1 Moduli spaces of polarized Hodge structures.

First of all, we briefly summarize the classical theory of the moduli spaces of polarized Hodge structures.

1.1 The moduli space $M_h = \Gamma\backslash D_h$.

Let n be an integer, and let h be a sequence of positive integers $(h^{n,0}, h^{n-1,0}, \cdots, h^{0,n})$ satisfying $h^{p,q} = h^{q,p}$, called the *Hodge numbers*. Let $H_{\mathbb{Z}}$ be a free abelian group of rank $\sum h^{p,n-p}$, with a non-degenerate bilinear form $Q : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$, which is symmetric (resp. anti-symmetric) if n is even (resp. odd). Let $G_{\mathbb{Z}}$ be the group functor $\mathrm{Aut}(H_{\mathbb{Z}}, Q)$ on rings, sending a ring R to the group of automorphisms on the free R-module $H_R := H_{\mathbb{Z}} \otimes R$ preserving the bilinear form Q. It is clearly a group scheme over $\mathbb{Z}$. Let Γ be an arithmetic subgroup of $G_{\mathbb{Z}}(\mathbb{Z})$ ([5], §3).

The set of Hodge structures of weight n on $H_{\mathbb{R}}$ with prescribed Hodge numbers h, such that Q induces a *polarization* on $H_{\mathbb{R}}$ (i.e. it induces a morphism $H_{\mathbb{R}} \otimes H_{\mathbb{R}} \to \mathbb{R}(-n)$ of Hodge structures, and the bilinear form $Q_C(u,v) := Q(u, Cv)$, where C is the Weil operator, is symmetric and positive definite), is parameterized by the homogeneous space $D_h = G_{\mathbb{R}}/K$, where K is the stabilizer group of a fixed polarized Hodge structure F_0 on $H_{\mathbb{R}}$.

This homogeneous space $D = D_h = G_{\mathbb{R}}/K$ has a complex structure defined as follows. It is clear that $Q : H_{\mathbb{R}} \otimes H_{\mathbb{R}} \to \mathbb{R}(-n)$ is a morphism of Hodge structures if and only if $Q(F^p, F^{n-p+1}) = 0$ for all p. Let $f^p = \sum_{r \geq p} h^{r,n-r}$, and let $D^\vee$, the *compact dual of* D, to be the subspace of the product of the Grassmannians $\prod_p \mathrm{Gr}(f^p, H_{\mathbb{R}})$ consisting of all flags $F^\bullet$

$$\cdots \subset F^{p+1} \subset F^p \subset \cdots$$

such that $Q(F^p, F^{n-p+1}) = 0$. Then $D^\vee = G_{\mathbb{C}}/P$, where P is a parabolic subgroup corresponding to a fixed flag. This gives $D^\vee$ a complex structure. We see that $D \subset D^\vee$ is the locus of flags satisfying

(i) $F^p \cap \overline{F}^{n-p+1} = 0$ (so that $F^p \oplus \overline{F}^{n-p+1} \cong H_{\mathbb{C}}$), and

(ii) $Q(\overline{u}, Cu) > 0$ for $u \neq 0$ in $H_{\mathbb{C}}$.

They are both open conditions, so $D \subset D^\vee$ is an open complex submanifold. The group Γ acts on D_h properly discontinuously, and the quotient $M_h = \Gamma\backslash D_h$ is the moduli space of Γ*-equivalence classes of* Q*-polarized Hodge structures on* $H_{\mathbb{C}}$ *with Hodge type* h. See ([3], 0.3.6, 0.3.7).

1.2 Variations of Hodge structure.

Definition 1.2.1. *Let S be a complex manifold. A* variation of Hodge structure $\mathscr{H}$ of weight n on S *is given by*

- *a local system $\mathscr{H}_{\mathbb{Z}}$ of free abelian groups of finite rank on S;*
- *a finite decreasing filtration $F^\bullet \mathscr{H}_{\mathscr{O}}$ of the vector bundle $\mathscr{H}_{\mathscr{O}} := \mathscr{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathscr{O}_S$ by holomorphic sub-bundles,*

such that the following conditions are satisfied:

1) (Griffiths transversality) the natural flat connection $\nabla = d \otimes \mathrm{id}_{\mathscr{H}_{\mathbb{Z}}} : \mathscr{H}_{\mathscr{O}} \to \Omega^1_S \otimes \mathscr{H}_{\mathscr{O}}$ takes $F^p\mathscr{H}_{\mathscr{O}}$ into $\Omega^1_S \otimes F^{p-1}\mathscr{H}_{\mathscr{O}}$, for every p;

2) for each point $s \in S$, the fiber $F^\bullet(s)$ over s is a Hodge structure of weight n.

