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Logarithmic degeneration and Dieudonne theory.

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To the memory of Dr. Osamu Hyodo.

§0. Introduction.

To describe the motivation of this paper, let A be a complete discrete valuation ring with residue field k and with field of fractions K, and let E be the "Tate elliptic curve" over K corresponding to a prime element π of A which is usually denoted as "$G_m/\{\pi^n\ ;\ n \in Z\}$". Then E does not extend to a smooth proper scheme over A and ${}_nE = \mathrm{Ker}(n : E \longrightarrow E)$ $(n \geq 1)$ does not extend to a finite flat group scheme over A. However the main claim of this paper is that, in a certain category which is obtained as an amplification of the category of schemes by means of the theory of "logarithmic structures", E extends to a smooth proper group object $\mathfrak{E}$ over A and ${}_nE$ extends to a finite flat group object ${}_n\mathfrak{E}$ over A having an exact sequence

$$0 \longrightarrow Z/nZ(1) \longrightarrow {}_n\mathfrak{E} \longrightarrow Z/nZ \longrightarrow 0$$

$(Z/nZ(1) = \mathrm{Ker}(n : G_m \longrightarrow G_m))$.

Assume k is perfect of positive characteristic p and let n be a power of p. In this paper (§5), we give a Dieudonne theory for group objects over k like ${}_n\mathfrak{E} \otimes_A k$ (resp. over A like ${}_n\mathfrak{E}$ when $A = W(k)$). The new aspect is that our Dieudonne module involves a monodromy operator N besides the old operators F and V.

The precise statement of our Dieudonne theory over k is the following theorem (0.1). (Cf. §5.2 where this theorem and a similar result on the Dieudonne theory of group objects over $W(k)$ are proved.) In (0.1), I do not explain the special terminologies used there which will be defined in the text of the paper. I hope the reader feels what is done in this paper from the statement of the theorem.

Theorem (0.1). Let k be a perfect field of positive characteristic p, and T be the logarithmic scheme whose underlying scheme is $\mathrm{Spec}(k)$ and whose logarithmic structure is associated to the homomorphism $\mathbf{N} \longrightarrow k$; $1 \longrightarrow 0$. Then, there exists an anti-equivalence between the following two categories (a)(b) which extends the classical Dieudonne theory.

(a) The category of finite flat commutative group objects G in the category of algebraic valuative logarithmic spaces over T satisfying the following conditions (i)(ii). (i) G is annihilated by some power of p. (ii) If G^{et} denotes the maximal logarithmically etale quotient of G, the underlying scheme of G^{et} is etale over k in the usual sense.

(b) The category of $W(k)$-modules D of finite length endowed with additive operators $F, V, N : D \longrightarrow D$ satisfying the following relations.

$F(ax) = \varphi(a)F(x)$, $V(\varphi(a)x) = aV(x)$, $N(ax) = aN(x)$ $(a \in W(k),$ $x \in D)$,

$FV = VF = p$, $FNV = N$.

Here φ denotes the standard frobenius of $W(k)$.

Usual commutative finite group schemes over k annihilated by some power of p correspond to Dieudonne modules with $N = 0$. The object ${}_n\mathfrak{E} \otimes_A k$ with $n = p^m$ belongs to the category (a), and its Dieudonne module is a free $W_m(k)$-module of rank 2 with basis e_i ($i = 1, 2$) on which F, V and N acts by

$$Fe_1 = e_1, \quad Fe_2 = pe_2, \quad Ve_1 = pe_1, \quad Ve_2 = e_2,$$
$$Ne_1 = 0, \quad Ne_2 = e_1.$$

A logarithmic structure in the sense of Fontaine-Illusie is defined to be a sheaf of commutative monoids on a local ringed space having some additional structure. (Deligne and Faltings have different formulations of logarithmic structures; [De][Fa_2]. We use the formulation of Fontaine-Illusie in this paper.) A logarithmic structure is a "magic" by which a degenerate scheme begins to behave as being non-degenerate. A large part of this paper (§1, §3, §4) is devoted to the foundation of algebraic geometry with logarithmic structures. Flat morphisms, quasi-finite morphisms, etale sites, etc., in the logarithmic sense are introduced in §3 and §4 (one tries there to have pieces of EGA and SGA for "spaces with rings and monoids"). We already studied basic facts about logarithmic structures in [Ka] (smooth morphisms, etale morphisms and the crystalline cohomology in the logarithmic situation were studied in [Ka]; these subjects were also studied by Faltings independently, in his great works on p-adic Galois representations [Fa_1][Fa_2]). The materials in [Ka] were not sufficient when we consider group objects, etale sites and flat sites in the logarithmic situation, and a new idea in this paper is to introduce valuative logarithmic spaces

(1.2.5) (or, almost equivalently, to consider schemes with log. str.'s "modulo" blowing-ups along the log. str.'s; cf. §1.4.) In the other sections §2 and §5, we consider group objects. In §2, we give two examples of group objects, the compactification of G_m and the proper smooth model of the Tate curve, which can exist only in the logarithmic world. In §5, we study finite flat group objects and give the Dieudonne theory.

An important theme which is not studied in this paper is the compactification of moduli spaces using logarithmic structures. As the Tate curve is a smooth proper object from the logarithmic point of view, we imagine that there is a good notion of "logarithmic abelian varieties" which are smooth proper in the logarithmic sense but may degenerate in the classical sense. I imagine that the compactification over $\mathbf{Z}$ of the moduli space of abelian varieties obtained in [Ch-Fa] and [Fu] should be reformulated as the solution of the moduli problem of logarithmic abelian varieties. (This does not contradict the fact that in [Fa-Ch] [Fu], there is no canonical toroidal compactification of the moduli space of abelian varieties and (for this reason) the compactifications were not obtained as the solution of a moduli problem. Different toroidal compactifications become isomorphic modulo blowing-ups along log. str.'s.)

I am thankful to Professors J.-M. Fontaine and L. Illusie and the late Professor Osamu Hyodo for stimulating discussions. I studied with them p-adic monodromy operators ([Bu]). The question to find a real geometric group object which yields a Dieudonne module with a monodromy operator came from discussions with them.

Contents.

§1. Logarithmic spaces.

The main purpose of this section is to introduce "valuative logarithmic spaces (1.2.5)" and prove basic facts about them.

§1.1. Integral monoids, saturated monoids, and valuative monoids.

(1.1.1) In this paper, a monoid means a commutative monoid (= a commutative semi-group) having a unit element. A homomorphism of monoid is assumed to send the unit element to the unit element, and a submonoid is assumed to contain the unit element of the total monoid.

For a monoid P, let $P^{\times}$ be the subgroup of P consisting of all invertible elements. We denote by P^{gp} the commutative group associated to P, that is,

$$P^{gp} = \{ab^{-1} \; ; \; a, b \in P\}$$

$$ab^{-1} = cd^{-1} \iff \exists\, s \in P \text{ such that } sad = sbc.$$

Definition (1.1.2). A monoid P is called integral if the canonical map $P \longrightarrow P^{gp}$ is injective (in other words, if "$ab = ac \Rightarrow b = c$" holds in P).

For a monoid P, we denote by P^{int} the image of $P \longrightarrow P^{gp}$.

Definition (1.1.3). A monoid is called saturated if P is integral and satisfies the following condition: If $a \in P^{gp}$ and $a^n \in P$ for some $n \geq 1$, then $a \in P$.

For a monoid P, we define the saturated monoid P^{sat} by

$$P^{sat} = \{a \in P^{gp} \ ; \ a^n \in P^{int} \ \text{for some} \ n \geq 1\}.$$

Lemma (1.1.4). _If P is a finitely generated monoid, then P^{sat} is also finitely generated._

Cf. [KKMS] Ch. I §1 Lemma 2 for the proof.

Definition (1.1.5). A monoid P is called valuative if P is integral and satisfies the following condition: If $a \in P^{gp}$, then either $a \in P$ or $a^{-1} \in P$ holds.

The following notions are in analogy:

integral monoids $\longleftrightarrow$ local integral domain

saturated monoids $\longleftrightarrow$ integrally closed local domain

valuative monoids $\longleftrightarrow$ valuation ring.

(The reason why a monoid is "local" is that the complement of $P^{\times}$ is an ideal (cf. (1.4.1)) as the case of a local ring.)

The following lemmas (1.1.6)-(1.1.9) below are proved easily imitating the arguments in the proofs of the analogous results in commutative algebra ([Bo] Ch. VI).

Lemma (1.1.6). _A valuative monoid is saturated._

Lemma (1.1.7). _An integral monoid P is valuative if and only if it is a maximal element in the set of all submonoids of P^{gp} with respect to the order relation_

$$Q \leq Q' \iff Q \subset Q' \ \text{and} \ Q^{\times} = Q \cap (Q')^{\times}.$$

Lemma (1.1.8). _For an integral monoid P, there exists a_ valuative submonoid V of P^{gp} containing P such that $P^{\times} = P \cap V^{\times}$.

Lemma (1.1.9). For an integral monoid P, P^{sat} coincides with the intersection of all valuative submonoids V of P^{gp} containing P such that $P^{\times} = P \cap V^{\times}$.

§1.2. Logarithmic structures; basic definitions.

We introduce logarithmic structures of Fontaine-Illusie. There is a different definitions of log. str.'s found by Deligne [De] and Falting [Fa_2]. Most materials of this §1.2 are discussed in [Ka] in a slightly different formulation.

Definition (1.2.1). A pre-logarithmic structure on a local ringed space $(X, \mathcal{O}_X)$ is a sheaf of monoids M endowed with a homomorphism $\alpha : M \longrightarrow \mathcal{O}_X$. (Here $\mathcal{O}_X$ is regarded as a sheaf of monoids by the multiplication.)

A pre-log. str. M is called a logarithmic structure if α induces

$$\alpha^{-1}(\mathcal{O}_X^{\times}) \xrightarrow{\cong} \mathcal{O}_X^{\times} .$$

If M is a log. str., we regard $\mathcal{O}_X^{\times}$ as a subsheaf of M via α^{-1}.

For example, a submonoid M of $\mathcal{O}_X$ (for the multiplication) containing $\mathcal{O}_X^{\times}$ is a log. str. The particular case $M = \mathcal{O}_X^{\times}$ is called the trivial log. str. In general, $M \longrightarrow \mathcal{O}_X$ need not be injective.

Definition (1.2.2). A logarithmic space is a local ringed space endowed with a log. str. It is called a logarithmic scheme if the underlying local ringed space is a scheme.

The structural log. str. of a log. space X is usually denotes by

M_X .

Morphism of log. spaces is defined in the evident way.

If X and Y are log. spaces and Y is endowed with the trivial log. str., to give a morphism of log. spaces $X \longrightarrow Y$ is equivalent to giving just a morphism of local ringed spaces (forgetting log. str.'s) $X \longrightarrow Y$. Thus, "endowed with the trivial log. str." sounds as "without log. str.".

Definition (1.2.3). For a pre-log. str. M on a local ringed space X, the log. str. $M^{\sim}$ on X associated to M is defined to be the push out of $M \longleftarrow \alpha^{-1}(\mathcal{O}_X^{\times}) \xrightarrow{\alpha} \mathcal{O}_X^{\times}$ in the category of sheaves of monoids on X. (We endow $M^{\sim}$ with the homomorphism $M^{\sim} \longrightarrow \mathcal{O}_X$ induced by $M \longrightarrow \mathcal{O}_X$ and by the inclusion map $\mathcal{O}_X^{\times} \longrightarrow \mathcal{O}_X$.)

It is easily checked that $M^{\sim}$ is in fact a log. str., and that the canonical homomorphism $M \longrightarrow M^{\sim}$ is universal among homomorphisms from M to log. str.'s on X.

Definition (1.2.4). Let $f : Y \longrightarrow X$ be a morphism of local ringed spaces. Then, for a log. str. M on X, the inverse image f^*M of M is defined to be the log. str. on Y associated to the pre-log. str. $f^{-1}(M)$ which is endowed with the composite homomorphism $f^{-1}(M) \longrightarrow f^{-1}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y$. (Here we denoted the sheaf theoretic inverse image by f^{-1}, not by f^*.)

Definition (1.2.5). A log. str. M on a local ringed space X is called integral (resp. saturated, resp. valuative) if the stalk M_x is an integral (resp. saturated, resp. valuative) monoid for all $x \in X$.

Definition (1.2.6). A log. str. on a local ringed space X is called quasi-coherent if locally on X, there exists a monoid P and a homomorphism $P \longrightarrow \mathcal{O}_X$ (here we denote the constant sheaf associated to P by the same notation P) such that M is isomorphic to the log. str. associated to the pre-log. str. $P \longrightarrow \mathcal{O}_X$.

If we can take (locally) as P finitely generated (resp. finitely generated integral) monoids, M is called a coherent (resp. fine) log str.

We have the equivalence

fine $\Leftrightarrow$ integral and coherent.

In this paper, valuative log. spaces (abbreviation of "log. spaces whose log. str.'s are valuative") and fine log. schemes (abbreviation of "log. schemes whose log. str.'s are fine") are the most important two types of log. spaces.

(1.2.7) A standard example of a fine log. str. is the following. Let X be a regular locally noetherian scheme and let D be a reduced divisor on X with simple normal crossings. Then

$$M = \{f \in \mathcal{O}_X \ ; \ f \text{ is invertible outside } D\}$$

is a fine log. str. which is associated locally to $\mathbf{N}^r \longrightarrow \mathcal{O}_X$; $e_i \longrightarrow \pi_i$, where $(e_i)_i$ is the canonical basis of $\mathbf{N}^r$ and π_i are local sections of $\mathcal{O}_X$ such that $D = \bigcup_i$ "$\pi_i = 0$" and each "$\pi_i = 0$" is regular.

For example, if A is a discrete valuation ring, the closed point of $\mathrm{Spec}(A)$ is regarded as a reduced divisor with simple normal crossings on $\mathrm{Spec}(A)$. We call the corresponding log. str. of $\mathrm{Spec}(A)$ the canonical log. str. of $\mathrm{Spec}(A)$. The recipe is that, when $\mathrm{Spec}(A)$ is endowed with the canonical log. str., some degenerate objects over A, endowed with suitable log. str.'s, behave as being of good reduction.

Lemma (1.2.8). Let $f : Y \longrightarrow X$ be a morphism of local ringed spaces and let M be a log. str. on X. If (P) is one of the following properties (i)-(vi) of log. str.'s and M has (P), then f^*M also has (P).

(i) integral, (ii) saturated, (iii) valuative, (iv) quasi-coherent, (v) coherent, (vi) fine.

Proof. The cases (i) - (iii) follow from the fact $f^{-1}(M/\mathcal{O}_X^x) \cong (f^*M)/\mathcal{O}_Y^x$, and the cases (iv) - (vi) follow from the fact that if M is associated to $P \longrightarrow \mathcal{O}_X$, then f^*M is associated to the composite $P \longrightarrow f^{-1}(\mathcal{O}_X) \longrightarrow \mathcal{O}_Y$.

Proposition (1.2.9). Let X be a log. scheme whose log. str. is quasi-coherent. Then, there exists a scheme X^{int} (resp. X^{sat}) endowed with an integral (resp. saturated) log. str. and with a morphism of log. spaces $X^{int} \longrightarrow X$ (resp. $X^{sat} \longrightarrow X$) having the following universal property: If Y is a log. space whose log. str. is integral (resp. saturated), a morphism $Y \longrightarrow X$ of log. spaces factors uniquely through $X^{int} \longrightarrow X$ (resp. $X^{sat} \longrightarrow X$).

Locally on X, X^{int} (resp. X^{sat}) is described as follows. Assume the log. str. of X is associated to a homomorphism $P \longrightarrow \mathcal{O}_X$ for a monoid P. Then,

$$X^* = X \otimes_{Z[P]} Z[P^*]$$

($*$ = int (resp. sat); here $Z[P]$ and $Z[P^*]$ denote the monoid rings), which is endowed with the log. str. associated to the canonical map $P^* \longrightarrow \mathcal{O}_X \otimes_{Z[P]} Z[P^*]$.

Proof of (1.2.9). The above explicit local construction shows the existence of a universal object.

In §1.3, we shall treat the "valuative version" X^{val} of (1.2.9). Contrarily to the integral and saturated cases, X^{val} is merely a log. space and not a scheme in general.

Lemma (1.2.10). The category of log. schemes has finite inverse limits. Log. schemes with coherent log. str.'s are stable under finite inverse limits in the category of log. schemes. The category of fine log. schemes has finite inverse limits.

Indeed, for a finite inverse system of log. schemes $(X_\lambda)_\lambda$, the inverse limit X is found as follows. As a scheme, X is the inverse limit of the schemes X_λ. The log. str. M_X is the inductive limit in the category of log. str.'s on X of the inverse images of M_{X_λ}. Cf. [K_a] (2.6) for the proof of the statement concerning coherent log. str. 's in (1.2.10). (There is a difference between [K_a] and this paper in the formulation of log. str.'s (see (1.2.11)) but the proof there works in the present formulation.) Finally, the inverse limit in the category of fine log. schemes is obtained as $(\ )^{int}$ (1.2.9) of the finite inverse limit in the category of schemes with coherent log. str.'s.

Remark (1.2.11). In [Ka], we defined a log. str. on a scheme as a sheaf on the etale site of a scheme whereas we defined it in this paper as a sheaf on the Zariski site. (It is possible to formulate the theory of log str.'s on a ringed topos more generally as Fontaine and Illusie explained in their first discussion on a log. str. with the author.) An advantage of the etale site was that, with the similar definition of a fine log. str. on the etale site as (1.2.6), a reduced divisor with normal crossings, not necessarily with simple normal crossings, on a regular locally noetherian scheme X is regarded as a fine log. str. on the etale site of X. However as we shall see in (4.2.3)(4), when we are interested in valuative log. spaces (we shall be so in this paper), fine log. str.'s on the etale site (or on the flat site) can be "replaced" by fine log. str.'s on the Zariski site.

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§1.3. The valuative log. space associated to a scheme with a quasi-coherent log. str.

Theorem (1.3.1). <u>Let X be a scheme with a quasi-coherent log str. (1.2.6). Then there exists a valuative log. space X^{val} with a morphism of log. spaces $X^{val} \longrightarrow X$ having the following universal property: For any valuative log. space Y, a morphism $Y \longrightarrow X$ factors uniquely through $X^{val} \longrightarrow X$.</u>

To prove this theorem, we may work locally on X and hence we may assume that M_X is associated to a homomorphism $P \longrightarrow \mathcal{O}_X$ for a monoid P. Furthermore, by replacing P with P^{int} and X with X^{int} (1.2.9), we may assume that P is integral. Under these assumptions, X^{val} is constructed explicitly as in (1.3.4) below.

Definition (1.3.2). A subset I of a monoid P is called an ideal of P if the condition

$$a \in P \text{ and } x \in I \Rightarrow ax \in I$$

is satisfied. For ideals I, J of P, we denote by IJ the ideal of P generated by $\{xy \,;\, x \in I,\ y \in J\}$.

The empty set is an ideal of P, but we shall use only non-empty ideals.

(1.3.3) Let X be a scheme, P an integral monoid, $P \longrightarrow \mathcal{O}_X$ a homomorphism, and endow X with the log. str. associated to this homomorphism.

Let Φ be the set of all finitely generated non-empty ideals of P. For $I \in \Phi$, let

$$X_I = X \times_{\mathrm{Spec}(Z[P])} \mathrm{Proj}(\bigoplus_{n\geq 0} \langle I\rangle^n)$$

where $\langle I\rangle$ denotes the ideal of $Z[P]$ generated by I.

On the other hand, for a monoid Q with a homomorphism $P \longrightarrow Q$, let X_Q be the scheme $X \otimes_{Z[P]} Z[Q]$ endowed with the log. str. associated to the canonical map $Q \longrightarrow \mathcal{O}_{X_Q}$. (I believe there is no confusion of the notations X_I and X_Q ; the intersection of the use of them is the case $I = Q = P$, but $X_I = X_Q = X$ in this case.)

For $a \in I$, let $P[a^{-1}I]$ be the submonoid of P^{gp} generated by the set $a^{-1}I = \{a^{-1}x \ ; \ x \in I\}$. Then X_I is covered by the affine open subschemes $X_{P[a^{-1}I]}$. The log. str.'s on $X_{P[a^{-1}I]}$ coincide on their intersections and define a log. str. on X_I.

Note that if $a_1, \ldots, a_r \in I$ generate I, then $X_{P[a_i^{-1}P]}$ $(1 \leq i \leq r)$ already cover X_I.

We endow Φ with the following directed ordering:

$I' \geq I \iff I' = IJ$ for some finitely generated non-empty ideal J of P.

If $I' \geq I$, we have a canonical morphism of log. schemes

$f_{I,I'}$ $X_{I'} \longrightarrow X_I$.

Proposition (1.3.4). <u>In the situation of (1.3.3), the inverse limit $\varprojlim_{I\in\Phi} X_I$ in the category of log. spaces has the properties of X^{val} in (1.3.1).</u>

(Here, as a topological space, $\varprojlim_{I\in\Phi} X_I$ is the inverse limit of the topological spaces X_I, and the structural sheaves $\mathcal{O}$ and M of $\varprojlim_{I\in\Phi} X_I$ are the inductive limits of the inverse images of $\mathcal{O}_{X_I}$ and M_{X_I}, respectively.)

The author learned the method to consider the inverse limit of blowing-ups with respect to the ordering by product of ideals as above, from a lecture of O. Gabber.

The proof of (1.3.4) is straightforwards and we omit it.

(1.3.5) We give another discription of the topological space X^{val} in the situation of (1.3.3).

There is a canonical bijection between the set X^{val} and the set of all pairs (V, p) such that V is a valuative submonoid of P^{gp} containing P and p is a point of $X_V = X \otimes_{Z[P]} Z[V]$ satisfying the following condition: If $a \in V$ and the image of a in $\mathcal{O}_{X_V,p}$ is invertible, then $a \in V^{\times}$. Indeed, if (V, p) is such a pair and I is a finitely generated ideal of X, the image of (V, p) in X_I is obtained as follows. The ideal IV of V is generated by one element $a \in I$, and $P[a^{-1}I] \subset V$. We obtain a point of $X_{P[a^{-1}I]} \subset X_I$ to be the image of the induced map $X_V \longrightarrow X_{P[a^{-1}I]}$.

For a pair (V, p) as above, when we regard it as an element of X^{val}, we have the description of stalks:

$$\mathcal{O}_{X^{val},(V,p)} \cong \mathcal{O}_{X_V,p} ,$$

and $M_{X^{val},(V,p)}$ is isomorphic to the push out of $V \longleftarrow V^{\times} \longrightarrow \mathcal{O}_{X_V,p}^{\times}$ in the category of monoids.

For a submonoid Q of P^{gp} containing P which is finitely generated over P, and for an open subscheme U of X_Q, the morphism $U^{val} \longrightarrow X^{val}$ induces an isomorphism of U^{val} with an open log. subspace of X^{val}. These U^{val} form a basis of open sets of X^{val}. A pair (V, p) as above, regarded as a point of X^{val}, belongs to U^{val} if and only if $Q \subset V$ and the image of p under

$X_V \longrightarrow X_Q$ belongs to U.

Definition (1.3.6). We say a homomorphism of integral monoids $h : P \longrightarrow Q$ is exact if the inverse image of Q under $h^{gp} : P^{gp} \longrightarrow Q^{gp}$ coincides with P. We say a morphism $f : Y \longrightarrow X$ of log. spaces with integral log. str.'s is exact at $y \in Y$ if $M_{X,f(y)} \longrightarrow M_{Y,y}$ is exact. We say f is exact if it is exact at any $y \in Y$.

