak{Q}_{\mathcal{X},\Xi} \to \mathfrak{T}_{\mathcal{X},\Xi} \to \mathfrak{T}_{\mathcal{X}}.$$

We then define

$$\mathcal{X}_\Xi := \mathcal{X} \times_{\mathfrak{T}_{\mathcal{X}}} \mathfrak{T}_{\mathcal{X},\Xi}$$

and

$$\mathcal{X}^{\mathrm{red}}_\Xi := \mathcal{X} \times_{\mathfrak{T}_{\mathcal{X}}} \mathfrak{Q}_{\mathcal{X},\Xi}.$$

Example 1.5 (The standard degeneration situation)**.** Suppose we are given a flat log smooth family $g : X \to B$ over a one-dimensional base B, where the log structure on B is the divisorial one coming from $0 \in B$. Assume further that X_0 is reduced. Let β be a class of stable log map (as in [1], Definition 3.1) and let $\mathscr{M}(X/B, \beta)$ be the moduli space of basic stable log maps over B of class β. We have the map $\mathscr{M}(X/B,\beta) \to B$, and $\mathscr{M}(X_0/0^\dagger, \beta) = \mathscr{M}(X/B,\beta) \times_B 0^\dagger$. The basic example we would like to apply the above discussion to is $\mathscr{M} := \mathscr{M}(X_0/0^\dagger,\beta) \to 0^\dagger$. We have a commutative diagram of DM stacks

$$\begin{tikzcd}
\mathscr{V} \arrow[r, "f"] \arrow[d, "v"'] & X_0 \arrow[d, "g"] \\
\mathscr{M} \arrow[r, "\pi"'] & 0^\dagger
\end{tikzcd} \tag{1.1}$$

where $f : \mathscr{V} \to X_0$ is the universal stable log map of class β.

We will now explain a natural way to describe splittings of $\mathscr{M}$ in this context. We write

$$X_0 = \coprod_{Y \in S(X_0)} Y$$

where $S(X_0)$ is the set of *(open) strata* of X_0, maximal locally closed subschemes on which the sheaves $\overline{\mathcal{M}}_X|_Y$ are locally constant. In what follows, we shall assume that $\lim \mathrm{Trop}_{X_0}$ makes sense as a reasonably well-behaved topological space [MAKE PRECISE WHAT THE NECESSARY HYPOTHESES ARE?] so that there is a one-to-one correspondence between elements of $S(X_0)$ and cones of $\lim \mathrm{Trop}_{X_0}$ which map non-trivially to $\mathrm{Trop}_{0^\dagger} = \mathbb{R}_{\geq 0}$. Note that because of log smoothness, this is always true étale locally. Indeed, log smoothness implies that for $x \in |X_0|$, there is an étale neighbourhood U of $x \in X_0$ and a diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & \operatorname{Spec} \Bbbk[\overline{\mathcal{M}}_{X,x}] \arrow[d] \\
0^\dagger \arrow[r] & \operatorname{Spec} \Bbbk[\mathbb{N}]
\end{tikzcd}$$

where the right-hand vertical arrow is given by the map $\mathbb{N} \to \overline{\mathcal{M}}_{X,x}$, $1 \mapsto \rho$, and the induced map of ordinary schemes $U \to \operatorname{Spec} \Bbbk[\overline{\mathcal{M}}_{X,x}] \times_{\operatorname{Spec} \Bbbk[\mathbb{N}]} 0$ is smooth. Thus the strata of U are in one-to-one correspondence with the faces of $\overline{\mathcal{M}}_{X,x}$ not containing ρ. These in turn are in one-to-one inclusion reversing correspondence with the faces of $\mathrm{Trop}_{X_0}(x)$ which map to $\mathrm{Trop}_{0^\dagger}$ non-trivially.

In this discussion, we shall write Trop_{X_0} rather than $\lim \mathrm{Trop}_{X_0}$.

