THE GROSS-SIEBERT CATEGORY OF A LOG SMOOTH FAMILY IS STABLE UNDER FINITE COLIMITS

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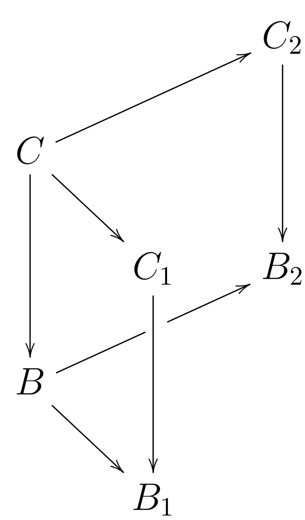
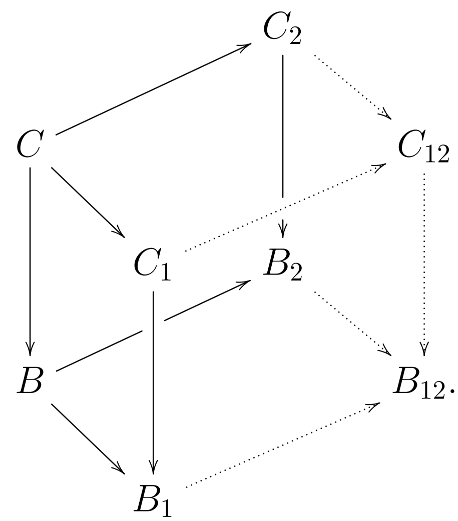
1. The diagram of log schemes

We consider a log smooth saturated $C \to B$, where $B = \operatorname{Spec}(\overline{M} \to \mathbb{C})$ and $C = (\underline{C}, N)$ where $\overline{M} \to \overline{N}$ is saturated with relative characteristic Q.

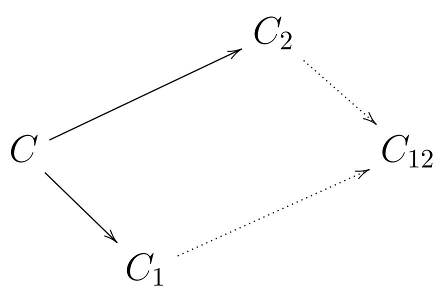
Theorem 1.1. *Assume we have a cartesian diagram*



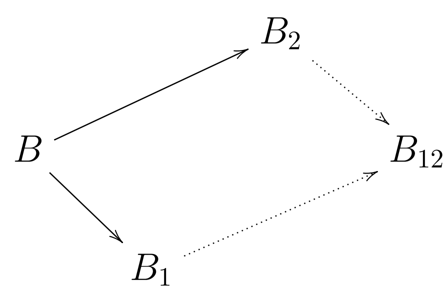
with log smooth saturated vertical arrows with $\underline{B}_i = \operatorname{Spec}\mathbb{C}$ *and relative characteristic* Q*. Then the diagram can be completed to a diagram with log smooth saturated vertical arrows as follows:*