Proposition (1.3.7). Let $f : Y \longrightarrow X$ be a morphism of log. schemes whose log. str.'s are integral and quasi-coherent. Let $y \in Y$ and $x = f(y) \in X$. Assume f is exact at y (1.3.6). Then for any point x' of X^{val} lying over x, there exists a point y' of Y^{val} which lies over x' and also over y.

Corollary (1.3.8). Let X be a scheme with quasi-coherent integral log. str. Then, $X^{val} \longrightarrow X$ is surjective.

Proof. Apply (1.3.7) by taking $\mathrm{Spec}(\mathbf{Z})$ with the trivial log. str. as X in (1.3.7) and taking X of (1.3.8) as Y of (1.3.7).

Proof of (1.3.7). We may assume $X = \mathrm{Spec}(k)$, $Y = \mathrm{Spec}(K)$ for fields k, K. Consider $M_{X,x}$ and $M_{Y,y}$ as "P" for X and for Y, respectively. Let $x' = (V, p) \in X^{val}$, where $M_{X,x} \subset V \subset M^{gp}_{X,x}$, $p \in X_V$. Then, by using the exactness assumption, we obtain

$$(VM_{Y,y})^{\times} \cap M_{X,x} = M^{\times}_{X,x} \quad (= k^{\times}) \quad \text{in} \quad M^{gp}_{Y,y},$$
$$(VM_{Y,y})^{\times} \cap V = V^{\times} \quad \text{in} \quad M^{gp}_{Y,y}.$$

Take a valuative submonoid W of $M^{gp}_{Y,y}$ containing $VM_{Y,y}$ such that $W^{\times} \cap VM_{Y,y} = (VM_{Y,y})^{\times}$ (1.1.9), and let I (resp. J) be the ideal of $\mathbf{Z}[V]$ (resp. $\mathbf{Z}[W]$) generated by $\{a \in V \; ; \; a \notin V^{\times}\}$ (resp. $\{a \in W \; ; \; a \notin W^{\times}\}$.) Then

$$K \otimes_{\mathbf{Z}[M_{Y,y}]} \mathbf{Z}[W]/J \cong K \otimes_{\mathbf{Z}[K^{\times}]} \mathbf{Z}[W^{\times}],$$

$k \otimes_{Z[M_{X,x}]} Z[V]/I \cong k \otimes_{Z[k^x]} Z[V^x]$

Since $K \otimes_{Z[K^x]} Z[W^x]$ is faithfully flat over $k \otimes_{Z[k^x]} Z[V^x]$, there is a prime ideal q of $K \otimes_{Z[M_{Y,y}]} Z[W]/J$ lying over the prime ideal of $k \otimes_{Z[M_{X,x}]} Z[V]/I$ corresponding to p. Then, $y' = (W, q) \in Y^{val}$ has the desired properties.

Proposition (1.3.9). Let X be a scheme with a quasi-coherent integral log. str.

(1) X^{val} is quasi-compact if and only if X is quasi-compact.

(2) If X is connected and the log. str. of X is saturated, then X^{val} is connected. Conversely, if X^{val} is connected, then X is connected.

(1.3.10) We prove (1.3.9)(1). The "only if" part follows from the surjectivity of $X^{val} \longrightarrow X$ (1.3.8). To prove the "if" part, we may assume X is as in (1.3.4). Though it is not true in general that the inverse limit of a projective system of quasi-compact spaces is quasi-compact, we can use the following (1.3.11) to affirm that $\varprojlim X_I$ is quasi-compact.

Lemma (1.3.11). Call a topological space T a good quasi-compact space if it satisfies the following two conditions. (i) T is quasi-compact. (ii) Define the "new topology" of T by taking the sets of the form $U \cap E$ with U an open set of T and E the complement of a quasi-compact open set of T, as a basis of open sets. Then T is compact with respect to this "new topology". (Our terminology "compact" includes "Hausdorff" as that of Bourbaki). We have:

(1) A quasi-compact scheme is a good quasi-compact space.

(2) Let $(T_\lambda)_{\lambda\in\Lambda}$ be an inverse system of topological spaces with directed index set Λ such that each T_λ is a good quasi-compact space and such that if $\lambda, \lambda' \in \Lambda$ and $\lambda' \geq \lambda$, then the inverse image of a quasi-compact open subset of T_λ in $T_{\lambda'}$ is quasi-compact. Then, $\varprojlim_\lambda T_\lambda$ is quasi-compact.

Proof. To prove (1), it is sufficient to consider the case of an affine scheme Spec(A). Let F be a ultra-filter on Spec(A). Let $p = \{a \in A ; D(a) \notin F\}$, where $D(a) = \{q \in \mathrm{Spec}(A) ; a \notin q\}$. Then p is a prime ideal of A and is a unique point to which F converges with respect to the "new topology".

Next we prove (2). If we endow T_λ with the "new" topologies, then $\varprojlim_\lambda T_\lambda$ is compact because it is an inverse limit of compact spaces. Since the original topologies are weaker than the new ones, we see that $\varprojlim_\lambda T_\lambda$ is quasi-compact.

(1.3.12) We prove (1.3.9)(2). The second statement follows from the surjectivity of $X^{val} \longrightarrow X$ (1.3.8). To prove the first statement, we may assume X is as in (1.3.4) with P saturated. It is sufficient to prove that each X_I is connected. Furthermore, by the fact $P = \cup Q$ where Q ranges over finitely generated saturated submonoids of P, we are reduced to the case P is a finitely generated saturated monoid. It is sufficient to show that the inverse image in X_I of each point $x \in X$ is connected. So we may assume that as a scheme, X is the Spec of a field k. Note $X_I = \mathrm{Spec}(k) \times_{\mathrm{Spec}(k[P])} Y$ with $Y = \mathrm{Proj}(\bigoplus_{n\geq 0} \langle I\rangle_k^n)$ where $\langle I\rangle_k$ is the ideal of k[P] generated by I. We have $\Gamma(Y, \mathcal{O}_Y) = k[P]$ (use the fact k[P] is normal [KKMS] Ch. I §1 Lemma 1). Since the morphism

$Y \longrightarrow \mathrm{Spec}(k[P])$ is proper and X_I is a fiber of this morphism, the theory of Stein factorization shows that X_I is connected.

§1.4. Algebraic valuative logarithmic spaces.

In this §1.4, let S be a scheme with the trivial log. str.

Definition (1.4.1). By an algebraic valuative logarithmic space over S, we mean a valuative log. space $\mathfrak{X}$ over S such that there exists an open covering

$$\mathfrak{X} = \bigcup_\lambda \mathfrak{U}_\lambda$$

of $\mathfrak{X}$ with the property that for each λ, there exists a fine log. scheme U_λ over S which is locally of finite presentation as an S-scheme such that $\mathfrak{U}_\lambda$ is isomorphic to $(U_\lambda)^{val}$ over S.

An open log. subspace of an alg. val. log. space over S is also an alg. val. log. space over S.

The following proposition says that the category of alg. val. log. spaces over S is regarded as a "localization" of the category of fine log. schemes over S which are locally of finite presentation as S-schemes.

Proposition (1.4.2). Let X, Y be log. schemes over S. Assume X is locally of finite presentation as an S-scheme and the log. str. of X is fine. Assume on the other hand that the log. str. of Y is associated to a homomorphism $P \longrightarrow \mathcal{O}_Y$ for an integral monoid P, and the underlying scheme of Y is quasi-compact and quasi-separated. Then,

$$\mathrm{Mor}_S(Y^{val}, X^{val}) \cong \varinjlim_I \mathrm{Mor}_S(Y_I, X)$$

where Mor_S denotes the set of morphisms of log. spaces over S, I is ranges over non-empty finitely generated ideals of P, and Y_I is as in (1.3.3).

Proof. Straightforward.

(1.4.3) The category of alg. val. log. spaces over S has finite inverse limits. In fact, by (1.4.2), a finite diagram in this category is locally the $(\ )^{val}$ of a finite diagram of the category of fine log. schemes. Take the finite inverse limit in the category of fine log. schemes (1.2.10) of the latter diagram, and take its $(\ )^{val}$. Then, it gives locally the finite inverse limit of the original diagram.

(1.4.4) Let S' be a scheme over S with the trivial log. str. Then, for an alg. val. log. space $\mathfrak{X}$ over S, the fiber product $\mathfrak{X} \times_S S'$ in the category of val. log. spaces exists. Indeed, $\mathfrak{X} = X^{val}$ locally, and the fiber product is $(X \times_S S')^{val}$ locally. This also shows that $\mathfrak{X} \times_S S'$ is an alg. val. log. space over S'. Thus we have the base change functor $(\ ) \times_S S'$ from the category of alg. val. log. spaces over S to that over S'.

When we talk about alg. val. log. spaces in this paper, the notation of the fiber product is used in the sense of (1.4.3) or of (1.4.4).

§2. Examples of group objects.

§2.1. Compactification of G_m.

(2.1.1) Let S be a scheme with the trivial log. str. Let $G_{m,S}^{cpt}$ be the scheme P_S^1 endowed with the valuative fine log. str. which is associated to $N \longrightarrow \mathcal{O}_S[t]$; $1 \longrightarrow t$ on $\mathrm{Spec}(\mathcal{O}_S[t]) \subset P_S^1$ and associated to $N \longrightarrow \mathcal{O}_S[t^{-1}]$; $1 \longrightarrow t^{-1}$ on $\mathrm{Spec}(\mathcal{O}_S[t^{-1}]) \subset P_S^1$. Here t denotes the standard coordinate on P_S^1. If S is a regular locally noetherian scheme, the log. str. of $G_{m,S}^{cpt}$ corresponds to the divisor "$t = 0$" $\cup$ "$t = \infty$" on P_S^1 in the sense of (1.2.7).

Proposition (2.1.2). <u>For a valuative log. space $\mathfrak{Y}$ over S, there exists a functorial isomorphism</u>

$$\mathrm{Mor}_S(\mathfrak{Y}, G_{m,S}^{cpt}) \cong \Gamma(\mathfrak{Y}, M_{\mathfrak{Y}}^{gp}).$$

In particular, $G_{m,S}^{cpt}$ is a group object in the category of alg. val. log. spaces over S.

Remark (2.1.3). If $\mathfrak{Y}$ is a log. space over S which is not necessarily valuative, $\mathrm{Mor}_S(\mathfrak{Y}, G_{m,S}^{cpt})$ is identified with $\Gamma(\mathfrak{Y}, M_{\mathfrak{Y}} \cup M_{\mathfrak{Y}}^{-1})$ where $M_{\mathfrak{Y}} \cup M_{\mathfrak{Y}}^{-1}$ denotes the push out of

$$M_{\mathfrak{Y}} \xleftarrow{\ i\ } \mathcal{O}_{\mathfrak{Y}}^{\times} \xrightarrow{\ j\ } M_{\mathfrak{Y}} \quad ;\ i(u) = u,\quad j(u) = u^{-1}$$

in the category of sheaves of sets. (Note $M_{\mathfrak{Y}} \cup M_{\mathfrak{Y}}^{-1} = M_{\mathfrak{Y}}^{gp}$ if and only if $M_{\mathfrak{Y}}$ is valuative.) Thus $G_{m,S}^{cpt}$ is not a group object when regarded as an object of the category of log. spaces over S (or the category of log. schemes over S) if $S \neq \phi$. Thus, to have many group objects, it is important to work in the category of valuative log. spaces.

§2.2. Tate curves.

Proposition (2.2.1). *Let S be a scheme with the trivial log. str., let $\mathfrak{X}$ be an alg. val. log. space over S, and let π be an element of* $\Gamma(\mathfrak{X}, M_{\mathfrak{X}})$ *whose image in* $\mathcal{O}_{\mathfrak{X}}$ *is locally nilpotent. Let* $\mathcal{A}_{\mathfrak{X}}$ *be the category of alg. val. log. spaces $\mathfrak{Y}$ over S endowed with a morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$ over S. Then, the functors*

$$\mathcal{A}_{\mathfrak{X}} \longrightarrow (\mathrm{Sets}) ;$$

$$\mathfrak{Y} \longmapsto \Gamma(\mathfrak{Y}, M_{\mathfrak{Y}}^{gp,\pi}), \qquad \mathfrak{Y} \longmapsto \Gamma(\mathfrak{Y}, M_{\mathfrak{Y}}^{gp,\pi}/\pi^{Z})$$

are representable, where $M_{\mathfrak{Y}}^{gp,\pi}$ *denotes the subsheaf of* $M_{\mathfrak{Y}}^{gp}$ *characterized by*

$$M_{\mathfrak{Y},y}^{gp,\pi} = \{a \in M_{\mathfrak{Y},y}^{gp} \ ; \ \pi^{m}|a \quad \text{and} \quad a|\pi^{n} \quad \text{for some} \quad m, n \in Z\}$$

for $y \in \mathfrak{Y}$ ($a|b$ means $a^{-1}b \in M_{\mathfrak{Y},y}$), *and π^{Z} denotes $\{\pi^{n} \ ; \ n \in Z\}$.*

We call the group object $\mathbb{G}^{\pi}$ of $\mathcal{A}_{\mathfrak{X}}$ which represents the second functor in (2.2.1), the Tate curve over $\mathfrak{X}$ corresponding to π.

Proof of (2.2.1). The first functor is represented by the open subspace $G_{m,\mathfrak{X}}^{cpt,\pi}$ of $G_{m,\mathfrak{X}}^{cpt} = G_{m,S}^{cpt} \times_{S} \mathfrak{X}$ consisting of all points x such that

$$\pi^{m}|t \quad \text{and} \quad t|\pi^{n} \quad \text{at} \quad x \quad \text{for some} \quad m, n \in Z.$$

Here t denotes the standard element in $\Gamma(G_{m,\mathfrak{X}}^{cpt}, M^{gp})$ corresponding to the projection $G_{m,\mathfrak{X}}^{cpt} \longrightarrow G_{m,S}^{cpt}$.

We show that the action of the group π^{Z} on $G_{m,\mathfrak{X}}^{cpt,\pi}$ has the following property: Each $x \in G_{m,\mathfrak{X}}^{cpt,\pi}$ has an open neighbourhood $\mathfrak{U}$ such that $\pi^{n}\mathfrak{U} \cap \mathfrak{U} = \emptyset$ for $n \in Z \setminus \{0\}$. Indeed, there exists $i \in Z$ such that $t^{2}|\pi^{i+1}$ and $\pi^{i}|t^{2}$ at x, and we can take

$\mathfrak{U} = \{y \in G_{m,\mathfrak{X}}^{cpt,\pi}$; $\pi^i | t^2$ and $t^2 | \pi^{i+1}$ at $y\}$.

Hence the quotient $G_{m,\mathfrak{X}}^{cpt,\pi}/\pi^{Z}$ exists and it represents the second functor.

(2.2.2) the relation with the classical theory of Tate curves is as follows.

Let E be the Tate curve $\bar{\mathfrak{G}}_m^q/q^Z$ over $Z[[q]]$ with the "q-invariant" q (where q is an indeterminate) in [D-R] VII 1.10. Let $E_0 \overset{=}{\text{def}} E \otimes_{Z[[q]]} Z$, where $Z[[q]] \longrightarrow Z$ is $q \longrightarrow 0$. To define a fine log. str., we blow up E taking the singular part of E_0 as the center. (E_0 is the quotient of P_Z^1 obtained by identifying the 0-section with the ∞-section, and the singular part of E_0 is the image of these two sections.) Let E' be the result of the blowing up, and let E'_0 be the reduced part of $E' \otimes_{Z[[q]]} Z$ ($q \longrightarrow 0$). We endow E' with the fine log. str. defined to be the sheaf of submonoids of $\mathcal{O}_E$ generated by $\mathcal{O}_E^{\times}$ and local sections of $\mathcal{O}_E$ which define some irreducible component of E'_0. We denote by T

E_0 E'_0

(resp. T_n for $n \geq 1$) the log. scheme $\mathrm{Spec}(Z[[q]])$ (resp. $\mathrm{Spec}(Z[q]/(q^n))$) whose log. str. is associated to $N \longrightarrow Z[[q]]$ (resp. $Z[q]/(q^n)$) ; $1 \longrightarrow q$. In the following, let $\mathcal{A}_T$ (resp. $\mathcal{A}_{T_n}$) be the category of log. spaces $\mathfrak{Y}$ over T (resp. T_n) such that $\mathfrak{Y}$ is an alg. val. log. space when considered over the scheme $\mathrm{Spec}(Z[[q]])$ (resp. $\mathrm{Spec}(Z[q]/(q^n))$) with the trivial log. str.

Proposition (2.2.3). Let E' be as above and let $\mathfrak{E} = (E')^{val}$. Then there exists on $\mathfrak{E}$ a structure of a group object of $\mathcal{A}_T$ having the following properties.

(i) For any $n \geq 1$, $\mathfrak{E} \times_T T_n$ is canonically isomorphic to the Tate curve (2.2.1) over T_n corresponding to q as a group object in $\mathscr{A}_{T_n}$.

(ii) It defines on the open subspace $\mathfrak{E} \otimes_{Z[q]} Z[q^{-1}] = E' \otimes_{Z[q]} Z[q^{-1}]$ the known group scheme str. of the elliptic curve $E' \otimes_{Z[q]} Z[q^{-1}]$ over $Z[[q]][q^{-1}]$.

Proof. Since we shall treat formal completions, we have to work with log. schemes before passing to val. log. spaces.

Let $\mathscr{C}_n$ be the category of log. schemes over T_n with fine and saturated log. str.'s. Let U_n, $U_{n,i}$ for $i \in Z$, and $U_{n,ijk}$ for $i, j, k \in Z$ be functors $\mathscr{C}_n \longrightarrow$ (Sets) defined by

$$U_n(X) = \Gamma(X, M_X^{gp}),$$

$$U'_n(X) = \{a \in U_n(X) \ ; \text{ for each } x \in X, \text{ there exists } i \in Z \text{ such that } q^i | a^2 \text{ and } a^2 | q^{i+1} \text{ at } x\},$$

$$V_n(X) = \{(a,b) \in U'_n(X) \times U'_n(X) \ ; \text{ for each } x \in X, \text{ there exists } i \in Z \text{ such that } \pi^k | (ab)^2 \text{ and } (ab)^2 | \pi^{k+1} \text{ at } x\}.$$

Then, U'_n and V_n are represented by objects of $\mathscr{C}_n$. We denote these objects by the same letters U'_n and V_n, respectively.

We have

$$(U'_n)^{val} = G_{m,T_n}^{cpt,q}, \qquad (V_n)^{val} = G_{m,T_n}^{cpt,q} \times_{T_n} G_{m,T_n}^{cpt,q}.$$

Furthermore, by the construction of the Tate curve in [D-R], we have

$$E' \times_T T_n = U'_n/q^Z.$$

Hence $\mathfrak{E} \times_T T_n = G_{m,T_n}^{cpt,q}/q^Z$. Furtheremore, the group law $U_n \times U_n \longrightarrow U_n$ induces $V_n \longrightarrow U'_n$, and a diagram of log. schemes

$$\begin{array}{ccc} V_n/(q^{\mathbf{Z}} \times q^{\mathbf{Z}}) & \longrightarrow & U'_n/q^{\mathbf{Z}} \\ \downarrow & & \\ U_n/q^{\mathbf{Z}} \times_{T_n} U_n/q^{\mathbf{Z}}. & & \end{array}$$

By taking the formal completion $\varinjlim_n$ and taking the algebraization of this diagram (it can be checkes that the formal completion is algebraizable), we have a diagram of fine log. schemes over T of finite presentation

$$\begin{array}{ccc} V & \longrightarrow & E' \\ \downarrow & & \\ E' \times_T E' & & \end{array}$$

such that $V^{val} \xrightarrow{\cong} (E' \times_T E')^{val}$. This defines the a group law on $\mathfrak{E} = (E')^{val}$. The property (i) in (2.2.3) is clear from the construction and (ii) follows from the well known uniqueness of the group law on an elliptic curve.

(2.2.4) The author believes that there is a good theory of "logarithmic abelian variety" generalizing the above case of the Tate curve. There should exist a moduli space of logarithmic abelian varieties with logarithmic polarization (and with a suitable level structure) in the category of alg. val. log. spaces. He further believes that a good theory of "logarithmic Picard variety" exists. The "logarithmic Jacobian variety" should be related to the works [O-S] and [Is].

Here he reports, without proof, a result of his computation of the "logarithmic Picard group" of the Tate curve in a special case, which may help a reader who likes to construct logarithmic Picard varieties.

Let $\mathfrak{X}$, π and $\mathfrak{E}^{\pi}$ be as in (2.2.1) and let $f : \mathfrak{E}^{\pi} \longrightarrow \mathfrak{X}$ be the structural morphism. then if the underlying local ringed space of $\mathfrak{X}$ is the Spec of a field and $M_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^{x}$ is generated by the image of π (these assumptions were put to make the computation possible for the author, and seem to be unnecessary), then

$$\text{(2.2.4.1)} \qquad R^1 f_*(M_{\mathfrak{E}^{\pi}}^{gp}) \cong (M_{\mathfrak{X}}^{gp}/\pi^{Z}) \oplus Z .$$

The map $M_{\mathfrak{X}}^{gp} \longrightarrow R^1 f_* M_{\mathfrak{E}^{\pi}}^{gp}$ is defined as the cup product with the generator of $H^1(E_0', Z) \cong Z$, and the map $Z \longrightarrow R^1 f_* M_{\mathfrak{E}^{\pi}}^{gp}$ sends 1 to the class of the line bundle on E' corresponding to the divisor "origin of E'". The proof of (2.2.4.1) is based on

Proposition (2.2.5). *Let X, P and Φ be as in (1.3.3), and assume that the underlying scheme of X is quasi-compact and quasi-separated. Assume we are given a sheaf of abelian groups $\mathcal{F}_I$ on X_I for each $I \in \Phi$, and a homomorphism $h_{I,I'} : f_{I,I'}^{-1}(\mathcal{F}_I) \longrightarrow \mathcal{F}_{I'}$ for each pair (I, I') such that $I' \geq I$, satisfying*

$$h_{I',I''} \circ f_{I',I''}^{-1}(h_{I,I'}) = h_{I,I''} \quad \text{if } I'' \geq I' \geq I.$$

Then, if f_I denotes the canonical morphism $X^{val} \longrightarrow X_I$, we have

$$H^m(X^{val}, \varinjlim_I f_I^{-1}(\mathcal{F}_I)) \cong \varinjlim_I H^m(X_I, \mathcal{F}_I).$$

Proof. By (1.3.9)(1), this follows from SGA 4 Exp. VI.

§3. Properties of morphisms of logarithmic spaces.

In this section, we discuss how the definitions of properties of morphisms of schemes like "flat", "smooth", ..., are generalized to morphisms of fine log. schemes and also to morphisms of alg. val. log. spaces.

§3.1. Local properties of morphisms of fine log. schemes.

Here we discuss flatness, smoothness, etaleness and quasi-finiteness for morphisms of fine log. schemes.

Definition (3.1.1). (1) For a fine log. str. M on a scheme X, a chart of M is a homomorphism $P \longrightarrow M$ with P a finitely generated integral monoid (regarded as a constant sheaf on X) such that M is the log. str. associated to the pre-log. str. P with $P \longrightarrow M \longrightarrow \mathcal{O}_X$.

(2) For a morphism $f : Y \longrightarrow X$ of fine log. schemes, a chart of f is a triple $(P \longrightarrow M_X,\ Q \longrightarrow M_Y,\ P \xrightarrow{h} Q)$ of charts $P \longrightarrow M_X$ and $Q \longrightarrow M_Y$, and a homomorphism h such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{h} & Q \\ \downarrow & & \downarrow \\ f^{-1}(M_X) & \longrightarrow & M_Y \end{array}$$

is commutative.

A chart of f exists locally.