Under these assumptions, if σ is a cone of Trop_{X_0}, we denote by Y_σ the corresponding stratum of X_0. Note that if $\sigma_1 \subseteq \sigma_2$, then $Y_{\sigma_2} \subseteq \mathrm{cl}(Y_{\sigma_1})$.

Recall that β consists of the usual data for class of stable map: homology class, genus, and number of marked points, and in addition strict closed embeddings $Z_1, \dots, Z_r \subseteq X_0$ along with sections $s_i \in \Gamma(Z_i, (\overline{\mathcal{M}}^{\mathrm{gp}}_{Z_i})^*)$ which are used to determine tangency conditions.

Definition 1.6. A *splitting* Ξ *of the class* β is the following data: A graph Γ, along with maps

$$\begin{aligned} S_\Xi :& \Gamma^{[0]} \cup \Gamma^{[1]} \to S(X_0), \\ \beta_\Xi :& \Gamma^{[0]} \to \coprod_{Y \in S(X_0)} H_2(\mathrm{cl}(Y), \mathbb{Z}) \\ g_\Xi :& \Gamma^{[0]} \to \mathbb{N}. \end{aligned}$$

For $\tau \in \Gamma^{[0]} \cup \Gamma^{[1]}$, let P_τ denote the stalk of $\overline{\mathcal{M}}_X$ at the generic point of the stratum $S_\Xi(\tau)$. Then we also specify for every oriented bounded edge e_q,

$$u_q \in (P_q^{\mathrm{gp}})^* \cap g^\flat(1)^\perp.$$

Here $1 \in \mathbb{N}$ is the generator of $\overline{\mathcal{M}}_{0^\dagger}$. This data satisfies:

1. $\beta_\Xi(v) \in H_2(\mathrm{cl}(S_\Xi(v)), \mathbb{Z})$.
2. The homology class of the type β is $\sum_{v \in \Gamma^{[0]}} i_{v*}\beta_{\gamma(v)}$, where $i_v : \mathrm{cl}(S_\Xi(v)) \to X_0$ is the inclusion.
3. The genus g of the type β is $b_1(\Gamma) + \sum_{v \in \Gamma^{[0]}} g_\Xi(v)$.
4. If the type β specifies k marked points, then Γ has k unbounded edges and for an unbounded edge e_{p_i}, $S_\Xi(e_{p_i}) \cap Z_i \neq \emptyset$.
5. The data of u_q define a monoid Q_Ξ as a saturated quotient of $\prod_{v \in \Gamma^{[0]}} P_v \times \prod_q \mathbb{N}$ as usual. We require that $Q_\Xi \cong \mathbb{N}$.

□

We will now see how a primitive splitting of $\mathscr{M}$ in the sense of Definition 1.3 gives rise to a splitting of the class β.

Given a point $x \in |\mathscr{M}|$, we obtain a stable map $f : \mathscr{V}_x \to X_0$, and a map

$$\mathrm{Trop}_v : \mathrm{Trop}_{\mathscr{V}_x} \to \mathrm{Trop}_{\mathscr{M}}(x) = \mathrm{Hom}(\overline{\mathcal{M}}_{\mathscr{M},x}, \mathbb{R}_{\geq 0}).$$

We can view a fibre $\Gamma_y := \mathrm{Trop}_v^{-1}(y)$ as a graph Γ, with set of vertices $\Gamma^{[0]}$ and edges $\Gamma^{[1]}$. We then obtain a continuous map

$$\mathrm{Trop}_f : \Gamma_y \to \mathrm{Trop}_{X_0}.$$

Suppose $y \in \mathrm{Trop}_{\mathscr{M}}(x)$, $y \in \mathrm{Trop}_\pi^{-1}(1)$. For each vertex $v \in \Gamma_y^{[0]}$, there is a minimal cone σ of Trop_{X_0} containing $\mathrm{Trop}_f(v)$. By commutativity of the tropicalization of the diagram (1.1), σ maps non-trivially via Trop_π, and hence σ corresponds to a stratum Y_σ. Set