A polarization *of the variation of Hodge structure $\mathscr{H}$ is a locally constant bilinear form*

$$\mathscr{Q} : \mathscr{H}_{\mathbb{Z}} \otimes \mathscr{H}_{\mathbb{Z}} \to \underline{\mathbb{Z}}$$

such that on each fiber over $s \in S$, it induces a polarization of the fiber Hodge structure.

Suppose we have a polarized family of Hodge structures $(\mathscr{H}, \mathscr{Q} : \mathscr{H}_\mathbb{Z} \otimes \mathscr{H}_\mathbb{Z} \to \mathbb{Z})$ of weight n on S, and a global section of the sheaf

$$\Gamma\backslash\underline{\mathrm{Isom}}((\mathscr{H}_\mathbb{Z}, \mathscr{Q}), (H_\mathbb{Z}, Q)),$$

where $H_\mathbb{Z}$ is regarded as a constant sheaf on S, and assume that all the monodromies of this family of Hodge structures on S are contained in Γ. Then there is a *period map*

$$\varphi : S \to M_h$$

inducing this family of Hodge structures. This map is locally liftable to D_h.

If $f : X \to S$ is a projective smooth morphism between quasi-projective complex algebraic manifolds, with a relative hyperplane section $\eta \in H^0(S, R^2 f_* \mathbb{Z})$, then the family of the primitive part $P^n(X_s, \mathbb{Z})$ of the cohomology groups $H^n(X_s, \mathbb{Z})$ modulo torsion form a polarized variation of Hodge structure of weight n on S, and it induces a period map $S \to M_h$. To be precise, the family of $H^n(X_s, \mathbb{C})$'s are the stalks of $R^n f_*(f^{-1}\mathscr{O}_S)$, and the Hodge filtration on $R^n f_*(f^{-1}\mathscr{O}_S)$ is given by the degenerate spectral sequence

$$E_1^{pq} = R^q f_* \Omega^p_{X/S} \Longrightarrow R^{p+q} f_*(f^{-1}\mathscr{O}_S),$$

which is induced from the resolution $\Omega^\bullet_{X/S}$ of $f^{-1}\mathscr{O}_S$ (the relative holomorphic Poincaré lemma, see ([1], 3.4)). Since η is a global section, the primitive part form a variation of sub-Hodge structures on S.

2 Logarithmic Hodge structures.

One can ask the following question. Let $f : X \to S$ be a family of projective manifolds, and let S be the complement of a normal crossing divisor D in some compact manifold $\overline{S}$, and suppose one can extend the family f to a family $\overline{f} : \overline{X} \to \overline{S}$ which is log smooth (here $\overline{S}$ has the log structure induced by the divisor D). Is it possible to enlarge the moduli space M_h to some $\overline{M}_h$ so that the period map extends to $\overline{\varphi} : \overline{S} \to \overline{M}_h$?

To study the degenerations of Hodge structures, Kato and Usui introduced the notion of logarithmic Hodge structures.

2.1 The ringed space $X^{\log}$.

Let $(X, \alpha : M_X \to \mathscr{O}_X)$ be an fs log analytic space over $\mathbb{C}$ (for instance the $\mathbb{C}$-points of an fs log scheme over $\mathbb{C}$), and let $X^{\log}$ be the set of pairs (x, u), where $x \in X$ and $u : M_{X,x} \to S^1$ is a homomorphism of monoids, such that $u(f) = f(x)/|f(x)|$ for $f \in \mathscr{O}^*_{X,x} \subset M_{X,x}$. Here S^1 is the unit circle in the complex plane. Let $\tau : X^{\log} \to X$ be the function $(x, u) \mapsto x$. For any open $U \subset X$ and $f \in M_X(U)$, there is a function $arg(f) : \tau^{-1}(U) \to S^1$ sending $(x, u) \mapsto u(f)$. We give $X^{\log}$ the weakest topology such that the functions τ and $arg(f)$ are continuous. Over the open set $X^* \subset X$ where the log structure is trivial, the map τ is a homeomorphism, and the section $j^{\log} : X^* \hookrightarrow X^{\log}$ is a homotopy equivalence. The map τ is proper, with fibers $\tau^{-1}(x)$ compact tori $(S^1)^m$, where m is the rank of $\overline{M}^{\mathrm{gp}}_{X,x}$.