Definition (3.1.2). Let $f : Y \longrightarrow X$ be a morphism of fine log.

schemes. We say f is flat (resp. smooth, resp. etale, resp. quasi-finite) if locally on X and Y for the fppf topology (resp. etale topology, resp. etale topology, resp. fppf topology) in the usual sense, there exists a chart $(P \longrightarrow M_X, Q \longrightarrow M_Y, P \xrightarrow{h} Q)$ of f such that the induced morphisms of schemes

(3.1.2.1) $\quad Y \longrightarrow X \times_{\mathrm{Spec}(\mathbb{Z}[P])} \mathrm{Spec}(\mathbb{Z}[Q])$

(3.1.2.2) $\quad \mathrm{Spec}(\mathcal{O}_Y[Q^{gp}]) \longrightarrow \mathrm{Spec}(\mathcal{O}_Y[P^{gp}])$

are flat (resp. smooth, resp. etale, resp. quasi-finite) in the usual sense.

Here, by "f has the property .... locally on X and Y for the fppf (resp. etale) topology in the usual sense", I mean "there exist coverings of schemes $\{Y_\lambda \longrightarrow Y\}_\lambda$ and $\{X_{\lambda\mu} \longrightarrow X \times_Y Y_\lambda\}_\mu$ for each λ for the fppf (resp. etale) topology such that if we endow $X_{\lambda\mu}$ and Y_λ with the inverse image log. str.'s of M_X and M_Y, respectively, then each morphism $X_{\lambda\mu} \longrightarrow Y_\lambda$ has the property in problem".

Clearly we have the implications

etale $\Rightarrow$ smooth $\Rightarrow$ flat,

etale $\Rightarrow$ quasi-finite.

The implication (smooth) + (quasi-finite) $\Rightarrow$ (etale) is also true though it is not so evident.

To avoid confusions, we sometimes write "(log) flat", etc. instead of "flat", etc. when we are discussing the property of morphisms of logarithmic objects, and write "(cl) flat" (i.e. flat in the classical sense) etc. when we are discussing the property of a morphism of schemes.

Remark (3.1.3). Concerning the morphism (3.1.2.2) : For a non-empty scheme S and a homomorphism $h : G \longrightarrow H$ of finitely generated abelian groups, $\mathcal{O}_S[G] \longrightarrow \mathcal{O}_S[H]$ is flat (resp. smooth, resp. etale, resp. quasi-finite) if and only if;

the kernel of h is a finite group whose order is invertible on S

(resp. the kernel and the torsion part of cokernel of h are finite groups whose orders are invertible on S,

resp. the kernel and the cokernel of h are finite groups whose orders are invertible on S,

resp. the cokernel of h is finite).

Lemma (3.1.4). (1) Flat (resp. smooth, resp. etale, resp. quasi-finite) morphisms are stable under compositions, and under base changes using the fiber products (1.2.10) in the category of fine log. schemes.

(2) Let $f : Y \longrightarrow X$ be a morphism of fine log. schemes and assume $f^*M_X \xrightarrow{\cong} M_Y$. Then, f is flat (resp. smooth, resp. etale, resp. quasi-finite) if and only if the underlying morphism of schemes $Y \longrightarrow X$ is (cl) flat (resp. (cl) smooth, resp. (cl) etale, resp. (cl) quasi-finite).

(3) Let S be a scheme and let $P \longrightarrow Q$ be a morphism of finitely generated integral monoids. Endow $\mathrm{Spec}(\mathcal{O}_S[P])$ and $\mathrm{Spec}(\mathcal{O}_S[Q])$ with the log. str.'s associated to the canonical maps $P \longrightarrow \mathcal{O}_S[P]$ and $Q \longrightarrow \mathcal{O}_S[Q]$, respectively. Then, the morphism of fine log. schemes $\mathrm{Spec}(\mathcal{O}_S[Q]) \longrightarrow \mathrm{Spec}(\mathcal{O}_S[P])$ is flat (resp. smooth, resp. etale, resp. quasi-finite) if and only if the morphism of schemes $\mathrm{Spec}(\mathcal{O}_S[Q^{gp}]) \longrightarrow \mathrm{Spec}(\mathcal{O}_S[P^{gp}])$ is (cl) flat (resp. (cl) smooth, resp. (cl) etale, resp. (cl) quasi-finite).

Proof. All statements are proved easily except the one concerning the composition in (2) which follows from (3.1.6) below.

(3.1.5) By [Ka] (3.5), we have the following characterization of smooth (resp. etale) morphisms. (The formulation of log. str. is slightly different but the proof of (3.5) works in our present formulation.) A morphism $f : Y \longrightarrow X$ of fine log. schemes is smooth (resp. etale) if and only if it has the following properties (i)(ii).

(i) The underlying morphism of schemes $Y \longrightarrow X$ is locally of finite presentation.

(ii) For any commutative diagram of fine log. schemes

$$\begin{array}{ccc} T' & \xrightarrow{s} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle f} \\ T & \xrightarrow{t} & X \end{array}$$

such that T' is a closed subscheme of T defined by a nilpotent ideal of $\mathcal{O}_T$ and such that $i^*M_T \longrightarrow M_{T'}$ is an isomorphism, there exists locally on T a morphism (resp. there exists a unique morphism) $g : T \longrightarrow Y$ such that $gi = s$ and $fg = t$.

For examples of smooth morphism related to toroidal embeddings ([KKMS][Od]) or to semi-stable redution, see [Ka] (3.7).

Lemma (3.1.6). Let $f : Y \longrightarrow X$ be a morphism of fine log. schemes and let $\beta : P \longrightarrow M_X$ be a chart. Assume f is flat (resp. smooth, resp. etale, resp. quasi-finite). Then, fppf (resp. etale, resp. etale, resp. fppf) locally on X and on Y in the usual sense, there exists chart $(P \longrightarrow M_X, Q \longrightarrow M_Y, P \longrightarrow Q)$ including β satisfying the condition in (3.1.2). We can require further that $P^{gp} \longrightarrow Q^{gp}$ is injective.

Proof. Let $(P' \xrightarrow{\gamma} M_X, Q' \longrightarrow M_Y, P' \longrightarrow Q')$ be a chart satisfying (i)(ii) in (3.1.2). Fix $y \in Y$, $x = f(y) \in X$. By replacing P' with the inverse image P'' of $M_{X,x}$ under

$$P^{gp} \oplus (P')^{gp} \longrightarrow M^{gp}_{X,x} \quad ; \quad (a, b) \longrightarrow ab,$$

and by replacing Q' with the $(\ )^{int}$ of the push out of $P'' \longleftarrow P' \longrightarrow Q'$, we may assume that $\beta : P \longrightarrow M_X$ factors as $P \longrightarrow P' \xrightarrow{\gamma} M_X$. We may assume

$$P'/(P')^{\times} \xrightarrow{\cong} M_{X,x}/\mathcal{O}^{\times}_{X,x}, \qquad Q'/(Q')^{\times} \xrightarrow{\cong} M_{Y,y}/\mathcal{O}^{\times}_{Y,y}.$$

In the flat case and in the quasi-finite case (resp. In the smooth case and in the etale case), we can replace Q' with a push out Q'' of $Q' \longleftarrow (Q')^{\times} \longrightarrow G$ for a finitely generated abelian group G including $(Q')^{\times}$ such that $G/(Q')^{\times}$ is a finite group (resp. a finite group whose order is invertible on Y). After this replacing, we find a commutative diagram

$$\begin{array}{ccc} P^{gp} & \longrightarrow & H \\ \downarrow & & \downarrow \\ (P')^{gp} & \longrightarrow & (Q')^{gp} \end{array}$$

such that H is a finitely generated abelian group, $P^{gp} \longrightarrow H$ is injective, and

$$\mathrm{Coker}(P^{gp} \longrightarrow H) \xrightarrow{\cong} \mathrm{Coker}((P')^{gp} \longrightarrow (Q')^{gp}).$$

Let Q be the inverse image of $M_{Y,y}$ under $H \longrightarrow M^{gp}_{Y,y}$. Then, $(P \xrightarrow{\beta} M_X,\ Q \longrightarrow M_Y,\ P \longrightarrow Q)$ is a desired local chart at (y, x).

§3.2. Local properties of morphisms of alg. val. log. spaces.

In this §3.2, S denotes a scheme with a trivial log. str. We discuss flatness, smoothness, etaleness, and quasi-finiteness of morphisms of alg. val. log. spaces over S.

Definition (3.2.1). Let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of alg. val. log. spaces over S. We say f is flat (resp. smooth, resp. etale, resp. quasi-finite) if locally on $\mathfrak{X}$ and $\mathfrak{Y}$, there exist fine log. schemes X, Y which are S-schemes locally of finite presentation, and a (log) flat (resp. (log) smooth, resp. (log) etale, (log) quasi-finite) morphism $g : Y \longrightarrow X$ over S such that $g^{val} : Y^{val} \longrightarrow X^{val}$ coincides with $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ upto isomorphism over S.

It is easily seen that flat morphisms, smooth morphisms, etale morphisms, quasi-finite morphisms are stable under compositions and base changes.

We have the following log. version of the well known openness of a (cl) flat morphisms locally of finite presentation (EGA IV 2.4.6).

Theorem (3.2.2). <u>A flat morphism between alg. val. log. spaces over S is an open map.</u>

Proof. First, if $f : Y \longrightarrow X$ is a flat morphism of fine log. schemes whose underlying morphism of schemes is locally of finite presentation, and if $f^*M_X \xrightarrow{\cong} M_Y$, then $Y^{val} \longrightarrow X^{val}$ is an open map by (1.3.7), (3.1.4)(2) and the classical (without log) version of (3.2.2).

By this, we are reduced to proving the following fact. Let X be a fine log. scheme whose log. str. is associated to a homomorphism $P \longrightarrow \mathcal{O}_X$ for a finitely generated integral monoid P. Let Q be a finitely generated integral monoid and let $P \longrightarrow Q$ be a homomorphism such that $P^{gp} \longrightarrow Q^{gp}$ is injective. Let $\mathfrak{X} = X^{val}$, $\mathfrak{Y} = (X_Q)^{val}$ with X_Q as in (1.3.3). Then, $f : \mathfrak{Y}^{val} \longrightarrow \mathfrak{X}^{val}$ is an open map.

We are easily reduced to the following two cases.

(i) $Q = P \oplus N$ and $P \longrightarrow Q$ is a $\longrightarrow (a, 0)$.

(ii) Q^{gp}/P^{gp} is finite.

We consider the case (i). Let t be the generator of $N \subset Q$. Let $y \in \mathfrak{Y}$. Then one of the folowing three conditions is satisfied.

$(i\text{-}1)_y$ $a | t$ in $M_{\mathfrak{Y},y}$ for any $a \in P$.

$(i\text{-}2)_y$ $a | t$ and $t | b$ in $M_{\mathfrak{Y},y}$ for some $a, b \in P$.

$(i\text{-}3)_y$ $t | a$ in $M_{\mathfrak{Y},y}$ for any $a \in P$.

The argument for the case $(i\text{-}3)_y$ is similar to that for $(i\text{-}1)_y$, and so we give here the arguments for $(i\text{-}1)_y$ and $(i\text{-}2)_y$. Assume we are in the case $(i\text{-}1)_y$ (resp. $(i\text{-}2)_y$). The following fact is proved easily: If $\mathfrak{U}$ is an open neighbourhood of y in $\mathfrak{Y}$, there exist a finitely generated submonoid P' of P^{gp} containing P and a finitely generated submonoid Q' of Q^{gp} containing Q, and an open set U of $X_{Q'}$, having the following properties (a)(b).

(a) $y \in U^{val} \subset \mathfrak{U}$ and $f(y) \in (X_{P'})^{val}$.

(b) There exist $n \geq 1$ and $a \in P'$ (resp. $a, b \in P'$ such that $a | b$) for which Q' is generated over P' by t and $t^n a^{-1}$ (resp. by t, $t^n a^{-1}$ and bt^{-n}).

Then, the underlying morphism of schemes $X_{Q'} \longrightarrow X_{P'}$ is (cl) flat of finite presentation, and hence the image $V \subset X_{P'}$ of $U \subset X_{Q'}$ is open. Furthermore $X_{P'} \longrightarrow X_{Q'}$ is exact (1.3.6) as is checked easily, and hence $f(U^{val}) = V^{val}$ by (1.3.7). Hence $f(\mathfrak{U})$ contains an open neighbourhood V^{val} of $f(y)$ in $\mathfrak{X}$.

Next we consider the case (ii). It is sufficient to show that for any finitely generated submonoid Q' of Q^{gp} containing Q and any open set U of $\mathrm{Spec}(A_{Q'})$, $f(U^{val})$ is open in $\mathfrak{X}$. By replacing P

by $P^{gp} \cap Q'$ (then $\mathfrak{X}$ is replaced by its open set $(X_{(P^{gp} \cap Q')})^{val}$), we are reduced to the case where $Q' = Q$ and $P \longrightarrow Q$ is exact. Then from the exactness of $P \longrightarrow Q$ and the fact Q^{gp}/P^{gp} is of torsion, we can easily deduce that $Y \longrightarrow X$ is exact. By (1.3.7), it is sufficient to show that $X_Q \longrightarrow X$ is an open map.

Lemma (3.2.3). Let $P \longrightarrow Q$ be an exact homomorphism of finitely generated integral monoids such that $P^{gp} \longrightarrow Q^{gp}$ is an injection with finite cokernel. Let X be a scheme and let $P \longrightarrow \mathcal{O}_X$ be a homomorphism. Then, $g : X_Q \longrightarrow X$ is an open map.

Proof. The map g is surjective as is seen by the reduction to the case where X is the Spec of a field. Since g is entire it is a closed map and hence the topology of X is the quotient topology of the topology of X_Q. Let U be an open set of X_Q. Consider the action of the Z-group scheme $\mathrm{Spec}(Z[Q^{gp}/P^{gp}])$ on X_Q

$$(*) \qquad \mathrm{Spec}(Z[Q^{gp}/P^{gp}]) \times_Z X_Q \longrightarrow X_Q$$

defined by $\mathcal{O}_{X_Q} \longrightarrow \mathcal{O}_{X_Q}[Q^{gp}/P^{gp}]$; $a \longrightarrow a \otimes a$ $(a \in Q)$. Then g is equivaliant with respect to the trivial action of $\mathrm{Spec}(Z[Q^{gp}/P^{gp}])$ on X, and it is seen easily that $g^{-1}(g(U))$ coincides with the orbit of U, i.e. the image of $\mathrm{Spec}(Z[P^{gp}/Q^{gp}]) \times_Z U \longrightarrow X_Q$. But the last map is an open map since the morphism (*) above is flat and of finite presentation.

We end §3.2 by giving remarks on flatness and quasi-finiteness.

Proposition (3.2.4). If $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ is a flat morphism of alg. val. log. spaces over S, the ring homomorphism $\mathcal{O}_{\mathfrak{X},x} \longrightarrow \mathcal{O}_{\mathfrak{Y},y}$ is flat for any $y \in \mathfrak{Y}$ and $x = f(y)$.

This follows from the fact that if $V \longrightarrow W$ is an injective homomorphism between valuative monoids, then $Z[V] \longrightarrow Z[W]$ is flat.

The converse of this is not true. For example, let $\mathfrak{X} = S$ with the trivial log. str., and let $\mathfrak{Y}$ be the scheme S endowed with the log. str. associated to $N \longrightarrow \mathcal{O}_S$; $1 \longrightarrow 0$. Then, the canonical morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$ is not flat if $S \neq \phi$.

Proposition (3.2.5). A morphism $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ of alg. val. log. spaces over S is quasi-finite if and only if for any alg. val. log. space $\mathfrak{X}'$ over S and any morphism $\mathfrak{X}' \longrightarrow \mathfrak{X}$ over S, the inverse image of any element of $\mathfrak{X}'$ under $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}' \longrightarrow \mathfrak{X}'$ is discrete.

We omit the proof of this fact.

§3.3. Global properties of morphisms of alg. val. log. spaces.

We fix a scheme S with the trivial log. str.

Definition (3.3.1). Let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of alg. val. log. spaces over S.

(1) We say f is quasi-compact if for any quasi-compact open subset $\mathfrak{U}$ of $\mathfrak{X}$, $f^{-1}(\mathfrak{U})$ is quasi-compact.

(2) We say f is quasi-separated if the diagonal morphism $\mathfrak{Y} \longrightarrow \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ is quasi-compact.

(3) We say f is a closed immersion if locally on $\mathfrak{X}$ (not locally on $\mathfrak{Y}$), there exist fine log. schemes X, Y over S which are locally of finite presentation as an S-schemes, and a morphism $i : Y \longrightarrow X$ over S having the following properties: The underlying morphism of schemes $Y \longrightarrow X$ is a closed immersion, $i^*M_X \xrightarrow{\cong} M_Y$, and $i^{val} : Y^{val} \longrightarrow X^{val}$ coincides with $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ upto isomorphism over S.

(4) We say f is separated if the diagonal morphism $\mathfrak{Y} \longrightarrow \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ is a closed immersion.

(5) We say f is proper if it is quasi-compact and separated and the following condition (*) is satisfied.

(*) For any alg. val. log. space $\mathfrak{X}'$ over S and a morphism $\mathfrak{X}' \longrightarrow \mathfrak{X}$ over S, the image of any closed subset of $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ in $\mathfrak{X}'$ is closed.

As is seen easily, quasi-compact morphisms, quasi-separated morphisms, closed immersions, separated morphisms, and proper morphisms are stable under base changes. These properties of morphisms are local on the base $\mathfrak{X}$.

We give remarks on closed immersions and separate morphisms.

Lemma (3.3.2). Let X, Y be fine log. schemes over S which are locally of finite type over S, and let $i : Y \longrightarrow X$ be a morphism over S such that the underlying morphism of schemes $Y \longrightarrow X$ is a closed immersion and such that $i^*M_X \longrightarrow M_Y$ is surjective. Then $i^{val} : Y^{val} \longrightarrow X^{val}$ is a closed immersion.

Corollary (3.3.3). For any alg. val. log. spaces $\mathfrak{X}$, $\mathfrak{Y}$ over S and any morphism $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ over S, f is separated locally on $\mathfrak{Y}$. In particular, f is separated if and only if the image of $\mathfrak{Y}$ in $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ is closed.

Indeed, for any fine log. schemes X, Y over S which are locally of finite presentation over S and for any morphism $Y \longrightarrow X$ over S whose underlying morphism of schemes $Y \longrightarrow X$ is separated, the morphism $Y^{val} \longrightarrow X^{val}$ is separated, for: If Z denotes the fiber product of $Y \longrightarrow X \longleftarrow Y$ in the category of fine log. schemes, the diagonal morphism $Y \longrightarrow Z$ satisfies the assumption on $Y \longrightarrow X$ of (3.3.2).

Proof of (3.3.2). Working locally on X, we may assume that X is quasi-compact and there is a chart $P \longrightarrow M_X$. For each $y \in Y$, let $P(y)$ be the inverse image of $M_{Y,y}$ under $P^{gp} \longrightarrow M^{gp}_{Y,y}$. Then, on some open neighbourhood $U(y)$ of y, we have a chart $P(y) \longrightarrow M_{U(y)}$. Since Y is quasi-compact, there exists a finite number of points $y_1, \ldots, y_r$ of Y such that $\bigcup_{i=1}^{r} U(y_i) = Y$. For each i, take elements a_j, b_j $(1 \le j \le m(i))$ of P such that $P(y_i)$ is generated by $a_j b_j^{-1}$ $(1 \le j \le m(i))$ over P. Let I be the product of all ideals (a_i, b_i) $(1 \le i \le r,\ 1 \le j \le m(i))$. Then, if X' denotes X_I in (1.3.3) and Y' denotes the fiber product of $Y \longrightarrow X \longleftarrow X_I$ in the category of fine log. schemes, $f' : Y' \longrightarrow X'$ satisfies $(f')^*M_{X'} \xrightarrow{\cong} M_{Y'}$, and the underlying morphism of $Y' \longrightarrow X'$ is a closed immersion.

Proposition (3.3.4) (valuative criterion). Let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of alg. val. log. spaces over S. Then, f is proper (resp. separated) if and only if f is quasi-compact and separated (resp. f is quasi-separated) and the following condition (**) is satisfied.

(**) For any valuation ring V with field of fractions K and for any commutative diagram of local ringed spaces

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{s} & \mathfrak{Y} \\ j \downarrow & & \downarrow f \\ \mathrm{Spec}(V) & \xrightarrow{t} & \mathfrak{X} \end{array}$$

with j the canonical morphism, there exists at least one (resp. at most one) morphism $g : \mathrm{Spec}(V) \longrightarrow \mathfrak{Y}$ such that $gj = s$ and $fg = t$.

(Note that in the condition (**), we neglect the log. str.'s of $\mathfrak{X}$ and $\mathfrak{Y}$.)

Example (3.3.5). Let $\mathfrak{X}$, π, and the Tate curve $\mathfrak{E}^{\pi}$ over $\mathfrak{X}$ be as in (2.2.1). Then:

(1) $\mathfrak{E}^{\pi}$ is proper and smooth over $\mathfrak{X}$.

(The properness is proved by the valuative criterion.)

(2) $G^{cpt,\pi}_{m,\mathfrak{X}} \longrightarrow \mathfrak{X}$ is separated and satisfies the unique existence of g for any commutative diagram as in the above (**), but is not proper ($G^{cpt,\pi}_{m,\mathfrak{X}}$ is not quasi-compact) if $\mathfrak{X} \ne \phi$.

(3) Let $\mathfrak{Y}$ be the push out of $G^{cpt}_{m,\mathfrak{X}} \longleftarrow G^{cpt,\pi}_{m,\mathfrak{X}} \longrightarrow G^{cpt}_{m,\mathfrak{X}}$ where the two arrows are the inclusions. Then, $\mathfrak{Y} \longrightarrow \mathfrak{X}$ is quasi-compact

and satisfies the unique existence of g for any commutative diagram as in the above (**), but is not proper ($\mathfrak{Y} \longrightarrow \mathfrak{X}$ is not separated) if $\mathfrak{X} \neq \emptyset$. This example shows that the implication "g in (**) is at most one $\Rightarrow f$ is separated" does not hold.

Before we prove (3.3.4), we give some lemmas.

Definition (3.3.6). Let $\mathfrak{X}$ be an alg. val. log. space over S. We say $\mathfrak{X}$ is quasi-separated if for any quasi-compact open subsets $\mathfrak{U}$, $\mathfrak{V}$ of $\mathfrak{X}$, $\mathfrak{U} \cap \mathfrak{V}$ is quasi-compact.

The following results (3.3.7)-(3.3.9) are deduced from (1.3.9)(1).

Lemma (3.3.7). Let X be a fine log. scheme over S which is locally of finite presentation as an S-scheme. Then X^{val} is quasi-separated if and only if the underlying scheme of X is (cl) quasi-separated.

Lemma (3.3.8). Let $\mathfrak{X}$ be an alg. val. log. space over S. Then $\mathfrak{X}$ is quasi-separated if and only if for any alg. val. log. spaces $\mathfrak{Y}_1$, $\mathfrak{Y}_2$ over S which are quasi-compact as topological spaces and for any morphisms $\mathfrak{Y}_1 \longrightarrow \mathfrak{X}$, $\mathfrak{Y}_2 \longrightarrow \mathfrak{X}$ over S, $\mathfrak{Y}_1 \times_{\mathfrak{X}} \mathfrak{Y}_2$ is quasi-compact.

Lemma (3.3.9). Let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of alg. val. log. spaces over S. Then, f is quasi-compact if and only if for any alg. val. log. space $\mathfrak{X}'$ over S which is quasi-compact as a topological space and for any morphism $\mathfrak{X}' \longrightarrow \mathfrak{X}$ over S, $\mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ is quasi-compact as a topological space.

The proofs of the following (3.3.10) and (3.3.11) are easy and we omit them.