$S_y(v) = Y_\sigma$. For each edge $e \in \Gamma_y^{[1]}$, it is easy to see that $\mathrm{Trop}_f(e)$ is contained in a cone of Trop_{X_0}. Taking σ to be the smallest such cone, we set $S_y(e) = Y_\sigma$. Thus we obtain a map

$$S_y : \Gamma_y^{[0]} \cup \Gamma_y^{[1]} \to S(X_0).$$

For $y \in \mathrm{Int}(\mathrm{Trop}_{\mathscr{M}}(x))$, it follows from the construction of $\mathrm{Trop}_{\mathscr{V}_x}$ that the vertices $\Gamma_y^{[0]}$ are in one-to-one correspondence with the irreducible components of $\mathscr{V}_x$. The bounded edges of $\Gamma_y^{[1]}$ are in one-to-one correspondence with nodes of $\mathscr{V}_x$ and the unbounded edges with marked points of $\mathscr{V}_x$. In particular, the topology of Γ_y only depends on x, so we write this graph as Γ_x. In addition, S_y is independent of y, so we write S_x.

Note that if C_v is the irreducible component of $\mathscr{V}_x$ corresponding to $v \in \Gamma_x^{[0]}$, then $f(C_v) \subseteq \mathrm{cl}(S_x(v))$. Define

$$\beta_x(v) := f_*[C_v] \in H_2(\mathrm{cl}(S_x(v)), \mathbb{Z})$$

and $g_x(v) = g(\tilde{C}_v)$, where $\tilde{C}_v$ is the normalization of C_v.

It follows from the definition of basicness that $\overline{\mathcal{M}}_{\mathscr{M},x}$ is a saturated quotient of $\prod_\eta P_\eta \times \prod_q \mathbb{N}$, with η running over generic points of $\mathscr{V}_x$ and q running over double points of $\mathscr{V}_x$. Let ρ_q denote the image of the generator of the copy of $\mathbb{N}$ corresponding to q in $\overline{\mathcal{M}}_{\mathscr{M},x}$. It then follows easily that if $y \in \mathrm{Trop}_{\mathscr{M}}(x)$, then $\Gamma_y = \mathrm{Trop}_v^{-1}(y)$ is obtained from Γ_x by contracting all bounded edges with $\rho_q(y) = 0$. In particular, there is a contraction map $\phi_y : \Gamma_x \to \Gamma_y$.

Note that if there is a face σ of $\mathrm{Trop}_{\mathscr{M}}(x)$ which is the image of $\mathrm{Trop}_{\mathscr{M}}(x')$ for some generization $x' \in |\mathscr{M}|$ of x, then for $y \in \mathrm{Int}(\sigma)$, we have $\Gamma_{x'} = \Gamma_y$. Thus in this case Γ_y is the dual intersection graph of the curve $\mathscr{V}_{x'}$, but in general Γ_y need not be the dual intersection graph of any stable log map.

Given $y \in \mathrm{Trop}_{\mathscr{M}}(x)$, $y \in \mathrm{Trop}_\pi^{-1}(1)$, we can associate data S_y, g_y, β_y of Definition 1.6. We have already defined S_y. There is a map of graphs $\phi_y : \Gamma_x \to \Gamma_y$ as described above, and Γ_x is the dual intersection graph of $C := \mathscr{V}_x$. For $v \in \Gamma_y^{[0]}$, we define C_v to be a partial normalization of a subcurve of C. The subcurve is defined by taking components whose corresponding vertices appear in $\phi_y^{-1}(v)$, and a node of this subcurve is normalized if and only if the corresponding edge of Γ_x is not contracted by ϕ_y. We define $g_y(v) = g(C_v)$. Noting further that if $v' \in \phi_y^{-1}(v)$, one has $S_x(v') \subseteq \mathrm{cl}(S_y(v))$, so that $f(C_v) \subseteq \mathrm{cl}(S_y(v))$. Set $\beta_y(v) = f_*[C_v] \in H_2(\mathrm{cl}(S_y(v)), \mathbb{Z})$.