One can define a sheaf of rings $\mathscr{O}_{X^{\log}}$ on $X^{\log}$. Roughly speaking, this is the subsheaf of rings of $j^{\log}_* \mathscr{O}_{X^*}$ on $X^{\log}$ generated over $\tau^{-1}\mathscr{O}_X$ by "$\log(q)$", for all $q \in M^{\mathrm{gp}}_X$. See ([3], 2.2.4) for the precise definition.

For example, if $x \in X$ and $y \in \tau^{-1}(x)$, and the free abelian group $\overline{M}^{\mathrm{gp}}_{X,x}$ has rank m and is generated by $f_1, \cdots, f_m \in M^{\mathrm{gp}}_{X,x}$, then the stalk $\mathscr{O}_{X^{\log},y}$ is isomorphic to the polynomial ring $\mathscr{O}_{X,x}[\log(f_1), \cdots, \log(f_m)]$. This shows that in general, $(X^{\log}, \mathscr{O}_{X^{\log}})$ is not a locally ringed space.

Let Ω^1_X be the sheaf of log differential forms on the fs log analytic space X, i.e.

$$\Omega^1_X = (\Omega^1_{\underline{X}} \oplus (\mathscr{O}_X \otimes_{\mathbb{Z}} M^{\mathrm{gp}}_X))/\{(-d\alpha(f), \alpha(f) \otimes f) | f \in M_X\}.$$

For a morphism $f : X \to Y$ of fs log analytic spaces, define

$$\Omega^1_{X/Y} = \mathrm{Coker}(f^*\Omega^1_Y \to \Omega^1_X).$$

They are both coherent $\mathscr{O}_X$-modules. Let $\Omega^r_{X/Y}$ be the r-th exterior power of $\Omega^1_{X/Y}$, and let

$$\Omega^r_{X^{\log}/Y^{\log}} = \tau^*\Omega^r_{X/Y} = \tau^{-1}\Omega^r_{X/Y} \otimes_{\tau^{-1}\mathscr{O}_X} \mathscr{O}_{X^{\log}}.$$

One can define differential maps and have the log de Rham complex $(\Omega^\bullet_{X/Y}, d)$ (resp. $(\Omega^\bullet_{X^{\log}/Y^{\log}}, d)$) on X (resp. $X^{\log}$).

For $y \in X^{\log}$ and $x = \tau(y) \in X$, let $\mathrm{sp}(y)$ be the set of all ring homomorphisms $s : \mathscr{O}_{X^{\log},y} \to \mathbb{C}$ that extend the evaluation map $\mathrm{ev}_x : \mathscr{O}_{X,x} \to \mathbb{C}$. Since $\mathscr{O}_{X^{\log},y}$ is isomorphic to the polynomial ring over $\mathscr{O}_{X,x}$ generated by log of a basis for $\overline{M}_{X,x}$, if we fix an $s_0 \in \mathrm{sp}(y)$, then we have a bijection:

$$s \mapsto (f \mapsto s(\log(f)) - s_0(\log(f))) : \mathrm{sp}(y) \xrightarrow{\sim} Hom_{\mathrm{group}}(\overline{M}^{\mathrm{gp}}_{X,x}, \mathbb{C}).$$

2.2 Log variations of polarized Hodge structure.

Definition 2.2.1. *Let X be an fs log analytic space. A* log variation of polarized Hodge structure of weight n *on X is given by*

- *a local system of free abelian groups of finite rank $\mathscr{H}_{\mathbb{Z}}$ on $X^{\log}$,*
- *a bilinear form $\mathscr{Q} : \mathscr{H}_{\mathbb{Z}} \otimes \mathscr{H}_{\mathbb{Z}} \to \underline{\mathbb{Z}}$,*
- *a finite decreasing filtration $F^\bullet\mathscr{H}_{\mathscr{O}}$ of $\mathscr{H}_{\mathscr{O}} := \mathscr{H}_{\mathbb{Z}} \otimes \mathscr{O}_{X^{\log}}$ by $\mathscr{O}_{X^{\log}}$-submodules,*

such that the following conditions are satisfied:

1) there exist a locally free $\mathscr{O}_X$-module $\mathscr{E}$ and a finite decreasing filtration $F^\bullet\mathscr{E}$ by $\mathscr{O}_X$-submodules, such that $Gr_p(\mathscr{E})$ is locally free for each p, and

$$F^p\mathscr{H}_{\mathscr{O}} = \tau^*F^p\mathscr{E} = \tau^{-1}F^p\mathscr{E} \otimes_{\tau^{-1}\mathscr{O}_X} \mathscr{O}_{X^{\log}};$$