Lemma (3.3.10). Let T be a good quasi-compact topological space (1.3.11) and let F be a subset of T. Assume the following *(i)(ii)(iii). (i) Each point of T has a fundamental system of neighbourhoods consisting of quasi-compact subsets. (ii) If $x \in T$, $z \in F$ and $x \in \overline{\{z\}}$, then $x \in F$. (iii) F is closed in X with respect to the "new" topology (1.3.11). Then, F is closed in T with respect to the original topology.*

Lemma (3.3.11). *Let $\mathfrak{X}$ be a quasi-compact alg. val. log. space over S. Then, $\mathfrak{X}$ is a good quasi-compact space.*

(3.3.12) We prove (3.3.4). It is sufficient to prove that f is quasi-compact and satisfies the condition (*) in (3.3.1)(5) if and only if f is quasi-compact and satisfies the "at least one" version of the condition (**) in (3.3.4). Indeed, the statement of (3.3.4) for the proper morphisms then follows from it directly, and that for separated morphisms follows by applying it to the morphism $\mathfrak{Y} \longrightarrow \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ (note the "at least one" version of (**) for $\mathfrak{Y} \longrightarrow \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$ is equivalent to the "at most one" version of (**) for $\mathfrak{Y} \longrightarrow \mathfrak{X}$).

Assume f is quasi-compact and satisfies the "at least one" version of (**). We prove that for any closed subset E of $\mathfrak{Y}$, f(E) is closed. We may assume $\mathfrak{X}$ is quasi-compact. Then, $\mathfrak{Y}$ is quasi-compact. By (3.3.11), $\mathfrak{Y}$ and $\mathfrak{X}$ are compact for "new" topologies, and hence f(E) is closed with respect to the "new" topology of $\mathfrak{X}$. By (3.3.10), it remains to prove that if $x \in \mathfrak{X}$, $z \in f(E)$, $x \in \overline{\{z\}}$, then $x \in f(E)$. Take $y \in E$ such that $z = f(y)$. Let $K = \kappa(y)$, and let V be a valuation ring of K which dominates the image of $\mathcal{O}_{\mathfrak{X},x}$ in K. Then, we have a commutative square of local ringed spaces as in (**) in which the image of

$\mathrm{Spec}(K)$ in $\mathfrak{Y}$ is y and the image of the closed point of $\mathrm{Spec}(V)$ in $\mathfrak{X}$ is x. Let $u \in \mathfrak{Y}$ be the image of the closed point of $\mathrm{Spec}(V)$ under $g : \mathrm{Spec}(V) \longrightarrow \mathfrak{Y}$. Then, $f(u) = x$ and $u \in \overline{\{y\}} \subset E$. Hence we have $x \in f(E)$.

Conversely, assume f is quasi-compact and satisfies (*). We show that "at least one version" of (**) is satisfied. We may assume $\mathfrak{X}$ is quasi-compact and quasi-separated. Consider a commutative square as in (**). Let $\mathfrak{X}'$ be an alg. val. log. space over S endowed with a morphism of local ringed spaces $\mathrm{Spec}(V) \longrightarrow \mathfrak{X}'$ and a morphism of log. spaces $\mathfrak{X}' \longrightarrow \mathfrak{X}$ over S such that the composition of these two morphisms coincides with $t : \mathrm{Spec}(V) \longrightarrow \mathfrak{X}'$, $\mathfrak{X}'$ is quasi-compact and quasi-separated, and $M_{\mathfrak{X}'}$ coincides with the inverse image of $M_{\mathfrak{X}}$. We have a commutative diagram of local ringed spaces

$$\begin{array}{ccc} \mathrm{Spec}(K) & \xrightarrow{s'} & \mathfrak{Y}' \overset{\mathrm{def}}{=} \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}' \\ j \downarrow & & \downarrow f' \\ \mathrm{Spec}(V) & \xrightarrow{t'} & \mathfrak{X}' . \end{array}$$

Let $x(\mathfrak{X}')$ be the image of the closed point of $\mathrm{Spec}(V)$ in $\mathfrak{X}'$, and let $y(\mathfrak{X}')$ be the image of $\mathrm{Spec}(K)$ in $\mathfrak{Y}'$. Let

$$C(\mathfrak{X}') = \{u \in \mathfrak{Y}' \ ; \ u \in \overline{\{y(\mathfrak{X}')\}} \ \text{ and } \ f'(u) = x(\mathfrak{X}')\}.$$

Then $C(\mathfrak{X}')$ is not empty by the condition (*). Since $\mathfrak{Y}'$ and $\mathfrak{X}'$ are compact for the "new" topologies (3.3.11), $C(\mathfrak{X}')$ is compact with respect to the "new" topology of $\mathfrak{Y}'$. When $\mathfrak{X}'$ ranges, $\mathfrak{X}'$ form a filtered category. The transition maps are quasi-compact by (3.3.8) and hence continuous for the "new" topologies. From this we have $\varprojlim_{\mathfrak{X}'} C(\mathfrak{X}') \neq \emptyset$ since the filtered inverse limit of non-empty compact sets is non-empty. Let $u = (u(\mathfrak{X}'))_{\mathfrak{X}'}$ be an element of $\varprojlim_{\mathfrak{X}'} C(\mathfrak{X}')$. Since $\varinjlim_{\mathfrak{X}'} \mathcal{O}_{\mathfrak{X}',x(\mathfrak{X}')} \xrightarrow{\cong} V$, the image of $\varinjlim_{\mathfrak{X}'} \mathcal{O}_{\mathfrak{Y}',u(\mathfrak{X}')} \longrightarrow K$ is a subring of K dominating V, and hence it coincides with V. From this we obtain $\mathrm{Spec}(V) \longrightarrow \varprojlim \mathfrak{Y}'$ and hence $\mathrm{Spec}(V) \longrightarrow \mathfrak{Y}$.

Proposition (3.3.13). Let X, Y be fine log. schemes over S which are locally of finite presentation as S-schemes, and let $f : Y \longrightarrow X$ be a morphism over S. Then, $f^{val} : Y^{val} \longrightarrow X^{val}$ is quasi-compact (resp. quasi-separated, separated, proper) if and only if the underlying morphism of schemes $Y \longrightarrow X$ is (cl) quasi-compact (resp. (cl) quasi-separated, resp. (cl) separated, resp. (cl) proper).

Proof. The statements for quasi-compact morphisms and quasi-separated morphisms are deduced from (1.3.9)(1). The statements for separated morphisms and proper morphisms are deduced from the valuative criterion (we omit the details).

§3.4. Small morphisms, affine morphisms and finite morphisms.

In §3.4, S denotes a scheme with the trivial log. str.

Definition (3.4.1). Let $f : Y \longrightarrow X$ be a morphism of log. spaces. We say f is small if for any $y \in Y$, the cokernel of $(f^*M_X)^{gp}_y \longrightarrow M^{gp}_{Y,y}$ is a torsion group.

For example, a quasi-finite morphism between fine log. schemes, and quasi-finite morphisms between alg. val. log. spaces over S are small.

Lemma (3.4.2). Let $f : Y \longrightarrow X$ be a morphism of fine log.

schemes. Then, f is small if and only if $f^{val} : Y^{val} \longrightarrow X^{val}$ is small.

Proof. Let $\mathfrak{Y} = Y^{val}$ and let $y \in Y$. Then $M^{gp}_{Y,y}/\mathcal{O}^{\times}_{Y,y} \longrightarrow M_{\mathfrak{Y},y'}/\mathcal{O}^{\times}_{\mathfrak{Y},y'}$ is surjective for any point y' of $\mathfrak{Y}$ lying over y. This proves the "only if" part of (3.4.2). To prove the "if" part, it suffices to show that there exists $y' \in \mathfrak{Y}$ lying over y such that the kernel of $M^{gp}_{Y,y}/\mathcal{O}^{\times}_{Y,y} \longrightarrow M^{gp}_{\mathfrak{Y},y'}/\mathcal{O}^{\times}_{\mathfrak{Y},y'}$ is finite. This is reduced to

Lemma (3.4.3). Let P be a finitely generated integral monoid. Then, there exists a valuative submonoid V of P^{gp} containing P such that the kernel of $P^{gp}/P^{\times} \longrightarrow V^{gp}/V^{\times}$ is finite.

(If P is saturated, this property of V implies $P^{gp}/P^{\times} \xrightarrow{\cong} V^{gp}/V^{\times}$.)

Proof. Call an ideal p of P a prime ideal if $P \setminus p$ is a submonoid of P. Following the analogy with commutative algebra, we define $\dim(P)$ by using chains of prime ideals. Then,

(3.4.3.1) $\quad \dim(P) = \mathrm{rank}(P^{gp}/P^{\times})$.

(This is reduced to the case P is saturated. In this case, a prime ideal of P corresponds to a "face" of $P/P^{\times}$ in the geometry of $P^{gp}/P^{\times} \otimes \mathbb{R}$ ([KKMS][Od]), and (3.4.3.1) follows from this geometric interpretation.) For a prime ideal p of P, let

$$P_p = \{ab^{-1} \; ; \; a \in P, \quad b \in P \setminus p\} \subset P^{gp}.$$

Then $(P \setminus p)^{gp} = (P_p)^{\times}$, $\quad \dim(P \setminus p) + \dim(P_p) = \dim(P)$.

We prove (3.4.3) by induction on $\dim(P)$. We assume P is saturated without a loss of generality. If $\dim(P) = 0$, then P is a group. Assume $\dim(P) \geq 1$. Take a prime ideal p of P such that $\dim(P_p) = 1$. Then $P_p/(P_p)^{\times} \cong \mathbb{N}$. By induction on $\dim(P)$, there exists a valuative submonoid V' of $(P_p)^\times$ containing $P \setminus p$ such that $(P_p)^\times/(P \setminus p)^\times \xrightarrow{\cong} (V')^{gp}/(V')^\times$. Let $V = (P_p \setminus (P_p)^\times) \cup V'$. Then V has the desired property.

The main result of §3.4 is

Theorem (3.4.4). Let X be a fine log. scheme over S which is locally of finite presentation as an S-scheme and which has a chart $P \longrightarrow M_X$. Let $\mathfrak{X} = X^{val}$. Let $\mathcal{G}_{\mathfrak{X}}$ be the category of val. alg. log. space $\mathfrak{Y}$ over S endowed with a small morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$ over S such that $\mathfrak{Y}$ is quasi-compact and quasi-separated (3.3.6). On the other hand, for a non-empty ideal I of P, let X_I be as in (1.3.3) and let $\mathcal{G}_I$ be the category of fine log. schemes over X_I having the following properties: The morphism $Y \longrightarrow X_I$ is small, Y is locally of finite presentation as a scheme over S, and Y is quasi-compact and quasi-separated. Then, we have an equivalence of categories

$$\varinjlim_{I \in \Phi} \mathcal{G}_I \xrightarrow{\sim} \mathcal{G}_{\mathfrak{X}},$$

where Φ is as in (1.3.3). Here for $I, J \in \Phi$ such that $I \le J$ with respect to the ordering on Φ defined in (1.3.3), the functor $\mathcal{G}_I \longrightarrow \mathcal{G}_J$ is defined by sending Y to the $(\ )^{sat}$ (1.2.9) of the fiber product $Y \times_X X_I$ in the category of fine log. schemes, and the functor $\mathcal{G}_I \longrightarrow \mathcal{G}_{\mathfrak{X}}$ is defined by sending Y to Y^{val}.

We use the following lemma in the proof of (3.4.4).

Lemma (3.4.5). Let X be a fine log. scheme having a chart $P \longrightarrow M_X$. Let $Y \longrightarrow X$ and $Z \longrightarrow Y$ be small morphisms of fine log. schemes. Assume Z is quasi-compact.

(1) If I is an element of Φ which is sufficiently large for

the ordering of Φ,

$$(Z \times_X X_I)^{sat} \longrightarrow (Y \times_X X_I)^{sat}$$

is exact.

(2) If the log. str. of Z^{val} coincides with the inverse image of that of Y^{val}, then for an element I of Φ having the property in (1), the log. str. of $(Z \times_X X_I)^{sat}$ coincides with the inverse image of that of $(Y \times_X X_I)^{sat}$.

(3) If $Z^{val} \xrightarrow{\cong} Y^{val}$, then for an element I of Φ which is sufficiently large with respect to the ordering of Φ, the morphism $(Z \times_X X_I)^{sat} \longrightarrow (Y \times_X X_I)^{sat}$ is an isomorphism.

Proof of (1). The proof is similar to that of (3.3.2). For $z \in Z$, define $P(z) \subset P^{gp}$ just as there. Then, for some open neighbourhood $U(z)$ of z, th homomorphism $P(z) \longrightarrow M_{Z,z}$ extends to an exact homomorphism $P(z) \longrightarrow M_{U(z)}$. Take $z_1, \ldots, z_r \in Z$ such that $Z = \bigcup_{i=1}^{r} U(z_i)$, and take $a_{ij}, b_{ij} \in P$ and define I using them just as there. Then it is easily seen that any element I' of Φ such that $I' \geq I$ in Φ has the property stated in (1).

Proof of (2). By (3.4.3), for any point z of $(Z \times_X X_I)^{sat}$, there exists a point z' of Z^{val} lying over z such that $(M^{gp}/\mathcal{O}^{\times})_z \xrightarrow{\cong} (M^{gp}/\mathcal{O}^{\times})_{z'}$. From this we can deduce (2) easily.

Proof of (3). By (1)(2), we may assume that the log. str. of Z is the inverse image of that of Y, and that $Y \longrightarrow X$ is exact. By working locally on Y, and by using the fact Y is quasi-compact (1.3.9)(1), we may assume that $Y \longrightarrow X$ has a chart $(P \longrightarrow M_X, Q \longrightarrow M_Y, P \longrightarrow Q)$ which extends the given chart $P \longrightarrow M_X$ and for which $P \longrightarrow Q$ is exact and the cokernel of $P^{gp} \longrightarrow Q^{gp}/Q^{\times}$ is finite. We may also assume Y is quasi-separated. By (1.4.2), there exists a finitely generated non-empty ideal $J = (b_1, \ldots, b_m)$ of Q such that $Y_J \longrightarrow Y$ factors through $Z \longrightarrow Y$. Take $n \geq 1$ such that $b_i^n \bmod Q^x$ is the image of some $a_i \in P$ under $P \longrightarrow Q/Q^x$ for $i = 1, \ldots, m$, and let I be the ideal $(a_1, \ldots, a_m)$ of P.

Then

$$(Y_J \times_X X_I)^{sat} \xrightarrow{\cong} (Y \times_X X_I)^{sat}, \quad (Z_J \times_X X_I)^{sat} \xrightarrow{\cong} (Z \times_X X_I)^{sat}.$$

These show that $(Z \times_X X_I)^{sat} \xrightarrow{\cong} (Y \times_X X_I)^{sat}$.

(3.4.6) We prove (3.4.4). The functor $\varinjlim \mathcal{G}_I \longrightarrow \mathcal{G}_{\mathfrak{X}}$ is fully faithful by (1.4.2) and (3.4.5)(3). We prove that this functor is essentially surjective. Let $\mathfrak{Y}$ be an object of $\mathcal{G}_{\mathfrak{X}}$. take a finite open covering $\mathfrak{Y} = \bigcup_{i=1}^{r} (Y_i)^{val}$ where each Y_i is a fine log. scheme over S which is locally of finite presentation as an S-scheme and which is quasi-compact and quasi-separated as a scheme. By induction on r, $\bigcup_{i=1}^{r-1} (Y_i)^{val}$ comes from $\varinjlim_I \mathcal{G}_I$. Hence we may assume $r = 2$. Passing to $\mathcal{G}_I$ for a sufficiently large I in the ordered set Φ, we may assume that for $i = 1, 2$, there exists a quasi-compact and quasi-separated open subscheme U_i of Y_i $(i = 1, 2)$ such that $(U_i)^{val} = (Y_1)^{val} \cap (Y_2)^{val}$ in $(Y_i)^{val}$. By the fully faithfulness of $\varinjlim_I \mathcal{G}_I \longrightarrow \mathcal{G}_{\mathfrak{X}}$, passing to $\mathcal{G}_I$ for I sufficiently large, we may assume that $U_1^{val} \cong U_2^{val}$ comes from an isomorphism $U_1 \cong U_2$ over X. Then, $\mathfrak{Y} = Y^{val}$ where Y is the push out of $Y_1 \longleftarrow U_1 \cong U_2 \longrightarrow Y_2$.

Definition (3.4.7). Let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a morphism of alg. val. log. spaces over S. We say f is affine (resp. finite) if the following two conditions are satisfied.

(i) f is small.

(ii) Locally on $\mathfrak{X}$ (not on $\mathfrak{Y}$), there exist fine log. schemes X, Y over S which are locally of finite presentation over S and a morphism $g : Y \longrightarrow X$ over S such that $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ coincides with $g^{val} : Y^{val} \longrightarrow X^{val}$ up to isomorphism over S and such that the underlying morphism of $g : Y \longrightarrow X$ is (cl) affine (resp. (cl) finite).

Affine morphisms and finite morphisms are stable under compositions and base changes. (The stability under compositions is not clear, but is deduced from (3.4.4) and (3.4.5)(3) (we omit the details)).

Example (3.4.8). Let $\mathfrak{X}$, π, and the Tate curve $\mathfrak{E}^{\pi}$ over $\mathfrak{X}$ be as in (2.2.1). Then, for a non-zero integer n, the group object ${}_n\mathfrak{E}^{\pi} = \mathrm{Ker}(n : \mathfrak{E}^{\pi} \longrightarrow \mathfrak{E}^{\pi})$ over $\mathfrak{X}$ is finite and flat over $\mathfrak{X}$. In fact, we have a surjective homomorphism

$${}_n\mathfrak{E}^{\pi} \longrightarrow Z/nZ \times \mathfrak{X} \quad ; \quad a \longrightarrow r \in Z/nZ \quad \text{when} \quad a \in M^{gp}, \quad a^n = \pi^r,$$

and the inverse image of $(r \bmod Z/nZ) \times \mathfrak{X}$ $(r \in Z,\ r \geq 0)$ in ${}_n\mathfrak{E}^{\pi}$ is $\mathrm{Spec}(\mathcal{O}_S[t_1]) \times_{\mathrm{Spec}(\mathcal{O}_S[t_2])} \mathfrak{X}$ where t_1, t_2 are indeterminates, $\mathrm{Spec}(\mathcal{O}_S[t_i])$ $(i = 1, 2)$ are endowed with the log. str. associated to $N \longrightarrow \mathcal{O}_S[t_i]$; $1 \longrightarrow t_i$, and $t_2 \longrightarrow t_1^n$, $t_2 \longrightarrow \pi$ in $M_{\mathfrak{X}}$. It is easy to see that the morphism $\mathrm{Spec}(\mathcal{O}_S[t_1]) \longrightarrow \mathrm{Spec}(\mathcal{O}_S[t_2])$ is (log) finite and (log) flat.

Proposition (3.4.9). <u>The equivalence of the categories (3.4.4) induces an equivalence</u> $\varinjlim_I \mathcal{G}_I^{cl} \xrightarrow{\sim} \mathcal{G}_{\mathfrak{X}}^{cl}$ <u>and</u> $\varinjlim_I \mathcal{G}_I^{qf} \xrightarrow{\sim} \mathcal{G}_{\mathfrak{X}}^{qf}$ <u>(resp.</u> $\varinjlim_I \mathcal{G}_I^{aff} \xrightarrow{\sim} \mathcal{G}_{\mathfrak{X}}^{aff}$ <u>and</u> $\varinjlim_I \mathcal{G}_I^{fin} \xrightarrow{\sim} \mathcal{G}_{\mathfrak{X}}^{fin}$, <u>if X is quasi-compact and quasi-separated), where:</u> $\mathcal{G}_I^{cl}$ <u>(resp.</u> $\mathcal{G}_{\mathfrak{X}}^{cl}$) <u>denotes the full</u>

subcategory of $\mathcal{Y}_I$ (resp. $\mathcal{Y}_{\mathfrak{X}}$) consisting of objects whose log. str. coincides with the inverse image of the log. str. of $(X_I)^{sat}$ (resp. $\mathfrak{X}$), $\mathcal{Y}_I^{qf}$ (resp. $\mathcal{Y}_I^{aff}$, resp. $\mathcal{Y}_I^{fin}$) denotes the full subcategory of $\mathcal{Y}_I$ consisting of objects Y such that the underlying morphism of schemes $Y \longrightarrow X$ is (cl) quasi-finite (resp. (cl) affine, resp. (cl) finite) and $\mathcal{Y}_{\mathfrak{X}}^{qf}$ (resp. $\mathcal{Y}_{\mathfrak{X}}^{aff}$, resp. $\mathcal{Y}_{\mathfrak{X}}^{fin}$) denotes the full subcategory of $\mathcal{Y}_{\mathfrak{X}}$ consisting of objects $\mathfrak{Y}$ such that $\mathfrak{Y} \longrightarrow \mathfrak{X}$ is quasi-finite (resp. affine, resp. finite).

Proof. The statement concerning $\mathcal{Y}^{cl}$ (resp. $\mathcal{Y}^{qf}$, resp. $\mathcal{Y}^{aff}$, resp. $\mathcal{Y}^{fin}$) follows from (3.3.4) and (3.4.5)(2) (resp. (3.3.4), (3.4.5)(1) and (3.4.10) below, resp. (3.4.4) easily, resp. (3.4.4) easily).

Proposition (3.4.10). Let $f : Y \longrightarrow X$ be a morphism of fine log. schemes.

(1) Assume f is exact. Then, f is (log) quasi-finite if and only if f is small and the underlying morphism of schemes $Y \longrightarrow X$ is (cl) quasi-finite.

(2) Assume X and Y are S-schemes locally of finite presentation. Then f is quasi-finite if and only if $f^{val} : Y^{val} \longrightarrow X^{val}$ is quasi-finite.

Proof. Excercise.

Remark (3.4.11). The analogues of (3.4.10)(2) for flatness, smoothness and etaleness are not true. For example, let $X = \mathrm{Spec}(\mathcal{O}_S[t_1, t_2]/(t_1^2, t_2^2))$, $Y = \mathrm{Spec}(\mathcal{O}_S[t_1, t_2]/(t_1^2, t_1t_2, t_2^2))$ where t_1, t_2 are indeterminates, and endow X (resp. Y) with the log. str. associated to $N^2 \longrightarrow \mathcal{O}_X$ (resp. $\mathcal{O}_Y$) ; $(m, n) \longrightarrow t_1^m t_2^n$. Then, $Y^{val} \longrightarrow X^{val}$ is an isomorphism but $Y \longrightarrow X$ is not (log) flat.

Proposition (3.4.12). A morphism between alg. val. log. spaces over S is finite if and only if it is proper and quasi-finite.

Proof. The "only if" part is easy, and the "if" part follows from the part concerning $\mathcal{P}^{qf}$ of (3.4.9), the part concerning properness of (3.3.13), and the "without log" version of (3.4.12) ([EGA] IV 8.11.1).

§4. Etale sites, flat sites, and fundamental groups.

Throughout this §4, we fix a scheme S with a trivial log. str. and an alg. val. log. space $\mathfrak{X}$ over S.

§4.1. Logarithmic etale (or flat) sites.

Definition (4.1.1). We define the logarithmic flat (resp. logarithmic etale) site of $\mathfrak{X}$ which we denote by $\mathfrak{X}_{et}^{log}$ (resp. $\mathfrak{X}_{fl}^{log}$) to be the following site. An object of $\mathfrak{X}_{et}^{log}$ (resp. $\mathfrak{X}_{fl}^{log}$) is an alg. val. log. space $\mathfrak{Y}$ over S endowed with an etale morphism (resp. endowed with a morphism) $\mathfrak{Y} \longrightarrow \mathfrak{X}$. A covering $\{f_\lambda : \mathfrak{Y}_\lambda \longrightarrow \mathfrak{Y}\}_\lambda$ is a family of etale (resp. flat and quasi-finite) morphisms such that $\bigcup_\lambda f_\lambda(\mathfrak{Y}_\lambda) = \mathfrak{Y}$.