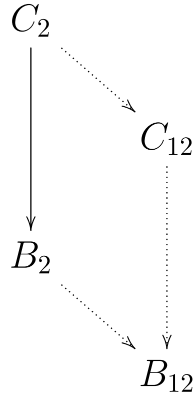
where the squares



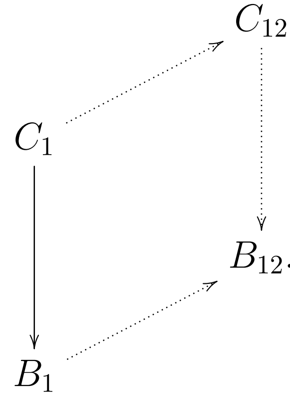
and



are co-cartesian and the squares



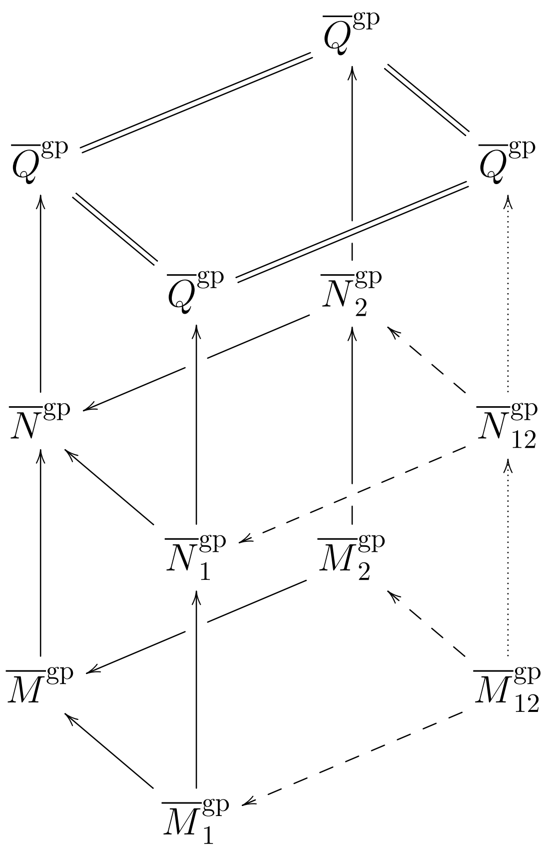
and



are cartesian.

2. The diagram of charactertistic groups

First we translate the original diagram to a diagram of characteristic groups



$$\overline{N}_{12}^{\mathrm{gp}} := \overline{N}_1^{\mathrm{gp}} \times_{\overline{N}^{\mathrm{gp}}} \overline{N}_2^{\mathrm{gp}}$$

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so all the solid vertical sequences are short exact sequences.[1] ←1

Lemma 2.1. *The dotted vertical sequence is a short exact sequence.*

Proof. Applying the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{M}_{12}^{\mathrm{gp}} & \longrightarrow & \overline{M}_1^{\mathrm{gp}} \oplus \overline{M}_2^{\mathrm{gp}} & \longrightarrow & \overline{M}^{\mathrm{gp}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{N}_{12}^{\mathrm{gp}} & \longrightarrow & \overline{N}_1^{\mathrm{gp}} \oplus \overline{N}_2^{\mathrm{gp}} & \longrightarrow & \overline{N}^{\mathrm{gp}} & \longrightarrow & 0 \end{array}$$

[1](Dan) explain why saturated implies injective $M \to N$

we get the vanishing of Ker and bottom exact sequence of cokernels:

$$
\begin{array}{ccccccccc}
 & & Ker & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{M}_{12}^{\mathrm{gp}} & \longrightarrow & \overline{M}_1^{\mathrm{gp}} \oplus \overline{M}_2^{\mathrm{gp}} & \longrightarrow & \overline{M}^{\mathrm{gp}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{N}_{12}^{\mathrm{gp}} & \longrightarrow & \overline{N}_1^{\mathrm{gp}} \oplus \overline{M}_2^{\mathrm{gp}} & \longrightarrow & \overline{N}^{\mathrm{gp}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Coker & \longrightarrow & Q^{\mathrm{gp}} \oplus Q^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} & \longrightarrow & 0
\end{array}
$$

as needed. ♠

Lemma 2.2. *The squares*

$$
\begin{array}{ccc}
\overline{N}^{\mathrm{gp}} & \longleftarrow & \overline{N}_i^{\mathrm{gp}} \\
\uparrow & & \uparrow \\
\overline{M}^{\mathrm{gp}} & \longleftarrow & \overline{M}_i^{\mathrm{gp}}
\end{array}
\quad \text{as well as} \quad
\begin{array}{ccc}
\overline{N}_i^{\mathrm{gp}} & \longleftarrow & \overline{N}_{12}^{\mathrm{gp}} \\
\uparrow & & \uparrow \\
\overline{M}_i^{\mathrm{gp}} & \longleftarrow & \overline{M}_{12}^{\mathrm{gp}}
\end{array}
\tag{2.0.3}
$$

are cartesian and co-cartesian.