2) for $y \in X^{\log}$ and $x = \tau(y) \in X$, let $s \in \mathrm{sp}(y)$ and let $f_1, \cdots, f_r \in M_{X,x} - \mathscr{O}^*_{X,x}$ generate the monoid $\overline{M}_{X,x}$. If the $|\exp(s(\log(f_i)))|$ are sufficient small for all i, then $(\mathscr{H}_{\mathbb{Z},y}, \mathscr{Q}, F^\bullet(s))$ is a polarized Hodge structure of weight n;

3) the connection $d \otimes \mathrm{id} : \mathscr{H}_{\mathscr{O}} \to \Omega^1_{X^{\log}} \otimes_{\mathscr{O}_{X^{\log}}} \mathscr{H}_{\mathscr{O}}$ takes $F^p\mathscr{H}_{\mathscr{O}}$ into $\Omega^1_{X^{\log}} \otimes F^{p-1}\mathscr{H}_{\mathscr{O}}$.

Here $F^\bullet(s)$, the *specialization of F at s*, is the decreasing filtration of $\mathscr{H}_{\mathbb{C},y} := \mathbb{C} \otimes_{\mathbb{Z}} \mathscr{H}_{\mathbb{Z},y}$ defined by $F^p(s) = \mathbb{C} \otimes_{s,\mathscr{O}_{X^{\log},y}} F^p\mathscr{H}_{\mathscr{O}}$. For a fixed point $y \in X^{\log}$, the family $(\mathscr{H}_{\mathbb{Z},y}, \mathscr{Q}, F^\bullet(s))_{s \in \mathrm{sp}(y)}$ is called a *polarized log Hodge structure* on the log point $(x, M_{X,x})$; this is the same as a log variation of polarized Hodge structure on the log point $(x, M_{X,x})$.

Log variations of polarized Hodge structure arise from geometry in the following way. Let $f : X \to Y$ be a projective log smooth morphism between fs log analytic spaces, and we fix a line bundle on X which is relatively ample over Y. By a theorem of Kajiwara and Nakayama, for every integer n, the sheaf $R^n f^{\log}_* \mathbb{Z}$ is a local system on $Y^{\log}$. We take $\mathscr{H}_Z$ to be $R^n f^{\log}_* \mathbb{Z}$ modulo torsion, take $\mathscr{Q}$ to be the pairing induced by the fixed ample line bundle, take $\mathscr{E}$ to be $R^n f_*(\Omega^\bullet_{X/Y})$, with filtration $F^p\mathscr{E} = R^n f_*(\Omega^{\geq p}_{X/Y}) \subset \mathscr{E}$, and take $F^p\mathscr{H}_{\mathscr{O}}$ to be $\tau^*F^p\mathscr{E}$. Then by a theorem of Kato, Matsubara and Nakayama, this is a log variation of polarized Hodge structure on Y.

3 Kato-Usui spaces.

We fix $n, h, H_{\mathbb{Z}}, Q, G_{\mathbb{Z}}, D$ and $D^\vee$ as in (1.1). Let $\mathfrak{g}_R = \mathrm{Lie}(G_R)$. A subset $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ is called a *nilpotent cone* if it is a cone

$$\sigma = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} N_i$$

generated by mutually commutative nilpotent operators $N_i \in \mathfrak{g}_{\mathbb{R}} \subset \mathrm{End}(H_{\mathbb{R}})$. Let Γ be a *neat subgroup* of $G_{\mathbb{Z}}(\mathbb{Z})$, i.e. for every element $\gamma \in \Gamma$, its eigenvalues on $H_{\mathbb{C}}$ generate a torsion-free subgroup of $\mathbb{C}^*$.

3.1 Nilpotent orbits.

Definition 3.1.1. *Let $\sigma = \sum_i \mathbb{R}_{\geq 0} N_i$ be a nilpotent cone. A subset $Z \subset D^\vee$ is called a* σ-nilpotent orbit*, if there exists an $F_0 \in D^\vee$ such that*

- $Z = \exp(\sum_i \mathbb{C} N_i) F_0$,
- $N F_0^p \subset F_0^{p-1}$ *for all* $p \in \mathbb{Z}$ *and* $N \in \sigma$,
- $\exp(\sum_i z_i N_i) F_0 \in D$ *if* $\mathrm{Im}(z_i) \gg 0$ *for all* i.

We also call the pair (σ, Z) a *nilpotent orbit.*

Let Σ be a *fan* in $\mathfrak{g}_{\mathbb{Q}}$, i.e. $\Sigma \neq \emptyset$ is a set of rational nilpotent cones in $\mathfrak{g}_{\mathbb{R}}$ (namely, those generated by nilpotent operators in $\mathfrak{g}_{\mathbb{Q}}$) such that

- if $\sigma \in \Sigma$, then all faces of σ are in Σ,
- for $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ',
- for every $\sigma \in \Sigma$, we have $\sigma \cap (-\sigma) = 0$.