We shall prove

Theorem (4.1.2). Let $\mathfrak{Z}$ be an alg. val. log. space over S endowed with a morphism $\mathfrak{Z} \longrightarrow \mathfrak{X}$ over S. Then the presheaf

$$\mathfrak{Y} \longmapsto \mathrm{Mor}_{\mathfrak{X}}(\mathfrak{Y}, \mathfrak{Z})$$

on $\mathfrak{X}_{fl}^{log}$ is a sheaf.

Since the flat topology is stronger than the etale topology, this theorem shows that the presheaf $\mathfrak{Y} \longmapsto \mathrm{Mor}_S(\mathfrak{Y}, \mathfrak{Z})$ on $\mathfrak{X}_{et}^{log}$ is also a sheaf.

Thm. (4.1.2) says that the presheaves

$$\mathfrak{Y} \longmapsto \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}), \quad \mathfrak{Y} \longmapsto \Gamma(\mathfrak{Y}, M_{\mathfrak{Y}}), \quad \mathfrak{Y} \longmapsto \Gamma(\mathfrak{Y}, M_{\mathfrak{Y}}^{gp})$$

on $\mathfrak{X}_{fl}^{log}$ are sheaves. (Indeed, these presheaves are the cases where $\mathfrak{Z}$ is $\mathfrak{X} \times_S A_S^1$ where A_S^1 is the affine line endowed with the trivial log. str., $\mathfrak{X} \times_S A^1_S$ where A^1_S is endowed with the log. str. associated to $N \longrightarrow \mathcal{O}_S[t]$; $1 \longrightarrow t$, and $G^{cpt}_{m,\mathfrak{X}}$, respectively.)

We denote sometimes these sheaves by $\mathcal{O}_{\mathfrak{X}}$, $M_{\mathfrak{X}}$, and $M^{gp}_{\mathfrak{X}}$, respectively, and we use the same notations for the corresponding sheaves on $\mathfrak{X}^{log}_{et}$.

We have a logarithmic version of the Kummer exact sequence:

Proposition (4.1.3). <u>For $n \in Z \setminus \{0\}$ (resp. For an integer n which is invertible on $\mathfrak{X}$), we have exact sequences</u>

$$0 \longrightarrow Z/nZ(1) \longrightarrow \mathcal{O}^{\times}_{\mathfrak{X}} \xrightarrow{n} \mathcal{O}^{\times}_{\mathfrak{X}} \longrightarrow 0$$

$$0 \longrightarrow Z/nZ(1) \longrightarrow M^{gp}_{\mathfrak{X}} \xrightarrow{n} M^{gp}_{\mathfrak{X}} \longrightarrow 0$$

<u>on</u> $\mathfrak{X}^{log}_{fl}$ <u>(resp.</u> $\mathfrak{X}^{log}_{et}$). <u>Here</u> $Z/nZ(1) = \mathrm{Ker}(\mathcal{O}^{\times}_{\mathfrak{X}} \xrightarrow{n} \mathcal{O}^{\times}_{\mathfrak{X}})$.

(4.1.4) We start the proof of (4.1.2). By using the fact that a finite inverse limit of sheaves in the category of presheaves is still a sheaf, we are reduced to showing that $\mathfrak{Y} \longmapsto \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ and $\mathfrak{Y} \longrightarrow \Gamma(\mathfrak{Y}, M_{\mathfrak{Y}})$ are sheaves on $\mathfrak{X}^{log}_{fl}$. Hence it is sufficient to prove the folowing

Lemma (4.1.5). <u>Let $\mathfrak{Y}$ be an alg. val. log. space over S, let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a flat quasi-finite morphism over S and let $y \in \mathfrak{Y}$, $x = f(y) \in \mathfrak{X}$. Then, the map from</u> $\mathcal{O}_{\mathfrak{X},x}$ <u>(resp.</u> $M_{\mathfrak{X},x}$) <u>to the equalizer of</u>

$$\mathcal{O}_{\mathfrak{Y},y} \rightrightarrows \varinjlim_{\mathfrak{U}} \Gamma(\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}, \mathcal{O}_{\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}})$$

<u>(resp.</u> $$M_{\mathfrak{Y},y} \rightrightarrows \varinjlim_{\mathfrak{U}} \Gamma(\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}, M_{\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}})$$

<u>is an isomorphism, where $\mathfrak{U}$ ranges over open neighbourhoods of y in $\mathfrak{Y}$.</u>

Proof. We may assume f comes from a flat quasi-finite morphism of fine log. schemes $Y \longrightarrow X$ over S such that Y, X are locally of finite presentation as S-schemes. By considering (cl) fppf locally on the underlying schemes of X and of Y, we are reduced to the following case: The log. str. of X is associated to a homomorphism $P \longrightarrow \mathcal{O}_X$ with P a finitely generated integral monoid, $Y = X_Q$ for a homomorphism $P \longrightarrow Q$ with Q a finitely generated integral monoid such that the map $P^{gp} \longrightarrow Q^{gp}$ is injective with finite cokernel,

$$x = (V, p), \quad P \subset V \subset P^{gp}, \quad p \in X_V$$

$$y = (W, q), \quad Q \subset W \subset Q^{gp}, \quad q \in X_W \,.$$

Furthermore, by considering (cl) fppf locally we may assume $V^{\times} \xrightarrow{\cong} W^{\times}$. Then, $V = V^{gp} \cap W$, and

(4.1.5.1) $\quad \mathcal{O}_{\mathfrak{Y},y} = \mathcal{O}_{\mathfrak{X},x} \otimes_{Z[V]} Z[W]$

(4.1.5.2) $\quad \varinjlim_{\mathfrak{U}} \Gamma(\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}, \mathcal{O}_{\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}}) = \mathcal{O}_{\mathfrak{X},x} \otimes_{Z[V]} Z[W \oplus W^{gp}/V^{gp}]$

where $\mathfrak{U}$ ranges as in (4.1.5) and the two arrows from (4.1.5.1) to (4.1.5.2) are induced from $W \longrightarrow W \oplus W^{gp}/V^{gp}$; $a \longrightarrow (a, a)$ and $a \longrightarrow (a, 1)$, respectively. Hence, Lemma (4.1.6) below (applied by replacing A, P, Q there with $\mathcal{O}_{\mathfrak{X},x}$, V, W, respectively) proves the part of (4.1.5) for $\mathcal{O}$. It follows that the similar result as (4.1.5) for $\mathcal{O}^{\times}$ holds. Hence, to prove the part of (4.1.5) for M, it is sufficient to show

Sublemma (4.1.5.3). <u>The equalizer of</u>

$$M_{\mathfrak{Y},y} \rightrightarrows \varinjlim_{\mathfrak{U}} \Gamma(\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}, M_{\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{U}})$$

<u>is contained in</u> $\mathrm{Image}(V).\mathcal{O}^{\times}_{\mathfrak{Y},y}$.

The difference of the two arrows in (4.1.5.3) induces

(4.1.5.4) $\quad M_{\mathfrak{Y},y} \longrightarrow \mathcal{O}_{\mathfrak{Y},y}[W^{gp}/V^{gp}]^{\times}$,

and the comosition

$$W \longrightarrow M_{\mathfrak{Y},y} \xrightarrow{(4.1.5.4)} \mathcal{O}_{\mathfrak{Y},y}[W^{gp}/V^{gp}]^{\times}$$

coincides with the canonical projection $W \longrightarrow W^{gp}/V^{gp}$. Furthermore, since $\kappa(x) \xrightarrow{\cong} \kappa(y)$, the composition

$$M_{\mathfrak{Y},y} \xrightarrow{(4.1.5.4)} \mathcal{O}_{\mathfrak{Y},y}[W^{gp}/V^{gp}]^{\times} \longrightarrow \kappa(y)[W^{gp}/V^{gp}]^{\times}$$

annihilates $\mathcal{O}_{\mathfrak{Y},y}^{\times}$. These facts prove (4.1.5.3).

Lemma (4.1.6). Let P and Q be integral monoids and let $P \longrightarrow Q$ be an exact homomorphism such that $P^{gp} \longrightarrow Q^{gp}$ is injective. Then, for any ring A and for any homomorphism $g : P \longrightarrow A$, the sequence of A-modules

$$0 \longrightarrow A \xrightarrow{\iota_0} A \otimes_{Z[P]} Z[Q] \xrightarrow{\iota_1} A \otimes_{Z[P]} Z[Q \oplus Q^{gp}/P^{gp}]$$

is exact, where ι_0 is the canonical map and

$$\iota_1(1 \otimes a) = 1 \otimes (a, a) - 1 \otimes (a, 1) \qquad (a \in Q).$$

Proof. Let

$$s_0 : A \otimes_{Z[P]} Z[Q] \longrightarrow A$$

$$s_1 : A \otimes_{Z[P]} Z[Q \oplus Q^{gp}/P^{gp}] \longrightarrow A \otimes_{Z[P]} Z[Q]$$

be the A-homomorphisms defined by

$$s_0(1 \otimes a) = \begin{cases} g(a) & \text{if } a \in P \\ 0 & \text{if } a \notin P \end{cases} \qquad s_1(1 \otimes (b,c)) = \begin{cases} 1 \otimes b & \text{if } c \neq 1 \\ 0 & \text{if } c = 1 \end{cases}$$

$(b \in Q,\ c \in Q^{gp}/P^{gp})$. Then, $s_0\iota_0$ and $s_1\iota_1 + \iota_0 s_0$ are the identity maps. This proves the lemma.

§4.2. Classical etale (or flat) sites.

Definition (4.2.1). We define the site $\mathfrak{X}_{et}^{cl}$ (resp. $\mathfrak{X}_{fl}^{cl}$) to be the full subcategory of $\mathfrak{X}_{et}^{log}$ (resp. $\mathfrak{X}_{fl}^{log}$) consisting of objects $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ such that $f^{*}M_{\mathfrak{X}} \xrightarrow{\cong} M_{\mathfrak{Y}}$, equipped with the following topology. A covering in $\mathfrak{X}_{et}^{cl}$ (resp. $\mathfrak{X}_{fl}^{cl}$) is a covering in $\mathfrak{X}_{et}^{log}$ (resp. $\mathfrak{X}_{fl}^{log}$) consisting of objects of $\mathfrak{X}_{et}^{cl}$ (resp. $\mathfrak{X}_{fl}^{cl}$).

The aim of this section is to show that the "descent theory" works for coverings in $\mathfrak{X}^{cl}_{fl}$.

Theorem (4.2.2). *Let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$ be a covering in $\mathfrak{X}^{cl}_{fl}$. Then we have an equivalence of categories*

$$\mathcal{G}^{aff}_{\mathfrak{X}} \xrightarrow{\sim} \mathcal{G}^{aff}_{\mathfrak{Y}\to\mathfrak{X}} \quad ; \quad \mathfrak{Z} \longmapsto \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Y}$$

where $\mathcal{G}^{aff}_{\mathfrak{X}}$ denotes the category of alg. val. log. spaces $\mathfrak{Z}$ over $\mathfrak{X}$ endowed with an affine morphism (3.4.7) $\mathfrak{Z} \longrightarrow \mathfrak{X}$ over S, and $\mathcal{G}^{aff}_{\mathfrak{Y}\to\mathfrak{X}}$ denotes the category of alg. val. log. spaces $\mathfrak{Z}$ over S endowed with an affine morphism $\mathfrak{Z} \longrightarrow \mathfrak{Y}$ over S and with a glueing datum

$$\theta : \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Z}' \cong \mathfrak{Z} \times_{\mathfrak{X}} \mathfrak{Y} \qquad \text{over} \quad \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{Y}$$

satisfying the usual transitivity condition.

In the proof of (4.2.2), we shall use the following theorem concerning the descent of the log. str.

Theorem (4.2.3). *Let X and Y be schemes and let $f : Y \longrightarrow X$ be a faithfully flat quasi-compact morphism. Let $\mathcal{L}_{Y\to X}$ be the category of integral log. str.'s M on Y endowed with an isomorphism*

$$\theta : p_1^*(M) \xrightarrow{\cong} p_2^*(M) \quad \text{on} \quad Y \times_X Y,$$

which satisfies

$$p_{23}^*(\theta)p_{12}^*(\theta) = p_{13}^*(\theta) : q_1^*(M) \longrightarrow q_3^*(M) \quad \text{on} \quad Y \times_X Y \times_X Y.$$

Here $p_i : Y \times_X Y \longrightarrow Y$, $p_{ij} : Y \times_X Y \times_X Y \longrightarrow Y \times_X Y$, $q_i : Y \times_X Y \times_X Y \longrightarrow Y$ *denote the projections. On the other hand, let $\mathcal{L}_X$ be the category of integral log. str.'s on X. Then:*

(1) *The functor $f^* : \mathcal{L}_X \longrightarrow \mathcal{L}_{Y\to X}$ is fully faithful.*

(2) *An object M of $\mathcal{L}_{Y\to X}$ belongs to the essential image of f^* if and only if the following condition is satisfied.*

(*) *For any $y \in Y$ and $z \in p_1^{-1}(y) \cap p_2^{-1}(y)$, the automorphism*

σ_z of $M_y/\mathcal{O}^x_{Y,y}$ defined by

$$M_y/\mathcal{O}^x_{Y,y} \xrightarrow{\cong} p_1^*(M)_z/\mathcal{O}^x_{Y\times_X Y,z} \overset{\theta}{\cong} p_2^*(M)_z/\mathcal{O}^x_{Y\times_X Y,z} \xleftarrow{\cong} M_y/\mathcal{O}^x_{Y,y}$$

is the identity map.

(3) Assume f is an open map. Then, for a log. str. N on X, f^*N is fine if and only if N is fine.

(4) Assume X and Y are quasi-compact and f is universally open. Let M be an object of $\mathcal{L}_{Y\to X}$ which is fine. Then there exist a projective morphism of finite presentation $X' \longrightarrow X$ and a fine log. str. N on X' such that if we endow $Y' = Y \times_X X'$ with the inverse image of N, there exists a morphism of log. schemes $g : Y' \longrightarrow Y$ satisfying the following (i)(ii)(iii).

(i) The underlying morphism of schemes of g is the canonical projection.

(ii) $g^{val} : (Y')^{val} \longrightarrow Y^{val}$ is an isomorphism.

(iii) The diagram

$$\begin{array}{ccc} p_1'^*g^*M & \xrightarrow{\text{by } \theta} & p_2'^*g^*M \\ & \searrow \quad \swarrow & \\ & q^*N & \end{array}$$

is commutative where $p_i' : Y' \times_{X'} Y' \longrightarrow Y'$ and $q : Y' \times_{X'} Y' \longrightarrow X'$ denote the projections.

The above (4.2.3)(4) is a generalization of the fact that a divisor with normal crossings (which need not be simple) on a regular noetherian scheme X is "blowing-up locally" a divisor with simple normal crossings (cf. (1.2.11)). Indeed, a divisor with normal crossings on X is etale locally with simple normal crossings, and by (1.2.7), it defines an object of $\mathcal{L}_{Y\to X}$ for some covering

$Y \longrightarrow X$ in X_{et}.

We prove (4.2.3). We omit the proof of the following elementary

Lemma (4.2.4). Let T, T', T'' and X be topological spaces, let $p : T \longrightarrow X$, $p_1, p_2 : T' \longrightarrow T$, $p_{12}, p_{13}, p_{23} : T'' \longrightarrow T'$, q_1, q_2, $q_3 : T'' \longrightarrow T$ be continuous maps satisfying $pp_1 = pp_2$, $q_1 = p_1p_{12} = p_1p_{13}$, $q_2 = p_2p_{12} = p_1p_{23}$, $q_3 = p_2p_{13} = p_2p_{23}$. Assume that the maps $p : T \longrightarrow X$, $(p_1, p_2) : T' \longrightarrow T \times_X T$, $(p_{12}, p_{23}) : T'' \longrightarrow T' {}_{p_2}\!\!\searrow X \swarrow_{p_1} T'$ are surjective, and that the topology of S coincides with image of the topology of T. Let F be a sheaf on T and let $\theta : p_1^{-1}(F) \xrightarrow{\cong} p_2^{-1}(F)$ be an isomorphism satisfying

$$p_{23}^{-1}(\theta)p_{12}^{-1}(\theta) = p_{13}^{-1}(\theta) \quad : \quad q_1^{-1}(F) \longrightarrow q_3^{-1}(F) \quad \text{on} \quad T''.$$

Let G be the sheaf on S defined as the equalizer of $p_*(F) \rightrightarrows p_*p_{2*}p_2^{-1}(F)$ where one arrow is induced from the canonical map $F \longrightarrow p_{2*}p_2^{-1}(F)$ and the other is induced from the canonical map $F \longrightarrow p_{1*}p_1^{-1}(F)$ and $\theta : p_1^{-1}(F) \xrightarrow{\cong} p_2^{-1}(F)$. Then, for $t \in T$, we have

$$p^{-1}(G)_t \xrightarrow{\cong} \{a \in F_t \ ; \ \sigma_z(a) = a \ \text{for all} \ z \in p_1^{-1}(t) \cap p_2^{-1}(t)\}$$

where σ_z denotes the automorphism of F_t

$$F_t \xrightarrow{\cong} p_1^{-1}(F)_z \xrightarrow{\theta} p_2^{-1}(F)_z \xleftarrow{\cong} F_t \ .$$

In particular, $p^{-1}(G) \xrightarrow{\cong} F$ if and only if $\sigma_t : F_t \longrightarrow F_t$ is the identity map for any $t \in T$ and for any $z \in p_1^{-1}(t) \cap p_2^{-1}(t)$.

(4.2.5) The fully-faithfulness (4.2.3)(1) is proved easily. We prove (4.2.3)(2). By applying (4.2.4) to $F = M/\mathcal{O}_Y^X$ with X, T, T', T'' the schemes $X, Y, Y \times_X Y$, $Y \times_X Y \times_X Y$, respectively, we see that the sheaf $M/\mathcal{O}_Y^X$ descents to X if and only if the condition (*) in (4.2.3)(2) is satisfied. (Note $Y \times_X Y$ etc. mean the fiber products as schemes, but the fiber product $T \times_X T$ etc. mean the fiber products as sets.)

Assume $M/\mathcal{O}_Y^x$ descents to a sheaf $\bar{N}$ on X. Define N to be the equalizer of $f_*(M) \rightrightarrows f_*p_{2*}p_2^*(M)$ where one of the arrows is induced from $M \longrightarrow p_{2*}p_2^*(M)$ and the other is induced from $M \longrightarrow p_{1*}p_1^*(M)$ and θ. We show that

(4.2.5.1) $\quad N/\mathcal{O}_X^x \xrightarrow{\cong} \bar{N}$.

Then, $f^*N \xrightarrow{\cong} M$ will follow from it and from the exactness of $\mathcal{O}_X^x \longrightarrow f_*(\mathcal{O}_Y^x) \rightrightarrows f_*p_{2*}(\mathcal{O}_{Y\times_X Y}^x)$. The problem in (4.2.5.1) is the surjectivity. Let $x \in X$ and $a \in \bar{N}_x$. We show that a comes from N_x. Let U be an open neibourhood of x on which a is defined, an let $V = Y \times_X U$. On V, the inverse image of $f^{-1}(a) \in \Gamma(V, M/\mathcal{O}_V^x)$ under $M \longrightarrow M/\mathcal{O}_V^x$ is a principal homogeneous space over $\mathcal{O}_V^x$ endowed with a glueing datum on the induced principal homogeneous space over $\mathcal{O}_{V\times_U V}^x$ on $V \times_U V$. By the descent theory of line bundles ([SGA 1] Exp. VIII), we see that the inverse image of a in N is a principal homogeneous space over $\mathcal{O}_U^x$ on U which has a section $\tilde{a}$ on an open neighbourhood of x in U. Thus we find $\tilde{a} \in N_x$ which maps to a.

(4.2.6) We prove (4.2.3)(3). Assume f^*N is fine. Let $x \in X$, $y \in f^{-1}(x)$. Take a finitely generated integral monoid P and a homomorphism $P \longrightarrow N_x$ such that $P/P^x \xrightarrow{\cong} N_x/\mathcal{O}_{X,x}^x$. (For example, take $N^r \longrightarrow N_x$ which induces a surjection $N^r \longrightarrow N_x/\mathcal{O}_{X,x}^x$, and let P be the inverse image of N_x under $Z^r \longrightarrow N_x^{gp}$.) Then $P \longrightarrow N_x$ extends to a homomorphism $P \longrightarrow N$ on an open neighbourhood U of x. The facts $P/P^x \xrightarrow{\cong} f^*(N)_y/\mathcal{O}_{Y,y}^x$ and that $f^*(N)$ is fine show that $f^*(N)$ is associated to $P \longrightarrow f^*(N) \longrightarrow \mathcal{O}_Y$ on an open neighbourhood $V \subset f^{-1}(U)$ of y. It follows that N is associated to $P \longrightarrow N \longrightarrow \mathcal{O}_X$ on the image of V.

To prove (4.2.3)(4), we use the following general fact on finitely generated integral monoids.

Lemma (4.2.7). Let P be a finitely generated integral monoid such that $P^{\times} = \{1\}$. Call an element a of P irreducible if $a = xy$ $(x, y \in P)$ implies $x = 1$ or $y = 1$. Then, if E is a minimal system of generators of P, E coincides with the set of all irreducible elements of P. In particular, such a system is unique, and the set of all irreducible elements of P is finite.

Proof. Easy.

Corollary (4.2.8). For a finitely generated integral monoid P, $\mathrm{Aut}(P)$ is a finite group.

Indeed, an automorphism is determined by its action on the finite set of all irreducible elements (4.2.7).

(4.2.9) Proof of (4.2.3)(4). We may assume that there exist a finitely generated integral monoid P and a homomorphism $h : P \longrightarrow \mathcal{O}_Y$ to which M is associated. In fact, such (P, h) exists at least locally on Y. If $Y = \bigcup_i V_i$ is a finite open covering such that such (P, h) exists on each V_i, then such (P, h) exists on $\coprod_i V_i$, and (4.2.3)(4) for $Y \longrightarrow X$ is reduced to that for $\coprod_i V_i \longrightarrow X$.

Now assume Y is affine and (P, h) exists on Y. Take a representative E in P of the set of all irreducible elements (1.5.6) in $P/P^{\times}$, and let I be the product ideal of all ideals of the form (a, b) with $a, b \in E$.

Let A be the quasi-coherent graded ring over Y defined by

$$A = \mathcal{O}_Y \otimes_{Z[P]} (\bigoplus_{n\geq 0} \langle I\rangle^n) \xrightarrow{\cong} \mathcal{O}_Y \otimes_{Z[M]} (\bigoplus_{n\geq 0} \langle I\rangle_M^n)$$

where $\langle I\rangle$ (resp. $\langle I\rangle_M$) denotes the ideal of $Z[P]$ (resp. $Z[M]$) generated by I (resp. by the image of $I \longrightarrow M$). (So $\mathrm{Proj}(A)$ is Y_I in (1.3.3).) The ideal of $p_1^*(M)$ generated by $p_1^{-1}(I)$ maps via θ onto the ideal of $p_2^*(M)$ generated by $p_2^{-1}(I)$. This gives a glueing datum

$p_1^*(A) \cong p_2^*(A)$ on $Y \times_X Y$ satisfying the usual transitivity condition. By the fpqc descent theory for quasi-coherent sheaves ([SGA 1] Exp. VIII), A descents to a quasi-coherent graded ring B on X. Let

$$X' = \mathrm{Proj}(B), \quad Y' = \mathrm{Proj}(A) = Y_I = X' \times_X Y.$$

The log. str. M' of $Y' = Y_I$ defined in (1.3.3) is regarded as an object of $\mathcal{L}_{Y'\to X'}$. We claim that this object satisfies the condition (*) in (4.2.3)(2) (with X, Y replaced by X', Y', respectively) and hence descents to a log. str. N on X'. Since M' is fine, N is also fine (4.2.3)(3) and this concludes the proof of (4.2.3)(4).