Proof. For the first pair of squares, the snake lemma for

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{M}_i^{\mathrm{gp}} & \longrightarrow & \overline{N}_i^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \overline{M}^{\mathrm{gp}} & \longrightarrow & \overline{N}^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} & \longrightarrow & 0
\end{array}
$$

gives

$$
\begin{array}{ccccccccc}
 & & Ker & = & Ker & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \overline{M}_i^{\mathrm{gp}} & \longrightarrow & \overline{N}_i^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \overline{M}^{\mathrm{gp}} & \longrightarrow & \overline{N}^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} & \longrightarrow & 0 \\
 & & \rotatebox{-90}{$\twoheadrightarrow$} & & \rotatebox{-90}{$\twoheadrightarrow$} & & & & \\
 & & Coker & = & Coker & & & &
\end{array}
$$

which means the diagram is cartesian and cocartesian.

Using the previous lemma, we have a diagram of exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{M}_{12}^{\mathrm{gp}} & \longrightarrow & \overline{N}_{12}^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \overline{M}_i^{\mathrm{gp}} & \longrightarrow & \overline{N}_i^{\mathrm{gp}} & \longrightarrow & Q^{\mathrm{gp}} & \longrightarrow & 0
\end{array}
$$

and the snake lemma argument applies to the second pair of squares as well. ♠

3. Things to do: checking that log smooth saturated maps behave well

Deduce same result for characteristic monoids

Deduce same result for log structures and proof of Theorem 1.1

Define extended category in which limits can exist. Check that log smooth morphisms behave well under all coproducts hence under all colmits.

4. Constancy

Lemma 4.1. *Let* $X = \underline{X} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}(\overline{M} \to \mathbb{Z})$ *and* $X \to X'$ *a morphism of coherent log structures such that* $\underline{X} \to \underline{X}'$ *is an isomorphism. Then* $\overline{M}'_X$ *is locally constant.*

Proof. Let $P \to M'_X$ be a chart over some étale $U \to \underline{X}$ which is characteristic at some $x \in \underline{X}$. The homomorphism $P \to \overline{M}_X$ is sharp, and we have $\overline{M}_X \to \mathcal{O}_X$ sending $m \mapsto 0^m$, so $p \mapsto 0^p$ as well, hence $(U, M_X) = U \times \operatorname{Spec}(P \to \mathbb{Z})$, and $\overline{M}'_X$ is constant on U. ♠

Let (S, M_S) be a logarithmic scheme. For a morphism $f : T \to S$ and a logarithmic morphism $(T, f^*M_S) \to (T, M_T)$ define a log structure $f_!^{M_S} M_T := M_S \times_{f_* f^* M_S} f_* M_T$ on S, so that the following diagram is commutative and co-cartesian:

$$\begin{array}{ccc} (T, f^*M_S) & \longrightarrow & (S, M_S) \\ \downarrow & & \downarrow \\ (T, M_T) & \longrightarrow & (S, f_!^{M_S} M_T). \end{array}$$

This is a left adjoint to $f^* : \mathbf{LogStr}_S/M_S \to \mathbf{LogStr}_T/(f^*M_S)$.

Lemma 4.2. *Suppose the log structures are constant:* $(S, M^i) = S \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}(\overline{M}^i \to \mathbb{Z})$ *and let* $(S, M^1) \to (S, M^2)$ *a morphism. Suppose* S *is integral and* $j : \eta \to S$ *its generic point. Then the morphism* $j_!^{M^1} j^* M^2 \to M^2$[2] *is an isomorphism.*

2→

Proof. In this case $M^i \to j_* j^* M^i$ is an isomorphism. Since $M^1 \to j_* j^* M^1$ an isomorphism we have $j_!^{M^1} j^* M^2 = j_* j^* M^2$, and since $M^2 \to j_* j^* M^2$ an isomorphism we have $M^2 \to j_!^{M_1} j^* M^2$ an isomorphism. ♠

[2](Dan) I hope I got this direction right

5. Constructibility of minimality given stability under generization

Fix $\xi^1/(S, M^1) \to \xi^2/(S, M^2)$. Let $s \in S$ and assume $\xi^1_s/(s, M^1_s) \to \xi^2_s/(s, M^2_s)$ is a minimal object for ξ^1_s. Suppose η specializes to s. In the other note one (JW) showed that

Lemma 5.1. *$\xi^1_\eta/(\eta, M^1_\eta) \to \xi^2_\eta/(\eta, M^2_\eta)$ is minimal as well.*

We assume known that coherent minimal objects exist at all points.