If $F \subseteq \mathrm{Trop}_{\mathscr{M}}(x)$ is the smallest face containing y, then F can be described as follows. First note that for every bounded edge e_q corresponding to a node q of $\mathscr{V}_x$, we have a corresponding u_q defined as usual after orienting the edge e_q. By definition, $u_q \in \mathrm{Hom}(P_q, \mathbb{Z})$, and $P_q = P_\sigma$ where $\sigma \subset \mathrm{Trop}_{X_0}$ is the smallest cone containing $\mathrm{Trop}_f(e_q)$. If e_q is not contracted by ϕ_y, then necessarily $u_q \in \mathrm{Hom}(P_{\sigma'}, \mathbb{Z})$, where σ' is the smallest cone of Trop_{X_0}

containing $\mathrm{Trop}_f(\phi_y(e_q))$. Thus we can define a monoid Q_y as the saturated quotient of $\prod_{v\in\Gamma_y^{[0]}} P_v \times \prod_q \mathbb{N}$ as usual, and one sees as usual that $F = \mathrm{Hom}(Q_y, \mathbb{R}_{\geq 0})$.

From this one concludes that a one-dimensional face of $\mathrm{Trop}_{\mathscr{M}}(x)$ mapping non-trivially to $\mathrm{Trop}_{0^\dagger}$ defines a splitting of β. Equivalent one-dimensional faces in the sense of Definition 1.3 yield the same splitting of β. However, it is possible that non-equivalent one-dimensional cones in $\mathrm{Trop}_{\mathscr{M}}$ yield the same splitting of β. Thus a splitting of the type β yields a splitting in the sense of Definition 1.3, but it may not be a primitive splitting.

Example 1.7 (Mikhalkin's setup for toric varieties)**.** Let Σ be a complete fan in $M_{\mathbb{R}}$, with $M \cong \mathbb{Z}^n$, defining a toric variety X_Σ, and set $N = \mathrm{Hom}(M, \mathbb{Z})$. Consider the trivial family $X = X_\Sigma \times \mathbb{A}^1 \to \mathbb{A}^1$, with the canonical log structures on X and $\mathbb{A}^1$ as toric varieties, i.e., the divisorial log structures $((\partial X_\Sigma) \times \mathbb{A}^1) \cup (X_\Sigma \times \{0\}) \subset X$ and $0 \in \mathbb{A}^1$. Suppose we are given d sections $\sigma_1, \ldots, \sigma_d : \mathbb{A}^1 \to X$. Fix a class $\underline{\beta} \in H_2(X_\Sigma, \mathbb{Z})$ such that if $\rho \in \Sigma^{[1]}$ is a ray and $D_\rho \subseteq X_\Sigma$ is the corresponding divisor, $D_\rho \cdot \underline{\beta} = d_\rho$. Further, fix partitions $d_\rho = d_{\rho,1} + \cdots + d_{\rho,n_\rho}$ where the $d_{\rho,j}$ are positive integers. We can use this to define a type of stable log map β. There are $d + \sum_\rho n_\rho$ marked points, $p_1, \ldots, p_d, p_{\rho,j}, 1 \leq j \leq n_\rho$. For $p \in \{p_1, \ldots, p_d\}$, we take the corresponding scheme Z to be all of X and $u_p = 0$. For $p = p_{\rho,j}$, we take the corresponding scheme Z to be $D_\rho \times \mathbb{A}^1$, and take $u_{p_{\rho,j}} \in \Gamma(Z, (\overline{\mathcal{M}}_X^{\mathrm{gp}}|_Z)^*)$ to be the homomorphism given by restriction of a section to the stalk of $\overline{\mathcal{M}}_X^{\mathrm{gp}}$ at the generic point of $D_\rho \times \mathbb{A}^1$ followed by multiplication by $d_{\rho,j}$. This is specifying a tangency condition of order $d_{\rho,j}$ along D_ρ at the point $p_{\rho,j}$, but no condition at the points $p_1, \ldots, p_d$.