Now we prove our claim. If $y \in Y'$ and $z \in p_1'^{-1}(y) \cap p_2'^{-1}(y)$, $\sigma_z : M'_y/\mathcal{O}^\times_{Y',y} \longrightarrow M'_y/\mathcal{O}^\times_{Y',y}$ is the identity map for the following reason. Note $P^{gp} \longrightarrow (M'_y)^{gp}/\mathcal{O}^\times_{Y',y}$ is surjective. Let t (resp. s) be the image of z (resp. y) in $Y \times_X Y$ (resp. Y). Then, $P \longrightarrow M_s/\mathcal{O}^\times_{Y,s}$ is surjective, and the diagram

$$\begin{array}{ccc} M_s/\mathcal{O}^\times_{Y,s} & \longrightarrow & M'_y/\mathcal{O}^\times_{Y',y} \\ \downarrow \sigma_t & & \downarrow \sigma_z \\ M_s/\mathcal{O}^\times_{Y,s} & \longrightarrow & M'_y/\mathcal{O}^\times_{Y',y} \end{array}$$

is commutative. Let E be a minimal system of generators of $P/P^\times$.

Then, there is a subset E_1 of the image of E in $M_s/\mathcal{O}^{\times}_{Y,s}$ which is a minimal system of generators of $M_s/\mathcal{O}^{\times}_{Y,s}$. It is sufficient to show that for $a \in E_1$, the image of a in $M'_y/\mathcal{O}^{\times}_{Y',y}$ is fixed by σ_z. Since $\sigma_t(a)$ is also an element of E_1 by (4.2.7), there exists b, $c \in E$ such that $b \longrightarrow a$, $c \longrightarrow \sigma_t(a)$ in $M_s/\mathcal{O}^{\times}_{Y,s}$. Let d be the image of bc^{-1} in $(M'_y)^{gp}/\mathcal{O}^{\times}_{Y',y}$. By the definition of the ideal I, either d or d^{-1} belongs to $M'_y/\mathcal{O}^{\times}_{Y',y}$. Since σ_z is of finite order (4.2.8) say n, we have $d\sigma_z(d)\ldots\sigma_z^{n-1}(d) = 1$, and this shows that $d = 1$ in $(M'_y)^{gp}/\mathcal{O}^{\times}_{Y',y}$, that is, σ_z fixes the image of a in $M'_y/\mathcal{O}^{\times}_{Y',y}$.

(4.2.10) We prove (4.2.2). By (4.1.2), the functor $\mathcal{G}^{aff}_{\mathfrak{X}} \longrightarrow \mathcal{G}^{aff}_{\mathfrak{Y}\longrightarrow\mathfrak{X}}$ is fully faithful. It remains to show that this functor is essentially surjective. We may assume that there exist fine log. schemes X, Y over S and a morphism $g : Y \longrightarrow X$ having the following properties: X and Y are locally of finite presentation as S-schemes, X and Y are quasi-compact and quasi-separated as schemes, $\mathfrak{X} = X^{val}$, $\mathfrak{Y} = Y^{val}$, $f = g^{val}$, g is (log) flat (log) quasi-finite and surjective, and $g^*M_X \xrightarrow{\cong} M_Y$. We may assume there is a chart $P \longrightarrow M_X$. To prove that an object $\mathfrak{Z}$ of $\mathcal{G}^{aff}_{\mathfrak{Y}\longrightarrow\mathfrak{X}}$ comes from $\mathcal{G}^{aff}_{\mathfrak{X}}$, by (3.4.9), we may assume that $\mathfrak{Z} = Z^{val}$ for a fine log. scheme Z over Y such that the underlying morphism of schemes $Z \longrightarrow Y$ is affine and locally of finite presentation, and that Z is endowed with a glueing datum

$$Z \times_X Y \cong Y \times_X Z \quad \text{over} \quad Y \times_X Y.$$

By the usual fppf descent theory for (cl) affine morphisms [SGA], the Y-scheme Z descents to an X-scheme Z_0 which is affine over X. The problem is that the log. str. of $\mathfrak{Z}$ descents to $(Z_0)^{val}$. By

(4.2.3)(4), there exists a projective morphism of finite presentation $Z'_0 \longrightarrow Z_0$ and a fine log. str. N on Z'_0 having the following property: If we endow $Y \times_X Z'_0 = Z \times_{Z_0} Z'_0$ with the inverse image of N, there exists a morphism of fine log. schemes $Y \times_X Z'_0 \longrightarrow Z$ whose underlying morphism of schemes is the canonical projection, such that $(Y \times_X Z'_0)^{val} \xrightarrow{\cong} Z^{val} = \mathfrak{Z}$. We have

$$\mathfrak{Y} \times_{\mathfrak{X}} (Z'_0)^{val} = (Y \times_X Z'_0)^{val} \xrightarrow{\cong} \mathfrak{Z}.$$

We show $(Z'_0)^{val} \longrightarrow X^{val}$ is affine. There exists a finitely generated non-empty ideal I of P such that

$$Y \times_X (Z'_0 \times_X X_I)^{sat} \xrightarrow{\cong} (Z \times_X X_I)^{sat}$$

(3.4.5)(3). Since $(Z \times_X X_I)^{sat} \longrightarrow Y \times_X X_I$ is small and the underlying morphism of schemes of it is affine, we see $(Z'_0 \times_X X_I)^{sat} \longrightarrow X_I$ is small and the underlying morphism of schemes of it is affine. This shows $(Z'_0)^{val} \longrightarrow X^{val}$ is affine.

Remark(4.2.11). For a covering in the log. flat site (not the cl. flat site), the descent theory does not work well. For example (cf. (5.2.11)), it happens that the "Cartier dual" of a (log) finite (log) flat commutative group object, which is defined as a sheaf on $\mathfrak{X}^{log}_{fl}$, is not represented by a (log) finite (log) flat commutative group object though it is represented by such object on a log. flat covering of $\mathfrak{X}$.

§4.3. Fundamental groups.

We give a log. version of the theory of arithmetic fundamental groups of Grothendieck ([SGA 1] Exp. V), which is closely related to the theory of tame fundamental groups of Grothendieck-Murre [GM] (cf. (4.3.16)).

Theorem (4.3.1). *Assume S is locally noetherian and $\mathfrak{X}$ is connected. Then there exists a pro-finite group G such that the following two categories (a)(b) are equivalent.*

(a) *The category of alg. val. log. spaces $\mathfrak{Y}$ over S endowed with a finite etale morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$.*

(b) *The category of finite G-sets.*

(A G-set means a set endowed with an action of G which is continuous with respect to the discrete topology of the set.)

The proof of (4.3.1) is almost the repetition of arguments in the classical theory of fundamental groups in [SGA 1] Exp. V. Delicate points in the logarithmic case are the openness of etale morphisms (3.2.2) and the descent theory (4.2.2).

Definition (4.3.2). A geometric point of $\mathfrak{X}$ is a morphism $a \longrightarrow \mathfrak{X}$ where a is a log. scheme having the following properties (i)-(iii).

(i) The underlying scheme of a is the Spec of a separably closed field.

(ii) The log. str. of a is valuative.

(iii) $\Gamma(a, M_a^{gp})$ is n-divisible for any integer n which is invertible on a.

Lemma (4.3.3). *For $x \in \mathfrak{X}$, there exists a geometric point $\bar{x} \longrightarrow \mathfrak{X}$*

lying over x such that the field $\kappa(\bar{x})$ is a separable closure of the residue field $\kappa(x)$ of x and such that

$$\mathbf{Z}_{(p)} \otimes_{\mathbf{Z}} M_x^{gp}/\mathcal{O}_{X,x}^{\times} \xrightarrow{\cong} \Gamma(\bar{x}, M_{\bar{x}}/\mathcal{O}_{\bar{x}}^{\times})$$

where p is the characteristic of $\kappa(x)$. The geometric point $\bar{x}$ having these properties is unique upto isomorphism over $\mathfrak{X}$.

We call $\bar{x} \longrightarrow \mathfrak{X}$ a log. separable closure of x.

Definition (4.3.4). For a sheaf $\mathcal{F}$ on $\mathfrak{X}_{et}^{log}$ and for a geometric point $a \longrightarrow \mathfrak{X}$, let

$$\mathcal{F}_a = \varinjlim_{\mathfrak{Y}} \mathcal{F}(\mathfrak{Y})$$

where $\mathfrak{Y}$ ranges over objects of $\mathfrak{X}_{et}^{log}$ endowed with a lifting $a \longrightarrow \mathfrak{Y}$ of $a \longrightarrow \mathfrak{X}$.

Lemma (4.3.5). (1) For a geometric point $a \longrightarrow \mathfrak{X}$, the functor $\mathcal{F} \longmapsto \mathcal{F}_a$ is a "point" ([SGA 4] Exp. IV §6) of the topos of sheaves on $\mathfrak{X}_{et}^{log}$.

(2) A morphism of sheaves $\mathcal{F} \longrightarrow \mathcal{G}$ is an isomorphism if the induced maps $\mathcal{F}_{\bar{x}} \longrightarrow \mathcal{G}_{\bar{x}}$ are bijective for all $x \in \mathfrak{X}$.

(1) is reduced to the following fact which is proved easily: For a covering $\{\mathfrak{Y}_\lambda \longrightarrow \mathfrak{Y}\}_\lambda$ in $\mathfrak{X}_{et}^{log}$ and for a geometric point $a \longrightarrow \mathfrak{Y}$, there exist λ such that the geometric point lifts to $a \longrightarrow \mathfrak{Y}_\lambda$.

(2) is proved easily and we omit the proof.

(4.3.6) For a geometric point $a \longrightarrow \mathfrak{X}$, we define a profinite group $\pi_1^{log}(\mathfrak{X}, a)$ as follows. Let $\mathcal{E}_{\mathfrak{X}}$ be the category of sheaves on $\mathfrak{X}_{et}^{log}$ which are locally isomorphic to a constant sheaf of a finite set, and let $\Phi_a : \mathcal{E}_{\mathfrak{X}} \longrightarrow (\text{Sets})$ be the functor $\mathcal{F} \longrightarrow \mathcal{F}_a$. Define

$$\pi_1^{log}(\mathfrak{X}, a) = \mathrm{Aut}(\Phi_a)$$

which we regard as a pro-finite group by taking the subgroups

$\mathrm{Ker}(\pi_1^{\log}(\mathfrak{X}, a) \longrightarrow \mathrm{Aut}(\mathcal{F}_a))$ as a basis of neibourhoods of 1, where $\mathcal{F}$ ranges over objects of $\mathcal{E}_{\mathfrak{X}}$.

Proposition (4.3.7). Let the notations be as in (4.3.6). If $\mathfrak{X}$ is connected, Φ_a induces an equivalence of categories

$$\mathcal{E}_{\mathfrak{X}} \xrightarrow{\simeq} \{\pi_1^{\log}(\mathfrak{X}, a)\text{-sets}\}.$$

In general, let T be a connected topos (i.e. a topos whose final object is not a disjoint union of two non-empty objects) with a "point" p. Then, if $\mathcal{E}_T$ denotes the category of objects of T which are locally isomorphic to a constant object corresponding to a finite set, then the category $\mathcal{E}_T$ with the fiber functor $\Phi_p : \mathcal{F} \longmapsto \mathcal{F}_p$ satisfies the axioms of a Galois category in [SGA 1] Exp. V §4 (the proof of this fact is straightforward) and hence Φ_p induces an equivalence of categories $\mathcal{E}_T \xrightarrow{\sim} \{\mathrm{Aut}(\Phi_p)\text{-sets}\}$. What we have to check is that the topos of sheaves on $\mathfrak{X}_{et}^{\log}$ is connected. Assume that the final object is a disjoint union of two sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$. Let $\mathfrak{U}_i = \{x \in \mathfrak{X} \ ; \ (\mathcal{F}_i)_{\bar{x}} \neq \phi\}$. Then, $\mathfrak{U}_1 \cap \mathfrak{U}_2 = \phi$ and $\mathfrak{U}_1 \cup \mathfrak{U}_2 = \mathfrak{X}$. It remains to show that $\mathfrak{U}_i$ are open. If $x \in \mathfrak{U}_1$, there exists an etale morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$ and a lifting $\bar{x} \longrightarrow \mathfrak{Y}$ of $\bar{x} \longrightarrow \mathfrak{X}$ such that $\mathcal{F}_1(\mathfrak{Y}) \neq \phi$. Then the image of $\mathfrak{Y}$ in $\mathfrak{X}$ is an open neighbourhood of x by (3.2.2) and is contained in $\mathfrak{U}_1$. This shows that $\mathfrak{U}_1$ is open, and $\mathfrak{U}_2$ is open similarly.

For the proof of (4.3.1), by taking $G = \pi_1(\mathfrak{X}, a)$ for any geometric point $a \longrightarrow \mathfrak{X}$, we are reduced to

Proposition (4.3.8). Assume S is locally noetherian. Let $\mathcal{E}_{\mathfrak{X}}$ be as in (4.3.6), and let $\mathcal{E}'_{\mathfrak{X}}$ be the category of alg. val. log. spaces $\mathfrak{Y}$ over S endowed with a finite etale morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$. Then, for an object $\mathfrak{Y}$ of $\mathcal{E}'_{\mathfrak{X}}$, the sheaf $\mathrm{Mor}_{\mathfrak{X}}(\ , \mathfrak{Y})$ on $\mathfrak{X}_{et}^{\log}$ belongs to $\mathcal{E}_{\mathfrak{X}}$.

Furthermore, this correspondence gives an equivalence of categories

$$\mathcal{E}'_{\mathfrak{X}} \xrightarrow{\cong} \mathcal{E}_{\mathfrak{X}} .$$

(4.3.9) We prove the first statement of (4.3.8) that for an object $\mathfrak{Y}$ of $\mathcal{E}'_{\mathfrak{X}}$, $Mor_{\mathfrak{X}}(\ , \mathfrak{Y})$ belongs to $\mathcal{E}_{\mathfrak{X}}$. By (4.3.10) below, locally on $\mathfrak{X}^{log}_{et}$, $M_{\mathfrak{Y}}$ becomes the inverse image of $M_{\mathfrak{X}}$. Assume $M_{\mathfrak{Y}}$ has this property. By (3.4.9), locally on $\mathfrak{X}$, we find fine log. schemes X, Y over S and a morphism $Y \longrightarrow X$ over S having the following properties. $\mathfrak{X} = X^{val}$, $\mathfrak{Y} = Y^{val}$, $\mathfrak{Y} \longrightarrow \mathfrak{X}$ is induced from $Y \longrightarrow X$, M_Y is the inverse image of M_X, and the underlying morphism of schemes $Y \longrightarrow X$ is (cl) finite (cl) etale. Hence for a (cl) etale covering $X' \longrightarrow X$ of the underlying scheme of X, the scheme $Y \times_X X'$ over X' becomes a disjoint union of finite copies of X'. When we endow X' with the inverse image of M_X, $Mor_{\mathfrak{X}}(\ , \mathfrak{Y})$ becomes a constant sheaf on the etale covering $(X')^{val} \longrightarrow X^{val}$.

Lemma (4.3.10). <u>Let $\mathfrak{Y}$, $\mathfrak{X}'$ be alg. val. log. spaces over S, let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}$, $g : \mathfrak{X}' \longrightarrow \mathfrak{X}$ be morphisms, and let</u> $f' : \mathfrak{Y}' \underset{def}{=} \mathfrak{Y} \times_{\mathfrak{X}} \mathfrak{X}'$ <u>be the base change. If the cokernel of</u> $f^*(M^{gp}_{\mathfrak{X}}) \longrightarrow M^{gp}_{\mathfrak{Y}}$ <u>is a torsion sheaf annihilated by a non-zero integer n and if the image of</u> $g^*(M^{gp}_{\mathfrak{X}})$ <u>in</u> $M^{gp}_{\mathfrak{X}'}$ <u>is contained in</u> $(M^{gp}_{\mathfrak{X}'})^n \mathcal{O}^{\times}_{\mathfrak{X}'}$, <u>then</u> $(f')^* M_{\mathfrak{X}'} \xrightarrow{\cong} M_{\mathfrak{Y}'}$.

The proof is easy and is left to the reader.

(4.3.11) Next we prove the categorical equivalence in (4.3.8). The problem is to show that $\mathcal{E}'_{\mathfrak{X}} \longrightarrow \mathcal{E}_{\mathfrak{X}}$ is essentially surjective. By the descent theory (4.2.2), it is sufficient to prove that for an object $\mathcal{F}$ of $\mathcal{E}_{\mathfrak{X}}$, there is a covering $\{\mathfrak{X}_\lambda \longrightarrow \mathfrak{X}\}_\lambda$ in $\mathfrak{X}^{cl}_{et}$ such that $\mathcal{F}|_{\mathfrak{X}_\lambda}$ belongs to $\mathcal{E}'_{\mathfrak{X}_\lambda}$.

Let $x \in \mathfrak{X}$, let $\bar{x}$ be a log. separable closure (4.3.3) of x, and define the categories $\mathcal{E}(x)$ and $\mathcal{E}'(x)$ by

$$\mathcal{E}(x) = \varinjlim_{\mathfrak{U}} \mathcal{E}_{\mathfrak{U}} , \qquad \mathcal{E}'(x) = \varinjlim_{\mathfrak{U}} \mathcal{E}'_{\mathfrak{U}}$$

where $\mathfrak{U}$ ranges over objects of $\mathfrak{X}^{cl}_{et}$ endowed with a morphism $\bar{x} \longrightarrow \mathfrak{U}$ over $\mathfrak{X}$. The proof of (4.3.8) is reduced to

Lemma (4.3.12). <u>We have an equivalence of categories</u>

$$\mathcal{E}'(x) \xrightarrow{\cong} \mathcal{E}(x).$$

Let

$$G_{\bar{x}} = \varprojlim_n \mathrm{Hom}(M^{gp}_{\mathfrak{X},x}/\mathcal{O}^{\times}_{\mathfrak{X},x} , Z/nZ(1))$$
$$= \mathrm{Hom}(M^{gp}_{\mathfrak{X},x}/\mathcal{O}^{\times}_{\mathfrak{X},x} \otimes (Z_{(p)}/Z), \ (Z_{(p)}/Z)(1))$$

with $(Z_{(p)}/Z)(1) = \varinjlim_n Z/nZ(1)$ where n ranges over all integers which are invertible in $\kappa(x)$ and $Z/nZ(1)$ denotes $\{a \in \kappa(\bar{x})^{\times} ; a^n = 1\}$.

Lemma (4.3.13). <u>We have an exact sequence of profinite groups</u>

$$0 \longrightarrow G_{\bar{x}} \longrightarrow \mathrm{Aut}(\bar{x}/x) \longrightarrow \mathrm{Gal}(\kappa(\bar{x})/\kappa(x)) \longrightarrow 0.$$

Indeed, for $\sigma \in G_{\bar{x}}$ corresponding to a homomorphism

$$h_{\sigma} : (M_{\bar{x}})^{gp}/\kappa(\bar{x})^{\times} \cong M^{gp}_{X,x}/\mathcal{O}^{\times}_{X,x} \otimes Z_{(p)} \longrightarrow \varinjlim_n Z/nZ(1),$$

we have an element of $\mathrm{Aut}(\bar{x}/x)$ which acts on he scheme $\bar{x}$ trivially and acts on the log. str. by

$$(M_{\bar{x}})^{gp} \longrightarrow (M_{\bar{x}})^{gp} ; \quad a \longrightarrow ah_{\sigma}(a).$$

The exactness of the sequence in (4.3.13) is easily seen.

The proof of (4.3.8) is now reduced to showing the following two lemmas.

Lemma (4.3.14). <u>We have an equivalence of categories</u>

$$\mathcal{E}(x) \xrightarrow{\cong} \{\text{finite } G_{\bar{x}}\text{-sets}\} ; \quad \mathcal{F} \longmapsto \mathcal{F}_{\bar{x}}$$

<u>where the action of</u> $G_{\bar{x}}$ <u>on</u> $\mathcal{F}_{\bar{x}}$ <u>is given by</u> $G_{\bar{x}} \longrightarrow \mathrm{Aut}(\bar{x}/x)$

(4.3.13).

Lemma (4.3.15). The functor $\mathcal{E}'(x) \longrightarrow \{\text{finite } G_{\bar{x}}\text{-sets}\}$ is essentially surjective.

Proof. It is easily seen that $\mathcal{E}(x)$ is a Galois category with a fiber functor $\Psi : \mathcal{F} \longmapsto \mathcal{F}_{\bar{x}}$. It follows that

$$\mathcal{E}(x) \xrightarrow{\cong} \{\text{finite } \mathrm{Aut}(\Psi)\text{-sets}\} \; ; \; \mathcal{F} \longmapsto \mathcal{F}_{\bar{x}} .$$

We have to prove that the inclusion $G_{\bar{x}} \subset \mathrm{Aut}(\Psi)$ is an equality. In the following, we denote $G_{\bar{x}}$ by G and we regard $\mathcal{E}'(x)$ as a subcategory of $\mathcal{E}(x)$ in the natural way.

For each open subgroup H of G, we shall define an object $\mathfrak{D}_H$ of $\mathcal{E}'(x)$ and $y_H \in \Psi(\mathfrak{D}_H)$, and for each pair (H, H') of open subgroups of G such that $H' \subset H$, we shall define a morphism $\mathfrak{D}_{H'} \longrightarrow \mathfrak{D}_H$, satisfying the following conditions.

(i) For (H, H') such that $H' \subset H$ as above, the morphism $\mathfrak{D}_{H'} \longrightarrow \mathfrak{D}_H$ sends $y_{H'} \in \Psi(\mathfrak{D}_{H'})$ to y_H.

(ii) For any object $\mathcal{F}$ of $\mathcal{E}(x)$, we have a surjection

$$\varinjlim_H \mathrm{Mor}_{\mathcal{E}(x)}(\mathfrak{D}_H , \mathcal{F}) \longrightarrow \Psi(\mathcal{F})$$

which is defined by sending $h : \mathfrak{D}_H \longrightarrow \mathcal{F}$ to $\Psi(h)(y_H) \in \Psi(\mathcal{F})$.

(iii) The map $G \longrightarrow \Psi(\mathfrak{D}_H)$; $\sigma \longrightarrow \sigma(y_H)$ induces $G/H \xrightarrow{\cong} \Psi(\mathfrak{D}_H)$.

(iv) The map

$$\mathrm{Mor}_{\mathcal{E}'(x)}(\mathfrak{D}_H , \mathfrak{D}_H) \longrightarrow \mathrm{Mor}_G(\Psi(\mathfrak{D}_H), \Psi(\mathfrak{D}_H)) = \mathrm{Mor}_G(G/H, G/H) = G/H$$

is surjective. (Note this map is injective by the property of the fiber functor.)

Then the map (ii) is in fact bijective, for it is injective by

$$\varinjlim_H \mathrm{Mor}_{\mathcal{E}(x)}(\mathfrak{D}_H , \mathcal{F}) \cong \varinjlim_H \mathrm{Mor}_{\mathrm{Aut}(\Psi)}(G/H, \Psi(\mathcal{F})) \quad \text{(by (iii))}$$

$$\subset \varinjlim \mathrm{Mor}_G(G/H, \Psi(\mathcal{F})) \cong \Psi(\mathcal{F}).$$

Hence we have

$$\mathrm{Aut}(\Psi) = \varprojlim_H \mathrm{Mor}_{\mathcal{E}'(x)}(\mathfrak{Y}_H, \mathfrak{Y}_H) = \varprojlim_H G/H = G.$$

Since $\Psi(\mathfrak{Y}_H) = G/H$, the functor $\mathcal{E}'(x) \longrightarrow$ {finite G-sets} is essentially surjective.

