LOGARITHMIC PLANS

DAN ABRAMOVICH

All log schemes here should probably be fine and saturated.

1. Log points and their maps

1.1. **The stack of standard log points.**

Definition 1.2. A standard log points is a point with log structure such that the characteristic is $\mathbb{N}$. Define a family of standard log points over a log scheme S as follows: it is a log scheme S' with a morphism $S' \to S$ which is an isomorphism on underlying schemes, and such that that $\overline{M}'_S = \overline{M}_S[m_x]$. Arrows of such afmilies are defined by cartesian diagrams as usual.

Problem 1.3. Show the following:

This defines a stack over $\mathfrak{LogSch}$.

This stack is isomorphic to $\mathcal{B}\,\mathbb{G}_m$, with the trivial log structure. The universal family is $\mathcal{B}\,\mathbb{G}_m$ with the log structure inherited from the embedding $\mathcal{B}\,\mathbb{G}_m \subset [\mathbb{A}^1/\mathbb{G}_m]$.

Now let X be a log scheme. Define another category $\wedge X$ over $\mathfrak{LogSch}$ whose objects over S are $(S' \to S, S' \to X)$ where $S' \to S$ is a family of standard log points and $S' \to X$ a morephism, and arrows by cartesian diagrams.

Problem 1.4. This stack $\wedge X$ is a representable countable union of log schemes projective over X.

Problem 1.5. Give a concrete description of $\wedge X$. Show that when X is log smooth this thing is also log smooth.

1.6. **The stack of standard log nodes.**

Definition 1.7. Define a family of standard log nodes over a log scheme S as follows: it is a log scheme S' with a homomorphism $\phi : \mathbb{N} \to \overline{M}_S$ and a morphism $\pi : S' \to S$ which is an isomorphism on underlying schemes, and such that that $\overline{M}'_S = \overline{M}_S[m_x, m_y]/(m_x + m_y = \phi(1))$. Arrows defined by cartesian diagrams. This defines a category $\mathfrak{Nodes}$ over $\mathfrak{LogSch}$

Problem 1.8. Figure out the right definition, whether or not this is it. Show it is an algebraic log stack. Describe the stack $\mathfrak{Nodes}$ in a reasonable way.

Date: today.

Now let X be a log scheme. Define another stack $\wedge^{\mathfrak{Nodes}}X$ over $\mathfrak{LogSch}$ whose objects over S are $(S' \to S, S' \to X)$ where $S' \to S$ is a family of standard log nodes and $S' \to X$ a morephism, and arrows by cartesian diagrams.

Problem 1.9. Give a concrete description of $\wedge^{\mathfrak{Nodes}}X$.

2. Log smooth GW theory

2.1. **Evaluation maps.** Consider a log smooth scheme X. We have a stack $K_\Gamma(X)$ of log stable maps to X with numerical invariants Γ (including genus, types of markings, class etc.).

Problem 2.2. Show that the restriction of the universal map $C^{univ} \to X$ to the i-th marking gives a morphism $e_i : K_\Gamma(X) \to \wedge X$.

2.3. **Obstruction theory.**

Problem 2.4. Show that the log cotangent complexes of X and of C/K are their sheaves of log differentials, so locally free, and there is a morphism $f^*\Omega_X \to \Omega_{C/S}$.

Problem 2.5. Show that the standard formalism provides a perfect log obstruction theory $\mathbb{E} \to \mathbb{L}_{K_\Gamma(X)}$.

Problem 2.6. We have an isomorphism $\mathbb{L}_{K_\Gamma(X)} \simeq LL_{\underline{K_\Gamma(X)}/LOG}$, where $\underline{K_\Gamma(X)}$ is the stack underlying $K_\Gamma(X)$.

2.7. **Virtual fundamental class.**

Problem 2.8. Show that the image of $\underline{K_\Gamma(X)}$ in LOG lies in an open substack LOG^0, of finite type over $\mathbb{C}$, of pure dimension 0, satisfying Kresch's stratification conditions [1], so a fundamental class $[LOG^0]$ exists.

Problem 2.9. Show that the morphism $\underline{K_\Gamma(X)} \to LOG^0$ and the log obstruction theory $\mathbb{E} \to \mathbb{L}_{K_\Gamma(X)}$ satisfy Manolache's requirements [2], so a refined pull-back $f^!_{\mathbb{E}} : A_*(LOG^0) \to A_*(\underline{K_\Gamma(X)})$ exists.

Problem 2.10. Investigate the resulting virtual fundamental class $[K_\Gamma(X)]^{vir} = f^!_{\mathbb{E}}[LOG^0]$ and its properties.

2.11. **Invariants.**

Problem 2.12. Define log GW invariants as usual by

$$\langle \gamma_1 \cdots \gamma_n \rangle := \int_{[K_\Gamma(X)]^{vir}} \prod e_i^* \gamma_i.$$

Define analogous descendant invariants.
Investigate their basic properties.

2.13. **Localization.**

Problem 2.14. Extend the standard theory of virtual localization in the log smooth context.

3. Log nodal theory

Definition 3.1. Define a log prenodal curve to be a subcurve of a log smooth curve.

Define a log presemistable variety to be a subvariety which is a union of components of a semistable (or maybe log smooth) variety.

Problem 3.2. Define log stable maps of log prenodal curves into log schemes.

Problem 3.3. Compare log stable maps of log prenodal curves, maybe with log presemistable targets, with log stable maps of log smooth curves into log smooth targets.

Problem 3.4. Define prenodal evaluation maps.

Problem 3.5. Investigate when stable log prenodal maps admit a perfect log obstruction theory.

Problem 3.6. In such situations, define prenodal GW invariants. Compare with the log smooth GW invariants.

Problem 3.7. Define Gross-Hacking-Keel invariants of a normal crossings variety.

4. Degeneration formula

Problem 4.1. Investigate log degeneration formulas in the higher rank case (the case of a smooth divisor case this should be in Qile's thesis). Use wither the log prenodal formalism or through a connection with the log smooth theory.

5. Other aspects

Problem 5.1. In the smooth divisor case, compare the log GW theory with Kim's theory.

Problem 5.2. Prove an existence result for general log maps (source not necessarily a point or curve).

References

Kresch [1] Kresch's thesis.
Manolache [2] Manolache's thesis.