Lemma 5.2. *Assume the log structures are constant: $(S, M^i) = S \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}(\overline{M}^i \to \mathbb{Z})$. Let $s \in S$ and assume*

$$\xi^1_s/(s, M^1_s) \to \xi^2_s/(s, M^2_s)$$

is a minimal object. Then

$$\xi^1_t/(t, M^1_t) \to \xi^2_t/(t, M^2_t)$$

is minimal for all $t \in S$.

Proof. Fix t, and

$$\xi^1_t/(t, M^1_t) \to \xi^3_t/(t, M^3_t)$$

minimal.

Minimality of ξ^3_t gives a morphism

$$\xi^2_t/(t, M^2_t) \to \xi^3_t/(t, M^3_t).$$

It should follow by homogeneity / the stack property that this lifts to

$$\xi^1_V/(V, M^1_V) \to \xi^2_V/(V, M^2_V) \to \xi^3_V/(V, M^3_V)$$

on some étale neighborhood $V \to U$ of t and η. [3] ←3

By Lemma 5.1 we have that both

$$\xi^1_\eta/(\eta, M^1_\eta) \to \xi^2_\eta/(\eta, M^2_\eta)$$

[3](Dan) Another Lemma to prove? Maybe it is in the other note?

and

$$\xi^1_{\eta_V}/(\eta_V, M^1_{\eta_V}) \to \xi^3_{\eta_V}/(\eta_V, M^3_{\eta_V})$$

are minimal, so the morphism

$$\xi^2_{\eta_V}/(\eta_V, M^2_{\eta_V}) \to \xi^3_{\eta_V}/(\eta_V, M^3_{\eta_V})$$

is an isomorphism.

By Lemma 4.1 we have that (V, M^3_V) is locally constant. Passing to an étale neighborhood $V' \to V$ of t, η with generic point $j : \eta' \hookrightarrow V'$ we have that $(V', M^3_{V'})$ is constant. From above we have that

$$\xi^2_{\eta'}/(\eta', M^2_{\eta'}) \to \xi^3_{\eta'}/(\eta', M^3_{\eta'})$$

is an isomorphism. We denote its inverse ϕ.

By definition we have a diagram

$$\begin{array}{ccc} (\eta', M^3_{\eta'}) & \longrightarrow & (V', M^3_{V'}) \\ \downarrow{\scriptstyle \phi} & & \downarrow \\ (\eta', M^2_{\eta'}) & \longrightarrow & (V', j_!^{M^3_{V'}} M^2_{\eta'}). \end{array}$$

By homogeneity there is a compatible object on $(V', j_!^{M^3_{V'}} M^2_{\eta'})$. Lemma 4.2 applies, so the two arrows

$$j_!^{M^3} M^2_{\eta'} \quad = \quad j_!^{M^3} j^* M^2_{V'} \longrightarrow M^2_{V'}$$

$$j_!^{M^3} M^2_{\eta'} \quad \overset{\phi}{\simeq} \quad j_!^{M^3} j^* M^3_{V'} \longrightarrow M^3_{V'}$$

are isomorphisms. This gives $M^2_{V'} \simeq M^3_{V'}$ and in particular $M^2_t \simeq M^3_t$ as needed. ♠

Corollary 5.3. *The locus where $\xi^1/(S, M^1) \to \xi^2/(S, M^2)$ is minimal is constructible.*

Proof. It is a union of strata of (S, M^2). ♠

Theorem 5.4. *The locus where* $\xi^1/(S, M^1) \to \xi^2/(S, M^2)$ *is minimal is open.*

Proof. It is constructible and stable under generization. ♠

6. Things to do: representability of logarithmic moduli

State representability criteria for log moduli.
Describe representing object of the GS category
Check that it extends from points to schemes