It remains to construct $\mathfrak{Y}_H$, y_H , $\mathfrak{Y}_{H'} \longrightarrow \mathfrak{Y}_H$.

The following fact is proved easily. There exists a fine log. scheme X with a chart $P \longrightarrow M_X$ which is locally of finite presentation over S as a scheme, such that X^{val} is an open neighbourhood of x in $\mathfrak{X}$ and such that the map $P^{gp} \longrightarrow M^{gp}_{\mathfrak{X},x}/\mathcal{O}^{\times}_{\mathfrak{X},x}$ is an isomorphism.

For an open subgroup H of G, define a subgroup D_H of $P^{gp} \otimes Z_{(p)}$ to be the intersection of the kernels of all the homomorphisms

$$P^{gp} \otimes Z_{(p)} \cong M^{gp}_{\mathfrak{X},x}/\mathcal{O}^{\times}_{\mathfrak{X},x} \otimes Z_{(p)} \longrightarrow (Z_{(p)}/Z)(1)$$

defined by elements of H. Let

$$Q_H = \{a \in D_H \; ; \; a^n \in P \text{ for some } n \geq 1\}.$$

Then, $Q_H^{gp} = D_H$, and

$$G/H \cong \mathrm{Hom}(Q_H^{gp}/P^{gp}, \; (Z_{(p)}/Z)(1)).$$

Let $\mathfrak{Y}_H = (X_{H})^{val}$ where X_{H} is as in (1.3.3). If $H' \subset H$, then $Q_H \subset Q_{H'}$, and we obtain $\mathfrak{Y}_{H'} \longrightarrow \mathfrak{Y}_H$. By fixing $P^{gp} \otimes Z_{(p)} \longrightarrow (M_{\bar{x}})^{gp}$ which lifts $P^{gp} \otimes Z_{(p)} \xrightarrow{\cong} (M_{\bar{x}})^{gp}/\kappa(\bar{x})^{\times}$, we obtain a morphism $\bar{x} \longrightarrow \mathfrak{Y}_H$ satisfying the condition (i). It is easily seen that the conditions (ii)(iii)(iv) are also satisfied.

Remark (4.3.16). Let X be a regular locally noetherian connected scheme, D a reduced divisor with simple normal crossings on X, and endow X with the log. str. associated to D. Assume X is locally of finite presentation over a scheme S, and let $\mathfrak{X} = X^{val}$ be the

Q_H

associated alg. val. log. space over S. Then,

$$\pi_1^{\log}(\mathfrak{X}) \cong \pi_1^D(X)$$

where $\pi_1^D(X)$ is the tame fundamental group of the scheme X relative to D in [GM] (2.4.4). Next let Y be a connected closed subscheme of X which is endowed with the inverse image of the log. str. of X, and let $\mathfrak{Y} = Y^{val}$ be the associated alg. val. log. space over S. Then,

$$\pi_1^{\log}(\mathfrak{Y}) \cong \pi_1^D(\hat{X})$$

where $\hat{X}$ denotes the formal completion of X along Y and $\pi_1^D(\hat{X})$ denotes the tame fundamental group of the formal scheme $\hat{X}$ relative to $\hat{X} \times_X D$ defined in [GM] (4.2.4). Here we omitted the geometric points from the notation π_1 to avoid some complicated argument how to choose geometric points.

§5. Finite flat group objects.

§5.1. Computation of H^1.

In §5.1, we fix a log. scheme T such that $N \cong M_T/\mathcal{O}_T^{\times}$ and such that the underlying scheme of T is locally noetherian.

Let S be the underlying scheme of T which we endow with the trivial log. str. We denote by $\mathcal{A}_T$ the category of log. spaces $\mathfrak{X}$ over T such that $\mathfrak{X}$ is an alg. val. log. space when regraded as a log. space over S. we denote by $\mathcal{A}_T^{cl}$ the full subcategory of $\mathcal{A}_T$ consisting of objects $\mathfrak{X}$ such that $M_{\mathfrak{X}}$ coincides with the inverse image of M_T. As is easily seen, an object of $\mathcal{A}_T^{cl}$ is a scheme (cf. (3.4.4)), and the functor "forgetting log. str." induces an equivalence between $\mathcal{A}_T^{cl}$ and the category of S-schemes locally of finite presentation. In particular, we have isomorphisms of sites

$$T_{fl}^{cl} \cong S_{fl} \quad \text{and} \quad T_{et}^{cl} \cong S_{et}.$$

For example, let A be a discrete valuation ring with maximal ideal m_A. Then, for $n \geq 1$, the log. scheme $\mathrm{Spec}(A/m_A^n)$ endowed with the inverse image of the canonical log. str. of Spec(A) (1.2.7) satisfies $M/\mathcal{O}^{\times} \cong N$, and the image in $M/\mathcal{O}^{\times}$ of a prime element of A is a generator.

Theorem (5.1.1). <u>Let G be a commutative group scheme over S, and assume that one of the following two conditions is satisfied.</u>

(i) <u>G is (cl) smooth over S.</u>

(ii) <u>G is (cl) finite and (cl) flat over S.</u>

<u>Let</u> $\varepsilon : T_{fl}^{log} \longrightarrow T_{fl}^{cl}$ <u>be the canonical morphism. Then, we have</u>

(1)

(5.1.1.1) $$R^1\varepsilon_* G \cong \varinjlim_{n\neq 0} \mathcal{H}om_{T^{cl}_{fl}}(Z/nZ(1), G) .$$

Here we denoted the sheaves $\mathfrak{Y} \longmapsto Mor_S(\mathfrak{Y}, G)$ on T^{log}_{fl} and on T^{cl}_{fl} by the same letter G, $\mathcal{H}om_{T^{cl}_{fl}}$ means the Hom-sheaf on T^{cl}_{fl}, and the inductive system is defined by the homomorphisms

$$Z/mnZ(1) \longrightarrow Z/nZ(1) \quad ; \quad x \longrightarrow x^m.$$

The homomorphism from the r.h.s. to the l.h.s. of (5.1.1.1) is defined as follows. The boundary map $M^{gp}_T \longrightarrow R^1\varepsilon_*(Z/nZ(1))$ on T^{cl}_{fl} of the Kummer exact sequence (4.1.3) annihilates $\mathcal{O}^{\times}_T$, and we have a map $N \cong M_T/\mathcal{O}^{\times}_T \longrightarrow R^1\varepsilon_*(Z/nZ(1))$. To $h : Z/nZ(1) \longrightarrow G$, we assign the section of $R^1\varepsilon_*(G)$ which is defined to be the image of $1 \in N$ under

$$N \longrightarrow R^1\varepsilon_*(Z/nZ(1)) \xrightarrow{h} R^1\varepsilon_*(G).$$

Corollary (5.1.2). Let G be as in (5.1.1), and let H be a (cl) finite (cl) etale commutative group scheme over S. Assume there exists an element $\pi \in \Gamma(T, M_T)$ whose image in $M_T/\mathcal{O}^{\times}_T \cong N$ is the generator. Then, we have a splitting exact sequence

$$0 \longrightarrow Ext^1_{T^{cl}_{fl}}(H, G) \longrightarrow Ext^1_{T^{log}_{fl}}(H, G) \longrightarrow Hom_S(H(1), G) \longrightarrow 0$$

where Ext^1 mean the group of extensions as sheaves of abelian groups, and $H(1)$ is the group scheme over S defined to be the Cartier dual of the Pontrjagin dual $\mathcal{H}om(H, Q/Z)$.

Proof of (5.1.2) assuming (5.1.1). We have an exact sequence

$$0 \longrightarrow Ext^1_{T^{cl}_{fl}}(H, G) \longrightarrow Ext^1_{T^{log}_{fl}}(H, G) \longrightarrow Hom_{T^{cl}_{fl}}(H, R^1\varepsilon_*G).$$

By (5.1.1),

$$Hom_{T^{cl}_{fl}}(H, R^1\varepsilon_*G) = Hom_S(H(1), G).$$

The splitting $\mathrm{Hom}_S(H(1), G) \longrightarrow \mathrm{Ext}^1_{T^{\log}_{fl}}(H, G)$ is induced by an element θ^{π}_H of $\mathrm{Ext}^1_{T^{\log}_{fl}}(H, H(1))$ which is defined as follows. Let $\mathfrak{E}^{\pi}$ be the Tate curve over T corresponding to π (note the image of π in $\Gamma(T, \mathcal{O}_T)$ is locally nilpotent since it is non-invertible everywhere). Then for any non-zero integer n, we have an exact sequence

$$0 \longrightarrow Z/nZ(1) \xrightarrow{i} {}_n\mathfrak{E}^{\pi} \xrightarrow{j} Z/nZ \longrightarrow 0$$

on $T^{\log}_{fl}$ where ${}_n\mathfrak{E}^{\pi} = \mathrm{Ker}(n : \mathfrak{E}^{\pi} \longrightarrow \mathfrak{E}^{\pi})$ (cf. (3.4.8)). By taking locally n which annihilates H and taking the tensor product of the above exact sequence with H, we obtain

$$0 \longrightarrow H(1) \longrightarrow H \otimes {}_n\mathfrak{E}^{\pi} \longrightarrow H \longrightarrow 0 \qquad \text{(exact)}$$

which defines θ^{π}_H.

(5.1.3) The rest of §5.1 is devoted to the proof of (5.1.1).

The finite flat case (ii) in (5.1.1) is reduced to the smooth case since there exists an exact sequence of commutative group schemes over S

$$0 \longrightarrow G \longrightarrow G' \longrightarrow G'' \longrightarrow 0$$

with G', G'' smooth. (For example, let $G' = \mathcal{M}or_S(G^*, G_m)$ where G^* is the Cartier dual of G, and let $G'' = G'/G$.)

Lemma (5.1.4). <u>Let $\mathfrak{X}$ be an object of $\mathcal{A}_T$ and let $f : \mathfrak{X} \longrightarrow T$ be the structural morphism. Assume f is small (3.4.1).</u>

(1) Then, $\mathfrak{X}$ is a scheme, and is locally of finite presentation as S-scheme, and $M_{\mathfrak{X}}/\mathcal{O}^{\times}_{\mathfrak{X}} \cong N$.

(2) f is quasi-finite (resp. finite) as a morphism of alg. val. log. spaces over S if and only if the underlying morphism of schemes $\mathfrak{X} \longrightarrow T$ is (cl) quasi-finite (resp. (cl) finite).

(3) The following three conditions (i) - (iii) are equivalent.

(i) f is flat (resp. smooth, resp. etale) as a morphism of alg. val. log. spaces over S.

(ii) f is flat (resp. smooth, resp. etale) as a morphism of fine log. schemes.

(iii) Locally on the schemes $\mathfrak{X}$ and T for the classical fppf (resp. classical etale, resp. classical etale) topology, there exist $\pi \in \Gamma(T, M_T)$, $\tau \in \Gamma(\mathfrak{X}, M_{\mathfrak{X}})$ and an integer $e \geq 1$ (resp. an integer $e \geq 1$ which is invertible on $\mathfrak{X}$, resp. an integer $e \geq 1$ which is invertible on $\mathfrak{X}$) such that the image of π in $M_T/\mathcal{O}_T^\times$ and that of τ in $M_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^\times$ are the generators and $\pi = \tau^e$, and such that the morphism of schemes

$$\mathfrak{X} \longrightarrow \mathrm{Spec}(\mathcal{O}_T[t]/(t^e - \alpha(\pi))) \quad ; \quad t \longrightarrow \alpha(\tau)$$

(t is an indeterminate) is (cl) flat (resp. (cl) smooth, resp. (cl) etale).

(4) Assume f is (log) smooth (resp. (log) etale.) Then $\mathfrak{X}$ belongs to $\mathscr{A}_T^{cl}$ if and only if the underlying scheme of $\mathfrak{X}$ is (cl) smooth (resp. (cl) etale over S.

The proof is easy (use (3.4.4)) and we omit it.

(5.1.5) To prove (5.1.1), we consider the Cech cohomology of G for a special covering in the logarithmic flat site. Let $\mathfrak{X}$ be an object of T_{fl}^{cl} such that there is $\pi \in \Gamma(\mathfrak{X}, M_{\mathfrak{X}})$ whose image in $M_{\mathfrak{X}}/\mathcal{O}_{\mathfrak{X}}^\times \cong \mathbf{N}$ is the generator. Let

$$\mathfrak{X}_{(n)} = \mathrm{Spec}(\mathcal{O}_{\mathfrak{X}}[t]/(t^n - \pi))$$

where t is an indeterminate, which we endow with the log. str.

associated to

$$N \longrightarrow \mathcal{O}_{\mathfrak{X}}[t]/(t^n - \pi) \quad ; \quad 1 \longrightarrow t.$$

Lemma (5.1.6). *Let $\mathfrak{X}$, π be as above, and assume G is smooth over S. For an object $\mathfrak{X}'$ of $\mathfrak{X}^{cl}_{fl}$, let $H^m(\mathfrak{X}'_{(n)}/\mathfrak{X}', G)$ be the Cech cohomology of G with respect to the covering $\mathfrak{X}_{(n)} \longrightarrow \mathfrak{X}$ in $\mathfrak{X}^{log}_{fl}$, and let $\mathcal{H}^m(\mathfrak{X}_{(n)}/\mathfrak{X}, G)$ be the sheaf on $\mathfrak{X}^{cl}_{fl}$ associated to the presheaf $\mathfrak{X}' \longrightarrow H^m(\mathfrak{X}'_{(n)}/\mathfrak{X}', G)$. Then*

$$\varinjlim_{n \neq 0} \mathcal{H}^1(\mathfrak{X}_{(n)}/\mathfrak{X}, G) \cong R^1\varepsilon_*(G) \quad \text{on} \quad \mathfrak{X}^{cl}_{fl}.$$

Proof. By the general theory of Cech cohomology, $\mathcal{H}^1(\mathfrak{X}_{(n)}/\mathfrak{X}, G)$ is isomorphic to the kernel of the canonical homomorphism from $R^1\varepsilon_*(G)$ to the sheaf on $\mathfrak{X}^{cl}_{fl}$ associated to the presheaf $\mathfrak{X}' \longmapsto H^1((\mathfrak{X}'_{(n)})^{log}_{fl}, G)$. Let $\mathfrak{X}'$ be an object of $\mathfrak{X}^{cl}_{fl}$, let $s \in H^1((\mathfrak{X}')^{log}_{fl}, G)$. It is sufficient to show that s dies in $H^1((\mathfrak{X}'_{(n)})^{log}_{fl}, G)$ locally on $\mathfrak{X}'$ for the classical fppf topology. We may assume that $\mathfrak{X}'$ is quasi-compact. Let $f : \mathfrak{Y} \longrightarrow \mathfrak{X}'$ be a covering in $(\mathfrak{X}')^{log}_{fl}$ which annihilates s such that $\mathfrak{Y}$ is quasi-compact. Then there exists $n \geq 1$ such that the cokernel of $f^*M^{gp}_{\mathfrak{X}'} \longrightarrow M^{gp}_{\mathfrak{Y}}$ is annihilated by n. Then, by (4.3.10), the log. str. of $\mathfrak{Y} \times_{\mathfrak{X}'} \mathfrak{X}'_{(n)}$ is the inverse image of the log. str. of $\mathfrak{X}'_{(n)}$. So, $\mathfrak{Y} \times_{\mathfrak{X}'} \mathfrak{X}'_{(n)} \longrightarrow \mathfrak{X}'_{(n)}$ is a covering in $(\mathfrak{X}'_{(n)})^{cl}_{fl}$, and hence the image of s in $H^1((\mathfrak{X}'_{(n)})^{log}_{fl}, G)$ belongs to $H^1((\mathfrak{X}'_{(n)})^{cl}_{fl}, G)$. Since G is smooth, a theorem of Grothendieck says $H^1((\mathfrak{X}'_{(n)})^{cl}_{fl}, G) = H^1((\mathfrak{X}'_{(n)})^{cl}_{et}, G)$. Since the direct image functor for the finite morphism $\mathfrak{X}'_{(n)} \longrightarrow \mathfrak{X}'$ is exact, the image of s dies locally for the classical etale topology on $\mathfrak{X}'$, and hence locally for the classical fppf topology on $\mathfrak{X}'$.

Now the proof of (5.1.1) is reduced to

Lemma (5.1.7). Let $\mathfrak{X}$, π be as in (5.1.5) and assume G is smooth over S. Then the map $\mathcal{H}om(Z/nZ(1), G) \longrightarrow R^1\varepsilon_*(G)$ induces $\mathcal{H}om(Z/nZ(1), G) \xrightarrow{\cong} \mathcal{H}^1(\mathfrak{X}_{(n)}/\mathfrak{X}, G)$.

Proof. It is easily seen that the image of $\mathcal{H}om(Z/nZ(1), G)$ in $R^1\varepsilon_*(G)$ is contained in $\mathcal{H}^1(\mathfrak{X}_{(n)}/\mathfrak{X}, G)$.

The group object $Z/nZ(1) = \mathrm{Spec}(\mathcal{O}_S[u]/(u^n-1))$ (u is an indeterminate, the log. str. of $Z/nZ(1)$ is trivial) acts on $\mathfrak{X}_{(n)}$ by

$$\mathcal{O}_{\mathfrak{X}}[t]/(t^n-\pi) \longrightarrow \mathcal{O}_{\mathfrak{X}}[t, u]/(t^n-\pi, u^n-1) \ ; \quad t \longrightarrow t \otimes u$$

and we have

$$(5.1.7.1) \quad Z/nZ(1) \times_S \mathfrak{X}_{(n)} \xrightarrow{\cong} \mathfrak{X}_{(n)} \times_{\mathfrak{X}} \mathfrak{X}_{(n)} \ ; \quad (a, x) \longrightarrow (x, ax)$$

(the fiber product on the right is taken in the category of alg. val. log. speces over S, not in the category of log. schemes or in that of fine log. schemes).

Let $\mathcal{F}$ be the sheaf $\mathfrak{X}' \longmapsto \mathrm{Mor}_S(\mathfrak{X}' \times_{\mathfrak{X}} \mathfrak{X}_{(n)}, G)$ on $\mathfrak{X}^{cl}_{fl}$. By (5.1.7.1), we see that the stadard complex which caluculates $H^m(\mathfrak{X}_{(n)}/\mathfrak{X}, G)$ is isomorphic to the standard complex which calculates the cohomology of the $Z/nZ(1)$-module $\mathcal{F}$ (here $Z/nZ(1)$ is regarded as a group sheaf on $\mathfrak{X}^{cl}_{fl}$). For $i \geq 0$, let Y_i be the closed subscheme of $\mathfrak{X}_{(n)}$ defined by the equation $t^i = 0$. Consider the sheaf

$$\mathcal{F}_i \ : \ \mathfrak{X}' \longmapsto \mathrm{Mor}_{\mathfrak{X}}(\mathfrak{X}' \times_{\mathfrak{X}} Y_i, G_{\mathfrak{X}})$$

on $\mathfrak{X}^{cl}_{fl}$. Since $Z/nZ(1)$ acts on Y_1 trivially, we have $H^1(Z/nZ(1), \mathcal{F}_1) = \mathrm{Hom}(Z/nZ(1), \mathcal{F}_1) = \mathrm{Hom}_{Y_1}(Z/nZ(1)_{Y_1}, G_{Y_1})$.

From the facts that Y_1 is identified with a closed subscheme of $\mathfrak{X}$

defined by a nilpotent ideal and that G is smooth, we can deduce easily that the last group is isomorphic to $\mathrm{Hom}_{\mathfrak{X}^{cl}_{fl}}(Z/nZ(1), G)$. It is easily seen that the composite

$$\mathcal{H}om_{\mathfrak{X}^{cl}_{fl}}(Z/nZ(1), G) \longrightarrow \mathcal{H}^1(\mathfrak{X}^{(n)}/\mathfrak{X}, G) \longrightarrow \mathcal{H}om_{\mathfrak{X}^{cl}_{fl}}(Z/nZ(1), G)$$

is the identity. It remains to show that

$$H^1(Z/nZ(1), \mathcal{F}_{i+1}) \longrightarrow H^1(Z/nZ(1), \mathcal{F}_i)$$

is injective for $i \geq 1$. But the kernel of $\mathcal{F}_{i+1} \longrightarrow \mathcal{F}_i$ is isomorphic to

$$I \otimes_{\mathcal{O}_S} \mathrm{Lie}_S(G)$$

with the trivial action of $Z/nZ(1)$ where I is the ideal of Y_i in Y_{i+1}, and we are reduced to $\mathrm{Hom}(Z/nZ(1), G_a) = 0$.

§5.2. Structure of finite flat commutative group objects.

Let T be a log. scheme such that $M_T/\mathcal{O}_T^{\times} \cong N$ as in §5.1, and we assume in §5.2 that the underlying scheme S of T is the Spec of a henselian noetherian local ring.

Let $\mathcal{A}_T$, $\mathcal{A}_T^{cl}$ be as in §5.1, let $\mathcal{G}_T$ be the category of group objects in $\mathcal{A}_T$ which are finite and flat over T, and let $\mathcal{G}_T^{cl}$ be the full subcategory of $\mathcal{G}_T$ consisting of objects $\mathfrak{G}$ such that $M_{\mathfrak{G}}$ coincides with the inverse image of M_T. By "forgetting the log. str.", $\mathcal{G}_T^{cl}$ is equivalent to the category of finite flat commutative group schemes over S.

We shall see soon (5.2.5) that an object $\mathfrak{G}$ of $\mathcal{G}_T$ has a largest (log) etale quotient $\mathfrak{G}^{et}$. Let $\mathcal{H}_T$ be the full subcategory of $\mathcal{G}_T$

consisting of objects $\mathfrak{G}$ such that the underlying scheme of $\mathfrak{G}^{et}$ is (cl) etale over S (i.e. such that $\mathfrak{G}^{et}$ belongs to $\mathcal{G}_T^{cl}$ (5.1.4)(4)).

We shall prove

Theorem (5.2.1). The category $\mathcal{H}_T$ is equivalent to the category of pairs (G, N) where G is a (cl) finite (cl) flat commutative group scheme over S and N is a homomorphism $G^{et}(1) \longrightarrow G^o$.

Here G^{et} denotes the largest (cl) etale quotient of G, $G^{et}(1)$ is as in (5.1.2), and G^o denotes the connected component of G containing the origin.

The equivalence in (5.2.1) is defined canonically once one fixes an element π of $\Gamma(T, M_T)$ whose image in $M_T/\mathcal{O}_T^x$ is the generator.

By combining with the classical Dieudonne theory, we obtain

Corollary (5.2.2). Let k be a perfect field of characteristic $p > 0$ and assume $S = Spec(k)$. Let W(k) be te ring of Witt vectors and let $\varphi : W(k) \longrightarrow W(k)$ be the frobenius map. Let $\mathcal{H}_T(p)$ be the full subcategory of $\mathcal{H}_T$ consisting of objects annihilated by some power of p. Then $\mathcal{H}_T(p)$ is anti-equivalent to the category of W(k)-modules D of finite length endowed with additive operators $F, V, N : D \longrightarrow D$ having the following properties:

$F(ax) = \varphi(a)F(x)$, $V(\varphi(a)x) = aV(x)$, $N(ax) = aN(x)$ $(a \in W(k), x \in D)$,

$FV = VF = p$, $FNV = N$.