One can then define the set $D_{h,\Sigma}$ (or just D_Σ, if there is no confusion) of *nilpotent orbits in the directions in* Σ to be the set of nilpotent orbits (σ, Z) where $\sigma \in \Sigma$. There is a natural injection

$$F \mapsto (0, \{F\}) : D \hookrightarrow D_\Sigma.$$

3.2 The moduli space M_Σ.

Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$ and let $\Gamma \subset G_{\mathbb{Z}}(\mathbb{Z})$ be a subgroup. Then we say that Γ *is compatible with* Σ if for every $\gamma \in \Gamma$ and $\sigma \in \Sigma$, we have $Ad(\gamma)(\sigma) \in \Sigma$. In this case, there is an action of Γ on D_Σ given by

$$(\sigma, Z) \xmapsto{\gamma} (Ad(\gamma)(\sigma), \gamma Z).$$

We say that Γ *is strongly compatible with* Σ if every cone $\sigma \in \Sigma$ is generated by elements in $\log \Gamma$. Kato and Usui showed that when Γ is strongly compatible with Σ and the arithmetic subgroup Γ is neat, the quotient set $\Gamma \backslash D_\Sigma$ can be given the structure of a log locally ringed space over $\mathbb{C}$, in fact a *log manifold* (see ([3], 3.5.7)). Roughly speaking, a log manifold is a log locally ringed space over $\mathbb{C}$, which is locally isomorphic to the "zero locus" of some log differential forms on a log smooth analytic space.

Informally speaking, Kato and Usui proved the following. First, there is a one-to-one correspondence between D_Σ and the set of polarized log Hodge structures of the given type. Second, if $\overline{X} \to \overline{S}$ is a log smooth family extending the projective smooth family $X \to S$, where $S \subset \overline{S}$ is the complement of a normal crossing divisor, then the period map extends to $\overline{S} \to M_\Sigma$. We briefly explain the first part in the following.

We shall show how to get a nilpotent orbit from a polarized log Hodge structure on a log point ([3], 0.4.24). Let x be an fs log point with log structure M_x. Then $\overline{M}_x$ is a sharp fs monoid and $\overline{M}_x^{\mathrm{gp}}$ if a free abelian group of finite rank, say r. Fix $y \in x^{\log}$. We have $x^{\log} = Hom(\overline{M}_x^{\mathrm{gp}}, S^1) \simeq (S^1)^r$ and hence $\pi_1(x^{\log}) = Hom(\overline{M}_x^{\mathrm{gp}}, \mathbb{Z}) \simeq \mathbb{Z}^r$. Let

$\pi_1^+(x^{\log}) \subset \pi_1(x^{\log})$ be the subset consisting of those homomorphisms $a : \overline{M}_x^{\mathrm{gp}} \to \mathbb{Z}$ that take $\overline{M}_x$ into $\mathbb{N}$; this subset is an fs monoid.

Let $(H_{\mathbb{Z}}, Q, F^\bullet H_{\mathscr{O}})$ be a polarized log Hodge structure on x. Let $(h_i)_{i=1}^n$ be a family of generators for $\pi_1^+(x^{\log})$ and $s_0 \in \mathrm{sp}(y)$. Let $z_1, \cdots, z_r$ be complex numbers, and let $s \in \mathrm{sp}(y)$ be such that

$$s\Big(\frac{\log(f)}{2\pi i}\Big) - s_0\Big(\frac{\log(f)}{2\pi i}\Big) = \sum_{i=1}^{r} z_i h_i(f), \quad \text{for } f \in \overline{M}_x^{\mathrm{gp}}.$$

Let $N_i : H_{\mathbb{Q},y} \to H_{\mathbb{Q},y}$ be the logarithm of h_i. Then we have

$$F(s) = \exp\Big(\sum_{i=1}^{n} z_i N_i\Big) F(s_0),$$

which shows that $(F(s))_{s\in\mathrm{sp}(y)}$ is an orbit of filtrations under $\exp(\sigma \otimes \mathbb{C})$ for $\sigma = \sum_i \mathbb{R}_{\geq 0} N_i$. Moreover, the condition 2) in (2.2.1) implies that $F(s) \in D$ if $\mathrm{Im}(z_i) \gg 0$ for all i, and the condition 3) in (2.2.1) implies that $NF(s_0)^p \subset F(s_0)^{p-1}$ for all $p \in \mathbb{Z}$ and $N \in \sigma$. In other words, the family $(F(s))_s$ is a σ-nilpotent orbit.

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