We then consider the moduli space $\mathscr{M}(X/\mathbb{A}^1, \beta)$ fibering over $\mathbb{A}^1$. We have evaluation maps

$$\mathrm{ev} : \mathscr{M}(X/\mathbb{A}^1, \beta) \to X^d := X \times_{\mathbb{A}^1} X \times_{\mathbb{A}^1} \cdots \times_{\mathbb{A}^1} X$$

where we take d copies of X, evaluating at the marked points $p_1, \ldots, p_d$. Similarly, we obtain a map

$$\sigma := \prod_{i=1}^{d} \sigma_i : \mathbb{A}^1 \to X^d.$$

Then we can construct the moduli space

$$\mathscr{M}(X/\mathbb{A}^1, \beta, \sigma) := \mathscr{M}(X/\mathbb{A}^1, \beta) \times_{X^d} \mathbb{A}^1,$$

where the two maps are ev and σ. Taking the fibre product

$$\mathscr{M} := \mathscr{M}(X/\mathbb{A}^1, \beta, \sigma) \times_{\mathbb{A}^1} 0^\dagger$$

gives another situation in which we would like to apply the splitting formalism.

Note this can be described more directly by restricting the sections σ_i to $0^\dagger$, giving d points $\sigma_i : 0^\dagger \to X_0$. This gives maps

$$\mathrm{ev} : \mathscr{M}(X_0/0^\dagger, \beta) \to X_0^d := X_0 \times_{0^\dagger} \cdots \times_{0^\dagger} X_0, \quad \sigma : 0^\dagger \to (X_0)^d,$$

and

$$\mathscr{M} = \mathscr{M}(X_0/0^\dagger, \beta) \times_{X_0^d} 0^\dagger.$$

Let us give a tropical interpretation for splittings in this case.

For a toric variety X_Σ, we can identify Trop_{X_Σ} with $M_\mathbb{R}$ as a topological space, with the fan Σ being the set of cones in Trop_{X_Σ}. In particular, Trop_X and also Trop_{X_0} can be identified with $M_\mathbb{R} \times \mathbb{R}_{\geq 0}$. In Trop_{X_0}, the set of cones which arise as $\mathrm{Hom}(\overline{\mathcal{M}}_{X_0,x}, \mathbb{R}_{\geq 0})$ for some $x \in |X_0|$ is $\{\sigma \times \mathbb{R}_{\geq 0} \,|\, \sigma \in \Sigma\}$. The tropicalization of the diagram

$$\begin{tikzcd} \mathscr{V} \arrow[r, "f"] \arrow[d, "v"'] & X_0 \arrow[d, "g"] \\ \mathscr{M}(X_0/0^\dagger, \beta) \arrow[r, "\pi"'] & 0^\dagger \end{tikzcd}$$

as before yields a family of tropical curves in Trop_{X_0}. Recall from [1], Example 6.7 that these are genuine tropical curves in $M_\mathbb{R} \times \mathbb{R}_{\geq 0}$ in the sense that the balancing condition of tropical geometry is satisfied. A fibre of $\mathrm{Trop}_{\mathscr{V}} \to \mathrm{Trop}_{\mathscr{M}(X_0/0^\dagger,\beta)}$ is a graph Γ with $d + \sum_\rho n_\rho$ unbounded edges. We have $\phi = \mathrm{Trop}_f : \Gamma \to M_\mathbb{R} \times \mathbb{R}_{\geq 0}$ a tropical curve with image contained in $M_\mathbb{R} \times \{r\}$ for some r. Unbounded edges e_{p_i} corresponding to the p_i's are contracted by Trop_f, since $u_{p_i} = 0$, and an edge $e_{p_{\rho,j}}$ corresponding to $p_{\rho,j}$ has $\phi(e_{p_{\rho,j}})$ parallel to the ray $\rho \in \Sigma$, taken with weight $d_{\rho,j}$ in the traditional sense of tropical geometry.