Remark (5.2.3). The operator N above satisfies $N^2 = 0$ automatically. Indeed, $N^2 = FNVFNV = pFN^2V$, and hence

$$N^2 = pFN^2V = p^2F^2N^2V^2 = \dots = p^rF^rN^2V^r = 0 \quad (r >> 0).$$

Proof of (5.2.2) assuming (5.2.1). Let G be a finite commutative group scheme over k annihilated by some power of p and let $D(G)$ be the Dieudonne module. Then, $D(G^{et})$ is the largest $W(k)$-submodule of $D(G)$ which is stable under F and on which F is an isomorphism. Furthermore, the underlying $W(k)$-module of $D(G^{et}(1))$ is identified with that of $D(G^{et})$, F on $D(G^{et}(1))$ is identified with pF on $D(G^{et})$, and V on $D(G^{et}(1))$ is identified with F^{-1} on $D(G^{et})$. A homomorphism $G^{et}(1) \longrightarrow G$ corresponds to a $W(k)$-homomorphism $N : D(G) \longrightarrow D(G^{et}(1))$ compatible with F, V and hence corresponds to a $W(k)$-homomorphism $N : D(G) \longrightarrow D(G^{et})$ such that $NF = pFN$ and $NV = F^{-1}N$. Hence it corresponds to a $W(k)$-homomorphism $N : D(G) \longrightarrow D(G)$ such that $FNV = N$.

Lemma (5.2.4). <u>Let $\mathscr{A}_T^{fin}$ (resp. $\mathscr{E}_T$) be the full subcategory of $\mathscr{A}_T$ consisting of objects which are finite (resp. finite etale) over T. Then, the inclusion functor $\mathscr{E}_T \longrightarrow \mathscr{A}_T^{fin}$ has a left adjoint. This left adjoint functor, which we shall denote by $\mathfrak{X} \longmapsto \mathfrak{X}^{et}$, preserves finite inverse limits. If $\mathfrak{X}$ is an object of $\mathscr{A}_T^{fin}$ which is flat over T, the canonical morphism $\mathfrak{X} \longrightarrow \mathfrak{X}^{et}$ is flat.</u>

Proof. Let $\mathfrak{X}$ be an object of $\mathscr{A}_T^{fin}$ and assume $\mathfrak{X}$ is connected. Then $\mathfrak{X}^{et}$ is constructed as follows.

Let k be the residue field of the closed point of S and let k' be the separable closure of k in the residue field of the closed point of $\mathfrak{X}$. Let S' be the (cl) finite (cl) etale scheme over S corresponding to the extension $k \longrightarrow k'$, and let M be the inverse image on S' of the log. str. of T. On the other hand, let e be the order of the cokernel of $M_T^{gp}/\mathcal{O}_T^{\times} \cong Z \longrightarrow M_{\mathfrak{X}}^{gp}/\mathcal{O}_{\mathfrak{X}}^{\times} \cong Z$ and let e' be the largest divisor of e which is invertible on $\mathfrak{X}$. Then, there exist $\pi' \in \Gamma(S', M)$ and $\tau \in \Gamma(\mathfrak{X}, M_{\mathfrak{X}})$ such that the image of π'

in $M/\mathcal{O}_S^\times$, is a generator and such that $\tau^{e'} = \pi'$. Let

$$\mathfrak{X}^{et} = \mathrm{Spec}(\mathcal{O}_S,[t]/(t^{e'} - \alpha(\pi')))$$

with t an indeterminate, and endow $\mathfrak{X}^{et}$ with the log. str. associated to $\mathbf{N} \longrightarrow \mathcal{O}_{\mathfrak{X}^{et}}$; $1 \longrightarrow t$. Then $\mathfrak{X}^{et}$ is an object of $\mathcal{E}_T$. We have a canonical morphism $\mathfrak{X} \longrightarrow \mathfrak{X}^{et}$; $t \longrightarrow \tau$, and it is easily checked that for any object $\mathfrak{Y}$ of $\mathcal{E}_T$, the induced map $\mathrm{Mor}_T(\mathfrak{X}^{et}, \mathfrak{Y}) \longrightarrow \mathrm{Mor}_T(\mathfrak{X}, \mathfrak{Y})$ is bijective. It is also checked easily that $\mathfrak{X} \longrightarrow \mathfrak{X}^{et}$ is flat if $\mathfrak{X}$ is flat over T.

Finally the fact the functor $(\ )^{et}$ preserves finite inverse limits is shown as follows. Take a geometric point $a \longrightarrow T$ lying over the closed point of T such that $\kappa(a)$ is algebraically closed and $\Gamma(a, M_a^{gp})$ is a divisible group. Then, for any object $\mathfrak{X}$ of $\mathcal{A}_T^{fin}$, we have

$$\mathrm{Mor}_T(a, \mathfrak{X}) \xrightarrow{\cong} \mathrm{Mor}_T(a, \mathfrak{X}^{et}) \xleftarrow{\cong} (\mathfrak{X}^{et})_a$$

where $(\mathfrak{X}^{et})_a$ is the stalk at a of the sheaf on T_{et}^{log} defined by $\mathfrak{X}^{et}$. Since the functor $\mathfrak{X} \longmapsto \mathrm{Mor}_T(a, \mathfrak{X})$ preserves finite inverse limits, $\mathfrak{X} \longmapsto (\mathfrak{X}^{et})_a$ and hence $\mathfrak{X} \longmapsto \mathfrak{X}^{et}$ preserve finite inverse limits.

(5.2.5) Let $\mathfrak{G}$ be an object of $\mathcal{G}_T$. Since the functor $(\ )^{et}$ preserves finite inverse limits, $\mathfrak{G}^{et}$ is an object of $\mathcal{G}_T$. Let $\mathfrak{G}^o$ be the kernel of $\mathfrak{G} \longrightarrow \mathfrak{G}^{et}$. Then,

(5.2.5.1) $$0 \longrightarrow \mathfrak{G}^o \longrightarrow \mathfrak{G} \longrightarrow \mathfrak{G}^{et} \longrightarrow 0$$

is exact when regarded as a sequence of group sheaves on T_{fl}^{log}.

Lemma (5.2.6). <u>Let $\mathfrak{G}$ be as above. Then $\mathfrak{G}^o$ coincides with the connected component containing the origin.</u>

Proof. By applying $(\ )^{et}$ to (5.2.5.1), we have $(\mathfrak{G}^o)^{et} = T$, and this shows that $\mathfrak{G}^{o}$ is connected. It remains to show that $\mathfrak{G}^{o}$ is open in $\mathfrak{G}$. To see this, it is sufficient to show that the origin of $\mathfrak{G}^{et}$ is open in $\mathfrak{G}^{et}$. In general, a morphism between etale objects is etale as is seen from (3.1.5). So the origin $T \longrightarrow \mathfrak{G}^{et}$ is etale and hence is an open map.

Lemma (5.2.7). <u>Let $\mathfrak{G}$ be an object of</u> $\mathscr{G}_T$ <u>and assume $\mathfrak{G}$ is connected. Then, $\mathfrak{G}$ belongs to</u> $\mathscr{G}_T^{cl}$.

Proof. Let $f : \mathfrak{G} \longrightarrow T$ be the structural morphism. The image of $1 \in N$ under $N \cong f^{-1}(M_T/\mathcal{O}_T^{\times}) \longrightarrow M_{\mathfrak{G}}/\mathcal{O}_{\mathfrak{G}}^{\times} \cong N$ is a locally constant function on $\mathfrak{G}$, and coincides with 1 at the origin of $\mathfrak{G}$. Since $\mathfrak{G}$ is connected, it is 1 on the whole $\mathfrak{G}$.

(5.2.8) Now we can prove (5.2.1). We fix an element π of $\Gamma(T, M_T)$ whose image in $M_T/\mathcal{O}_T^{\times}$ is the generator. Bt using π, we have a functorial direct decomposition

(5.2.8.1) $\quad \mathrm{Ext}^1_{T^{log}_{fl}}(H, H') \cong \mathrm{Ext}^1_{T^{cl}_{fl}}(H, H') \oplus \mathrm{Hom}_S(H(1), H')$

for (cl) finite (cl) flat commutative group schemes H, H' over S with H (cl) etale over S (5.1.2).

Let $\mathfrak{G}$ be an object of $\mathscr{H}_T$. Then $\mathfrak{G}^{et}$ and $\mathfrak{G}^{o}$ belongs to $\mathscr{G}_T^{cl}$. We denote the underlying S-group schemes of $\mathfrak{G}^{et}$ and $\mathfrak{G}^{o}$ by the same letters $\mathfrak{G}^{et}$ and $\mathfrak{G}^{o}$, respectively. The extension (5.2.5.1) defines an element of $\mathrm{Ext}^1_{T^{log}_{fl}}(\mathfrak{G}^{et}, \mathfrak{G}^{o})$. In the decomposition (5.2.8.1) with $H = \mathfrak{G}^{et}$ and $H' = \mathfrak{G}^{o}$, the image of this element in $\mathrm{Ext}^1_{T^{cl}_{fl}}(\mathfrak{G}^{et}, \mathfrak{G}^{o})$ defines a finite flat commutative group scheme G over S having an exact sequence

$$0 \longrightarrow \mathfrak{G}^{o} \longrightarrow G \longrightarrow \mathfrak{G}^{et} \longrightarrow 0,$$

and the image in $\mathrm{Hom}_S(\mathfrak{G}^{et}(1), \mathfrak{G}^o)$ defines $G^{et}(1) = \mathfrak{G}^{et}(1) \longrightarrow G^o = \mathfrak{G}^o$. Conversely if we are given a finite flat commutative group scheme G over S and a homomorphism $N : G^{et}(1) \longrightarrow G^o$, then the pair $(0 \to G^o \to G \to G^{et} \to 0, N)$ is an element of the r.h.s. of (5.2.8.1) with $H = G^{et}$ and $H' = G^o$, and hence we obtain an element of $\mathrm{Ext}^1_{T^{log}_{fl}}(G^{et}, G^o)$.

For the proof of (5.2.1), it remains to show the following fact:

(5.2.8.2) For any element $\theta = (0 \longrightarrow H' \longrightarrow \mathcal{F} \longrightarrow H \longrightarrow 0)$ of $\mathrm{Ext}^1_{T^{log}_{fl}}(H, H')$ with H, H' as in (5.2.8.1), $\mathcal{F}$ is represented by an object of $\mathcal{G}_T$.

By the fppf descent (4.2.2), it is sufficient to show that $\mathcal{F}$ is represented by an object of $\mathcal{G}_T$ on a finite flat covering $T' \longrightarrow T$ of T in T^{cl}_{fl}. Hence we may assume that in the decomposition (5.2.8.1), the image of θ in $\mathrm{Ext}^1_{T^{cl}_{fl}}(H, H')$ is zero. So θ comes from a homomorphism $N : H(1) \longrightarrow H'$. The image of N belongs to the largest subgroup scheme of G^o of multiplicative type, and hence N is a composition of two homomorphisms $H(1) \longrightarrow H''(1) \xrightarrow{i} H'$ with H'' a finite etale commutative group scheme over S and with i a closed immersion. Recall that θ comes from a canonical element θ^{π}_H defined in the proof of (5.1.2), via $N : H(1) \longrightarrow H'$. First we show that if $0 \longrightarrow H''(1) \longrightarrow \mathcal{F}' \longrightarrow H \longrightarrow 0$ denotes the image of θ^{π}_H under $H(1) \longrightarrow H''(1)$, then $\mathcal{F}'$ is represented by an object of $\mathcal{G}_T$. Indeed, $\mathcal{F}'$ is obtained from the extension $\theta^{\pi}_{H''} = (0 \longrightarrow H''(1) \longrightarrow \mathcal{F}'' \longrightarrow H'' \longrightarrow 0)$, to be the fiber product of $\mathcal{F}'' \longrightarrow H'' \longleftarrow H$. So it is sufficient to show $\mathcal{F}''$ is represented by an object of $\mathcal{G}_T$ and $\mathcal{F}'' \longrightarrow H''$ is flat. In fact, working (cl) etale locally, we may assume $H'' = Z/nZ(1)$. Then, $\mathcal{F}'' = {}_n\mathfrak{E}^{\pi}$ and we have done by (3.4.8). Denote the object of $\mathcal{G}_T$ which represents $\mathcal{F}'$ by the same letter $\mathcal{F}'$. By [SGA 3] Exp. V (7.1), the quotient of the underlying scheme of $H' \times_T \mathcal{F}'$ by the action of $H''(1)$ exists, and by (4.2.3)(3) and by $\mathrm{Aut}(N) = \{1\}$, the log. str. of $H' \times_T \mathcal{F}'$ descends to this quotient. This quotient log. scheme represents $\mathcal{F}'$.

Remark (5.2.9). In (5.2.1), let $\mathfrak{G}$ be an object of $\mathcal{H}_T$ and let (G, N) be the corresponding pair. Then, for an alg. val. log. space $\mathfrak{X}$ over T, $\mathrm{Mor}_T(\mathfrak{X}, \mathfrak{G})$ is described in terms of (G, N) as the follows. Take a non-zero integer n which annihilates the image of $N : G^{et}(1) \longrightarrow G^o$. Then, there is a canonical bijection between $\mathrm{Mor}_T(\mathfrak{X}, \mathfrak{G})$ and the set of morphisms $h : \mathfrak{X} \times_T T_{(n)} \longrightarrow G$ over T ($T_{(n)}$ is as in (5.1.5)) satisfying

$$h(x, \xi y) = h(x, y).N(h(x, y), \xi)$$

for functorial "points" x, y, ξ of $\mathfrak{X}, T_{(n)}, Z/nZ(1)$, respectively. Here, ξy is defined by the canonical action of $Z/nZ(1)$ on $T_{(n)}$ (cf. the proof of (5.1.7)), and $N(\, , \,)$ denotes the composite morphism $G \times Z/nZ(1) \longrightarrow G^{et}(1) \xrightarrow{N} G^o \longrightarrow G$.

This fact is proved by applying $\mathrm{Mor}_S(\, , \mathfrak{G})$ to the exact sequence

$$\mathfrak{X} \times_T T_{(n)} \times Z/nZ(1) \cong \mathfrak{X} \times_T T_{(n)} \times_T T_{(n)} \rightrightarrows \mathfrak{X} \times_T T_{(n)} \longrightarrow \mathfrak{X}.$$

Proposition (5.2.10). The functor $\mathcal{H}om(\, , G_m)$ considered on T^{log}_{fl} induces an autoduality of the category $\mathcal{H}_T$. For an object $\mathfrak{G}$ of $\mathcal{H}_T$ corresponding to a pair (G, N) as in (5.2.1), $\mathcal{H}om(\mathfrak{G}, G_m)$ corresponds to the pair (G^*, N^*), where G^* is the usual Cartier dual of G and N^* is the Cartier dual of N.

(For $N : G^{et}(1) \longrightarrow G^o$, the Cartier dual N^* of N is defined as should have a surjective morphism $H' \longrightarrow H$ and hence H must be connected. By (5.2.7), H belongs to $\mathcal{G}_T^{cl}$ and hence $\mathcal{H}om(H, G_m)$ is represented by an object in $\mathcal{G}_T^{cl}$. On the other hand, $\mathfrak{G} \xrightarrow{\cong} \mathcal{H}om(\mathcal{H}om(\mathfrak{G}, G_m), G_m)$ (checked on T'), a contradiction.

(5.2.12) Let A be a complete discrete valuation ring and endow Spec(A) with the canonical log. str. (1.2.7). Then, the log. scheme Spec(A) does not satisfy $M/\mathcal{O}^x \cong N$, but we define the categories $\mathcal{A}_{Spec(A)}$, $\mathcal{G}_{Spec(A)}$, $\mathcal{H}_{Spec(A)}$ in the same way as at the beginnings of §5.1, §5.2. By applying (5.1.2) to the log. schemes $Spec(A/m_A^n)$ $(n \geq 1)$ and by taking $\lim_n$, we have that;

$\mathcal{H}_{Spec(A)}$ is equivalent to the category of pairs (G, N) where G is a finite flat commutative group scheme over A in the usual sense and N is a homomorphism $G^{et}(1) \longrightarrow G^o$.

This equivalence is canonical once one fixes a prime element of A.

Now let k be a perfect field of positive characteristic $p > 0$, and consider the case $A = W(k)$ and p is the fixed prime element. By combining the above categorical equivalence with the Dieudonne theory of finite flat commutative group schemes over $W(k)$ ([Fo][F-L]), we have:

Asume $p \neq 2$, then $\mathcal{H}_T(p)$ in this case is equivalent to the category of $W(k)$-modules D with finite length endowed with a $W(k)$-submodule $L \subset D$, frobenius-linear operators $\varphi_0 : D \longrightarrow D$, $\varphi_1 : L \longrightarrow D$, and a linear operator $N : D \longrightarrow D$ satisfying the following conditions:

$\varphi_0|_L = p\varphi_1$, $\varphi_0(D) + \varphi_1(L) = D$,

$N\varphi_0 = p\varphi_0 N$, $N\varphi_1 = \varphi_0 N$ on L.

follows. We assume the characteristic p of the residue field of the closed point of S is positive (otherwise, N is always zero, and we define $N^* = 0$). Then, N factors through $G\{p\}^{et}(1) \longrightarrow G\{p\}^{mult}$ where $\{p\}$ denotes the p-primary part and $(\ )^{mult}$ denotes the largest subgroup scheme of multiplicative type. By taking the Cartier dual, we obtain $N^* : G^*\{p\}^{et}(1) \longrightarrow G^*\{p\}^{mult}$.)

Proof of (5.2.10). The fact $\mathfrak{G} \longrightarrow \mathcal{H}om(\mathcal{H}om(\mathfrak{G}, G_m), G_m)$ is an isomorphism is checked on a finite flat covering $T' \longrightarrow T$ in T^{log}_{fl} such that $\mathfrak{G} \times_T T'$ belongs to $\mathcal{G}^{cl}_{T'}$, by using the classical Cartier duality theory. The fact $\mathcal{H}om(\mathfrak{G}, G_m)$ is represented by an object of $\mathcal{G}_T$ is proved as follows. The sheaf $\mathfrak{G}/\mathfrak{G}^{mult}$ is represented by an object of $\mathcal{G}_T$ (5.2.8.2), and this object belongs to $\mathcal{G}^{cl}_T$ by (5.2.1) since the corresponding N is zero. We have an exact sequence

$$0 \longrightarrow \mathcal{H}om(\mathfrak{G}/\mathfrak{G}^{mult}, G_m) \longrightarrow \mathcal{H}om(\mathfrak{G}, G_m) \longrightarrow \mathcal{H}om(\mathfrak{G}^{mult}, G_m) \longrightarrow 0.$$

Then, $\mathcal{H}om(\mathfrak{G}/\mathfrak{G}^{mult}, G_m)$ is represented by an object in $\mathcal{G}^{cl}_T$ and $\mathcal{H}om(\mathfrak{G}^{mult}, G_m)$ is represented by an etale object in $\mathcal{G}^{cl}_T$. Hence by (5.2.8.2), $\mathcal{H}om(\mathfrak{G}, G_m)$ is represented by an object of $\mathcal{G}_T$.

Remark (5.2.11). Unlike the classical Cartier duality theory, if T is not a Q-scheme, the functor $\mathcal{H}om(\ , G_m)$ (considered on T^{log}_{fl}) does not give an autoduality of the total $\mathcal{G}_T$. For example, let $\mathfrak{G}$ be an etale object in $\mathcal{G}_T$ annihilated by a power of a prime p which is not invertible on T, and assume that $\mathfrak{G}$ does not belong to $\mathcal{G}^{cl}_T$. Then, the sheaf $\mathcal{H}om(\mathfrak{G}, G_m)$ on T^{log}_{fl} is not represented by an object of $\mathcal{G}_T$. To see this, take a finite flat connected non-empty object T' in $\mathcal{A}_T$ such that $\mathfrak{G} \times_T T'$ belongs to $\mathcal{G}^{cl}_{T'}$. Then $\mathcal{H}om(\mathfrak{G}, G_m)$ on $(T')^{log}_{fl}$ is represented by a connected object H' in $\mathcal{G}_{T'}$. If $\mathcal{H}om(\mathfrak{G}, G_m)$ is represented by an object H of $\mathcal{G}_T$, we

References.

[BBM] Berthelot, P., Breen, L. and Messing, W, Théorie de Dieudonné cristalline II, Lecture Notes in Math. 930, Springer (1982).

[Bo] Bourbaki, N., Algèbre commutative (1982, 1983, 1985,...) Masson.

[De] Deligne, P., Letter to Illusie, L., June, 1, 1988.

[Ch-Fa] Chai, C.-L. and Faltings, G., Semiabelian degeneration and compactification, preprint.

[D-R] Deligne, P and Rapoport, M., Les schémas de modules de courbes elliptiques, in Modular functions in one variable II, Lecture Notes in Math. 349, Springer (1973), 143-316.

[Fa_1] Faltings, G., Crystalline cohomology and p-adic [illegible] Galois-representations, to appear in Amer. J. Math.

[Fa_2] Faltings, G., F-isocrystals on open varieties, results and conjectures, preprint.

[Fo] Fontaine, J.-M., Groupes finis commutatifs sur les vecteurs de Witt, CR. Acad. Sc. Paris, 280, 1975, 1423-1425.

[F-L] Fontaine, J.-M. and Laffaille, G., Construction de représentations p-adiques, Ann. Sci. Ec. Norm. Sup. 15 (1982) 547-608.

[Fu] Fujiwara, K., Compactifications of Shimura varieties over Z, preprint.

[GM] Grothendieck, A. and Murre, J. P., The fundamental group of a formal neighbourhood of a divisor with normal crossings on a scheme, Lecture Notes in Math. 208, Springer (1971).

[Is] Ishida, M., Compactification of a family of generalized Jacobian varieties, in Proc. of the International Symp. on alg.

Problem (5.2.13). Construct a Dieudonne theory of (log) finite (log) flat group objects and "logarithmic p-divisible groups" over an arbitrary base, by using the method of logarithmic crystalline cohomology ([Fa_1][Fa_2][Ka]), as is done in [BBM] in the classical (without log) case. In this theory, logarithmic crystalline cohomology of degree one should be related to the Dieudonne module of the logarithmic p-divisible group associated to the "logarithmic Picard variety".

..o, 1977, 503-524, Kinokuniya (1978).

, K., Logarithmic structures of Fontaine-Illusie, to appear in Amer. J. Math.

[KKMS] Kempf, G., Knudsen, F., Mumford, D. and Saint-Donat, B., Toroidal embeddings I, Lecture Notes in Math. 339, Springer, (1973).

[Od] Oda, T., Convex bodies and algebraic geometry, Springer (1987).

[O-S] Oda, T. and Seshadri, C. S., Compactifications of the generalized Jacobian variety, Trans. Amer. Math. Soc. 253, (1979) 1-90.

[Bu] Proceedings of the seminar on p-adic monodromy at IHES in the spring of 1988 (Fontaine, J.-M. and others), in preparation.

[EGA] Grothendieck, A. and Diuedonné, J., Eléments de géométrie algébrique, Publ. Math. IHES. 4 (1960), 8, 11 (1961), 17 (1963), 20 (1064), 24 (1965), 28 (1966), 32 (1967).

[SGA 1] Revêtement étale et groupe fondamental, Grothendieck, A. (with two exposés by Mme Raynaud, M.), Lecture Notes in Math. 224 Springer (1971).

[SGA 3] Grothendieck, A and Demazure M. with the collaboration of Artin, M., Bertin, J. E., Gabriel, P., Raynaud, M. and Serre, J. P., Schémas en groupe, Lecture Notes in Math. 151, 152, 153 Springer (1970).

[SGA 4] Artin, M., Grothendieck, A., Verdier, J. L., Théorie des topos et cohomologie etale des schémas, Lecture Notes in Math. 269, 270 (1972), 305 (1973), Springer.