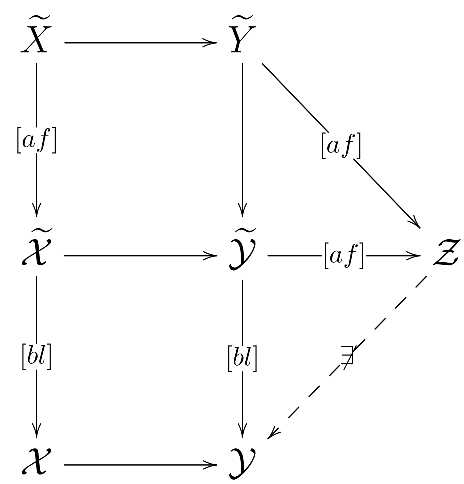
THE ARTIN FAN OF THE WHITNEY UMBRELLA

DAN

for Mark

1. The diagram

We are interested in the diagrams on page 24 of the survey, which I combine as follows:



The annotation $[af]$ stands for the Artin fan construction, $[bl]$ for blowing up an ideal sheaf. The map $\mathcal{X} \to \mathcal{Y}$ comes from the functoriality of Artin fans for the strict morphism $X \to Y$. The maps $\tilde{\mathcal{X}} \to \tilde{\mathcal{Y}}$ and $\tilde{X} \to \tilde{Y}$ are obtained by pullbacks of $\mathcal{X} \to \mathcal{Y}$, equivalently the fact that blowing up commutes with smooth morphisms. The map $\tilde{Y} \to \tilde{\mathcal{Y}}$ is similarly the pullback of $Y \to \mathcal{Y}$.

2. The Artin fan $\mathcal{Y}$

Let's go over the description of $\mathcal{Y}$ in the survey. Clearly $\mathcal{X} = \mathcal{A}^2$. We have $\mathcal{Y} = [R \rightrightarrows \mathcal{A}^2]$. Here $R = \mathcal{A}^2 \sqcup_{\mathcal{A}^0} \mathcal{A}^2$, where $\mathcal{A}^0$ is just the generic point. The first arrow $R \to \mathcal{A}^2$

is the identity on both copies of $\mathcal{A}^2$. The second arrow is the identity on the first copy and the flip on the second. The two arrows coincide on $\mathcal{A}^0$ since it is just a point! That really makes it hard to draw!

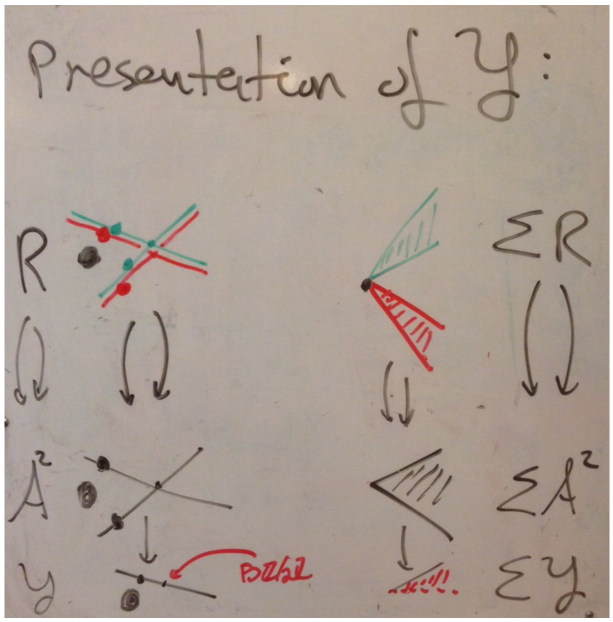


FIGURE 1. The Artin fan $\mathcal{Y}$ and its complex. That should be just $\mathbb{Z}/2\mathbb{Z}$ in red on the bottom.

In the survey the notation $\mathcal{Y} = \mathcal{A}^{[2]}$ is used. This is the image of $\mathcal{X}$ in Log - it has a unique 0-dimensional point, a unique copy of $\mathcal{B}\mathbb{G}_m$, and a codimension-2 point $[\mathcal{B}\mathbb{G}_m/(\mathbb{Z}/2\mathbb{Z})]$. The map to Log is an open embedding.

3. THE ARTIN FAN $\widetilde{\mathcal{Y}}$

Let's describe its blowing up $\widetilde{\mathcal{Y}}$. Write $\mathcal{B}$ be the blowing up of $\mathcal{A}^2$ at its closed gerbe $\mathcal{B}\mathbb{G}_m^2$, and $BR = \mathcal{B} \sqcup_{\mathcal{A}^0} \mathcal{B}$. The first induced arrow $BR \to \mathcal{B}$ is the identity on each open $\mathcal{B}$, whereas the second is the identity on the first copy and flips the coordinates on the second, and again we can glue uniquely because $\mathcal{A}^0$ is a point. Then $\widetilde{\mathcal{Y}} = [BR \rightrightarrows \mathcal{B}]$.