We wish to understand rays in $\mathrm{Trop}_{\mathscr{M}}$, so first we need to understand the monoids $\overline{\mathcal{M}}_{\mathscr{M},x}$ for $x \in |\mathscr{M}|$. Giving such a point means giving a point $x' \in \mathscr{M}(X_0/0^\dagger, \beta)$ such that $\mathrm{ev}(x') = \sigma(0)$, and we need to compute

$$\overline{\mathcal{M}}_{\mathscr{M},x} = \overline{\mathcal{M}}_{\mathscr{M}(X_0/0^\dagger,\beta),x'} \oplus_{\overline{\mathcal{M}}_{X_0^d,\sigma(0)}} \mathbb{N}.$$

Of course $Q_{x'} = \overline{\mathcal{M}}_{\mathscr{M}(X_0/0^\dagger,\beta),x'}$ is as usual a saturated quotient of $\prod_\eta P_\eta \times \prod_q \mathbb{N}$, where P_η is the stalk of $\overline{\mathcal{M}}_{X_0}$ at the image under f of a generic point η of $\mathscr{V}_{x'}$. The log structure on X_0 splits trivially as $\mathcal{M}_{X_\Sigma} \oplus \mathbb{N}$, and we can write each stalk $P_\eta = P'_\eta \oplus \mathbb{N}$. From the definition of this saturated quotient and the fact that the map f is defined over $0^\dagger$, one sees that all of the $\mathbb{N}$ factors in the various P_η are identified in $Q_{x'}$, so we can write $Q_{x'} = Q'_{x'} \oplus \mathbb{N}$. The map $x' \to 0^\dagger$ is given by $1 \mapsto (0, 1)$.

A point in the interior of $\mathrm{Hom}(Q'_{x'}, \mathbb{R}_{\geq 0})$ can be identified as usual with a tropical curve $\phi : \Gamma_{x'} \to M_\mathbb{R}$ such that for any vertex v_η of $\Gamma_{x'}$, $\phi(v_\eta)$ lies in the interior of the cone of Σ corresponding to $\mathrm{Hom}(P'_\eta, \mathbb{R}_{\geq 0})$. In particular, the interior of $\mathrm{Hom}(Q'_x, \mathbb{R}_{\geq 0})$ gives the moduli space of deformations of such a curve, with points on the boundary of $\mathrm{Hom}(Q'_x, \mathbb{R}_{\geq 0})$ corresponding to degenerations of these tropical curves where lengths of some edges go to zero or vertices $\phi(v_\eta)$ move to the boundary of $\mathrm{Hom}(P'_\eta, \mathbb{R}_{\geq 0})$. Finally, we can split

$$\mathrm{Hom}(Q_x, \mathbb{R}_{\geq 0}) = \mathrm{Hom}(Q'_{x'}, \mathbb{R}_{\geq 0}) \times \mathbb{R}_{\geq 0},$$

and interpret a point of $\mathrm{Hom}(Q_{x'},\mathbb{R}_{\geq 0})$ as a pair $(\phi:\Gamma_{x'}\to \mathrm{Trop}_{X_\Sigma}, r)$.

On the other hand, if $\sigma(0)=(y_1,\ldots,y_d)\in X_0^d$, then $\overline{\mathcal{M}}_{X_0^d,\sigma(0)}=(\prod_i P_i')\oplus\mathbb{N}$, where $P_i'=\overline{\mathcal{M}}_{X_\Sigma,y_i}$. Note that if $p_i\in \mathrm{cl}(\eta_i)$ for η_i a generic point of $\mathscr{V}_{x'}$, then $P_i'=P'_{\eta_i}$. Indeed, the morphism f yields a commutative diagram of monoids

$$\begin{tikzcd}
\overline{\mathcal{M}}_{\mathscr{V}_{x'},p_i}=Q_{x'}\oplus\mathbb{N} \arrow[d] & \overline{\mathcal{M}}_{X_0,y_i}=P_i'\oplus\mathbb{N} \arrow[l] \arrow[d] \\
\overline{\mathcal{M}}_{\mathscr{V}_{x'},\eta_i}=Q_{x'} & \overline{\mathcal{M}}_{X_0,f(\eta_i)}=P'_{\eta_i}\oplus\mathbb{N} \arrow[l]
\end{tikzcd}$$

where the vertical maps are generization and the horizontal maps induced by f must be sharp. However, since $u_{p_i}=0$, the only way for the top horizontal map to be sharp is for the right-hand vertical map to be an isomorphism.

