Addendum to
Logarithmic Geometry and Moduli

Tropical Geometry and Moduli Spaces
ICM 2018 Satellite
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Differentials

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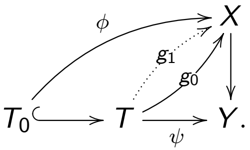
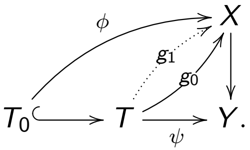


Differentials (continued)



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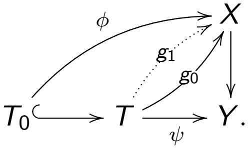
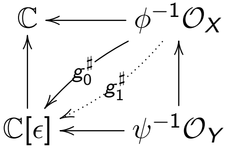


This translates to a diagram of groups



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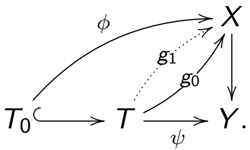
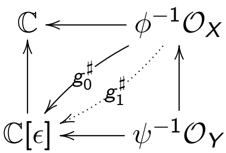
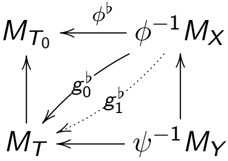


This translates to a diagram of groups and a diagram of monoids





Differentials (continued)

$$\begin{array}{ccc} \mathbb{C} & \longleftarrow & \phi^{-1}\mathcal{O}_X \\ \uparrow & \swarrow^{g_0^\sharp} \; \swarrow^{g_1^\sharp} & \uparrow \\ \mathbb{C}[\epsilon] & \longleftarrow & \psi^{-1}\mathcal{O}_Y \end{array} \qquad \begin{array}{ccc} M_{T_0} & \xleftarrow{\phi^\flat} & \phi^{-1}M_X \\ \uparrow & \swarrow^{g_0^\flat} \; \swarrow^{g_1^\flat} & \uparrow \\ M_T & \longleftarrow & \psi^{-1}M_Y \end{array}$$

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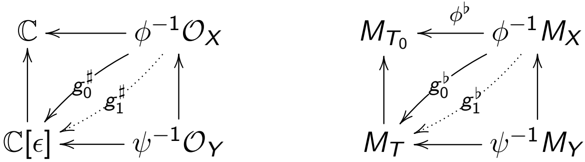

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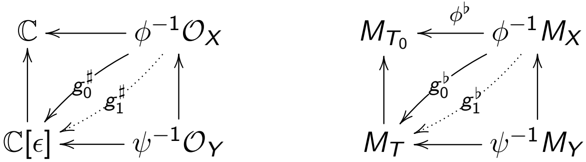

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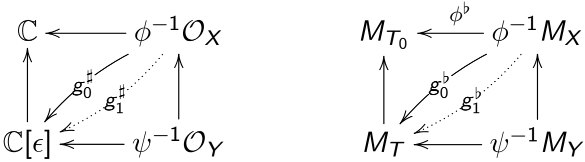

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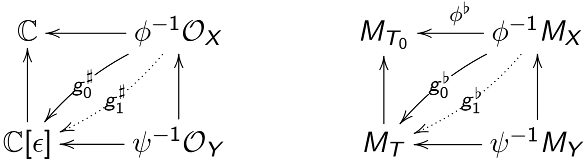

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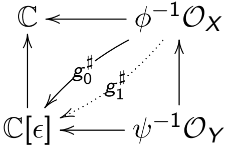

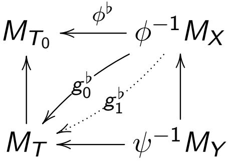

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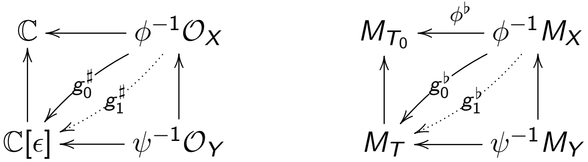

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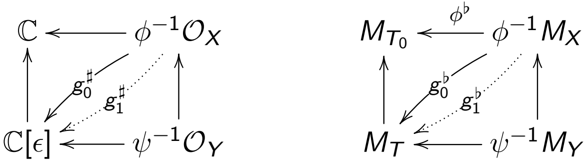

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Namely $D(m) =$ "$g_1^\flat(m) - g_0^\flat(m)$" $\in J$.

Key properties:

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A logarithmic derivation:

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The universal logarithmic derivation takes values in

$$\Omega^1_{X/Y} = \left(\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O} \otimes_{\mathbb{Z}} M^{\mathrm{gp}})\right)/\text{relations}$$

Prestable curves

A *prestable n-marked curve C/S* is a flat, proper morphism with connected reduced fibers of dimension 1, along with disjoint sections $s_i : S \to C$ for $i = 1, \ldots, n$ in the smooth locus of C/S. We require all fibers have at most nodes as singularities.

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We denote by p_i the images of s_i.

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A prestable curve C/S is *stable* if for every geometric fiber the automorphism group $\mathrm{Aut}(C_0, p_1, \ldots, p_n)$ is finite.

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Proposition

All three definitions coincide

Moduli of stable curves

Theorem (Deligne–Mumford–Knudsen)

Stable curves form a proper, smooth Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}$ *over* $\mathbb{Z}$ *with projective coarse moduli space. The universal curve is* $\overline{\mathcal{M}}_{g,n+1}$*.*

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These work sometimes, but often replaced by “coarse moduli spaces”, a compromise
The reason is that moduli functors dance around the problem instead of facing it directly - the problem of automorphisms.

Moduli as Categories

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With the forgetful functor $(X \to S) \mapsto S$ this is a category fibered in groupoids.

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This makes the category $\mathcal{M}$ into a stack.

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A stack is algebraic essentially if it admits a smooth map from a scheme.
For instance, $\overline{\mathcal{M}}_g$ is an algebraic stack since it has a smooth map from the Hilbert scheme of 3-canonically embedded stable curve.

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Often the crucial criterion is the existence of versal deformation spaces.

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- in fibers $\mathrm{Aut}(C_s \to X, s_i)$ is finite.

Gromov–Witten theory

We want to count curves on X of class $\beta \in H_2(X, \mathbb{Z})$ meeting cycles $\Gamma_1, \ldots, \Gamma_n$ corresponding to cohomology classes γ_i. For instance: lines through p_1, p_2.

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and one defines the *Gromov–Witten invariants*

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,\beta} = \int_{[M]^{\mathrm{vir}}} e_1^* \gamma_1 \cdots e_n^* \gamma_n.$$

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In order to define this one uses a *perfect obstruction theory*. In this case it is given by $R^\bullet \pi_* f^* T_X$, represented by a 2-term complex on S.