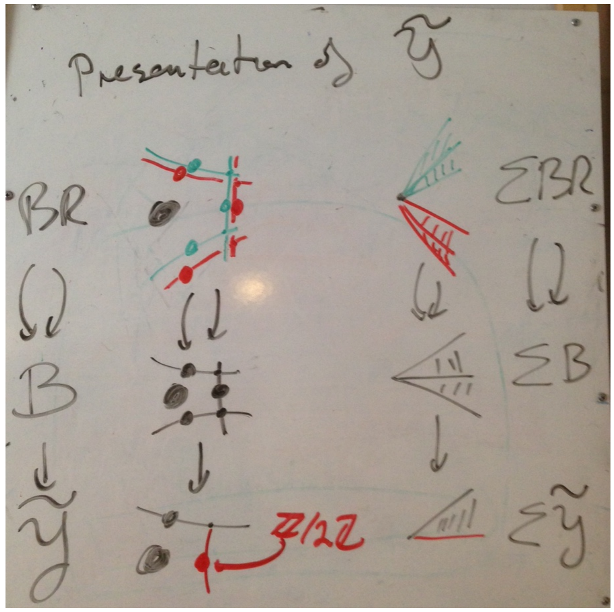


FIGURE 2. The Artin fan $\widetilde{\mathcal{Y}}$ and its complex

4. The Artin fan $\mathcal{Z}$

Since $\mathcal{Z}$ is the Artin fan of $\tilde{Y}$, which is smooth with two divisors intersecting transversally along an irreducible stratum, we know that $\mathcal{Z} = \mathcal{A}^2$. But let's explain how the maps work.

First, the map $\mathcal{B} \to \mathcal{Z} = \mathcal{A}^2$ is associated to the two divisors D and E on $\mathcal{B}$. Here D is the strict transform of the two coordinate axes and E is the exceptional. Let $U = \mathcal{B} \smallsetminus D \subset \mathcal{B}$. Here is an important observation: $U \simeq \mathcal{A}^1$.

Second, write $R_{\mathcal{Z}} = \mathcal{B} \times_{\mathcal{Z}} \mathcal{B}$. Then $R_{\mathcal{Z}} = \mathcal{B} \sqcup_U \mathcal{B}$. We have an evident map $R_{\mathcal{Z}} \to \mathcal{B}$ which is the identity on each of the two opens $\mathcal{B}$. We also have a map $R_{\mathcal{Z}} \to \mathcal{B}$ which is the identity on the first $\mathcal{B}$ and the flip on the second. The two maps coincide on $U = \mathcal{A}^1$ since they are both the unique automorphism of $\mathcal{A}^1$, namely the identity!

By construction $\mathcal{Z} = [R_{\mathcal{Z}} \rightrightarrows \mathcal{B}]$.

Note that there is a morphism $BR \to R_{\mathcal{Z}}$ obtained by the identity on each copy of $\mathcal{B}$. A moment thought will

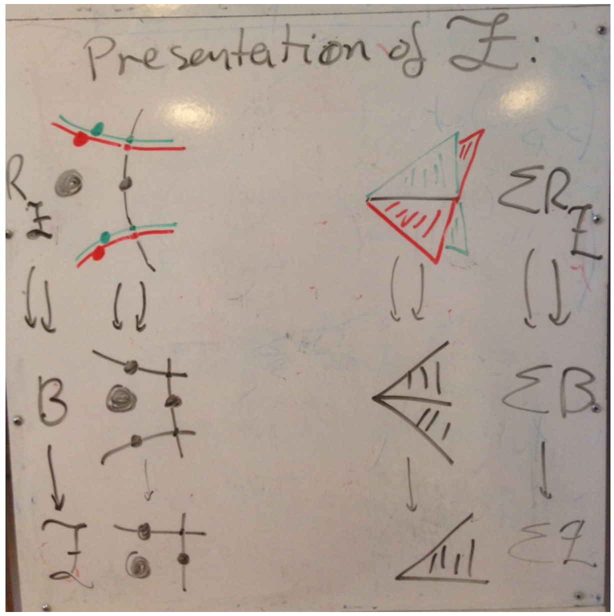


FIGURE 3. The Artin fan $\mathcal{Z}$ and its complex

hopefully convince you that this gives rise to a morphism of groupoids $[BR \rightrightarrows \mathcal{B}] \to [R_{\mathcal{Z}} \rightrightarrows \mathcal{B}]$. In other words we have a morphism $\widetilde{\mathcal{Y}} \to \mathcal{Z}$. One needs to check that it fits in the commutative diagram above, but things are sufficiently canonical that it works nicely.

We note that there is no map $\mathcal{Z} \dashrightarrow \mathcal{Y}$ fitting in the diagram since the two maps $\mathcal{U} \to R$ have different images of the closed gerbe.