One then sees that the map $\overline{\mathcal{M}}_{X_0^d,\sigma(0)}\to\overline{\mathcal{M}}_{\mathscr{M}(X_0/0^\dagger,\beta),x'}$ is induced by the natural inclusion $\prod_i P_{\eta_i}\to\prod_\eta P_\eta\times\prod_q\mathbb{N}$. On the other hand, the map $\overline{\mathcal{M}}_{X_0^d,\sigma(0)}\to\overline{\mathcal{M}}_{0^\dagger}=\mathbb{N}$ of course depends on the choice of sections $\sigma_1,\ldots,\sigma_d$, reflecting the orders of tangency of the sections $\sigma_i:\mathbb{A}^1\to X$ with $D_\rho\times\mathbb{A}^1$ for various ρ. However, the maps $P_i'\oplus\mathbb{N}\to\mathbb{N}$ are the identity on the $\mathbb{N}$ factor since the σ_i are sections. Thus σ_i specifies a point $q_i\in\mathrm{Hom}(P_i',\mathbb{N})\subseteq\mathrm{Hom}(P_i',\mathbb{R}_{\geq 0})\subseteq\mathrm{Trop}_{X_\Sigma}$.

As a consequence, it is easy to interpret $\mathrm{Hom}(\overline{\mathcal{M}}_{\mathscr{M},x},\mathbb{R}_{\geq 0})$ as a fibred product

$$\mathrm{Hom}(Q_{x'},\mathbb{R}_{\geq 0})\times_{(\prod_i\mathrm{Hom}(P_i',\mathbb{R}_{\geq 0}))\times\mathbb{R}_{\geq 0}}\mathbb{R}_{\geq 0}.$$

The first map can be interpreted geometrically as follows. The map $\mathrm{Hom}(Q_{x'},\mathbb{R}_{\geq 0})\to\mathrm{Hom}(P_i',\mathbb{R}_{\geq 0})$ takes (ϕ,r) to $r\cdot\phi(e_{p_i})$, where e_{p_i} is the edge of $\Gamma_{x'}$ corresponding to p_i. The map $\mathrm{Hom}(Q_{x'},\mathbb{R}_{\geq 0})\to\mathbb{R}_{\geq 0}$ is just $(\phi,r)\mapsto r$.

On the other hand, the second map $\mathbb{R}_{\geq 0}\to(\prod_i\mathrm{Hom}(P_i',\mathbb{R}_{\geq 0}))\times\mathbb{R}_{\geq 0}$ is given by $r\mapsto(rq_1,\ldots,rq_d,r)$. Thus the fibered product is

$$\{(\phi,r)\in\mathrm{Hom}(Q_{x'},\mathbb{R}_{\geq 0})\,|\,\phi(e_{p_i})=rq_i\}.$$

In other words, the cone $\mathrm{Hom}(\overline{\mathcal{M}}_{\mathscr{M},x},\mathbb{R}_{\geq 0})$ is the cone over the polyhedron which is the moduli space of those tropical curves of the correct type with marked edges $e_{p_1},\ldots,e_{p_d}$ mapping to $q_1,\ldots,q_d$ respectively.

In this way, a primitive splitting of $\mathscr{M}$ yields a rigid tropical curve in $M_\mathbb{R}$. Conversely, a rigid tropical curve in $M_\mathbb{R}$ with u_{p_i} mapping to q_i defines a splitting, not necessarily primitive as different primitive splittings may give rise to the same rigid tropical curve. Note that the notion of rigidity of tropical curves is not quite what one expects from traditional tropical geometry; the fan Σ plays a role here. A tropical curve is rigid if it cannot be deformed in such a way that the vertices do not leave the cones of Σ containing them.

REFERENCES

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