DEGENERATION THEORY FOR GROMOV-WITTEN INVARIANTS

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Abstract. This is a notes for the degeneration formula in a general setting.

Contents

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1. Introduction

2. Log maps punctured along markings

2.1. **Punctured log maps.** Denote by $\pi : X \to B$ a flat, proper, and integral log maps between two fs log schemes.

def:puncture-map

Definition 2.1.1. A *punctured log map* to X/B over an fs log scheme with genus g and n marked points is given by the data $f = (\underline{f}, f^\flat, C/S)$ such that

(1) $\phi : C \to S$ is a family of genus g, n-marked log curves.
(2) $\underline{f}$ is a family of usual pre-stable maps over $\underline{S}$ fits in the following commutative diagram:

$$\begin{tikzcd}
\underline{C} \arrow[r, "\underline{f}"] \arrow[d] & \underline{X} \arrow[d] \\
\underline{S} \arrow[r] & \underline{B}.
\end{tikzcd}$$

(3) $f^\flat : (\underline{f}^*\mathcal{M}_X)_{C^\circ} \to \mathcal{M}_{C^\circ}$ is a map of fs log structures, where $C^\circ \subset C$ is the open curve obtained by removing the marked points of C/B.
(4) The pair $(\underline{f}, f^\flat)_{C^\circ}$ defines a family of log maps to X/B:

$$\begin{tikzcd}
C^\circ \arrow[r, "(\underline{f}{,}f^\flat)"] \arrow[d] & X \arrow[d] \\
S \arrow[r] & B.
\end{tikzcd}$$

A punctured map is called *stable*, if the map $\underline{f}$ is stable in the usual sense. For simplicity, we will write $f : C/S \to X/B$ for a punctured maps, when there is no danger of confusion.

As before, we view the category of punctured maps as a category fibered over the category of fs log schemes.

def:puncture-arrow

Definition 2.1.2. An *arrow* between two punctured maps $f_1 : C_1/S_1 \to X/B$ and $f_2 : C_2/S_2 \to X/B$ is given by a commutative diagram of log schemes:

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[d] & C_2 \arrow[d] \\
S_1 \arrow[r] & S_2
\end{tikzcd}$$

such that the following diagram of log schemes commutes:

$$\begin{tikzcd}
C_1^\circ \arrow[rr] \arrow[dr, "f_1"'] \arrow[ddd] & & C_2^\circ \arrow[dl, "f_2"] \arrow[ddd] \\
& X \arrow[ddd] & \\
& & \\
S_1 \arrow[rr] \arrow[dr] & & S_2 \arrow[dl] \\
& B &
\end{tikzcd}$$

Remark 2.1.3. If we require f extends to all marked points, then this is the log map defined as before.

2.2. **Contact orders along the punctured markings.** We now discuss the behavior of punctured maps near the marked points. They behaves very much like the log maps except we allowing poles here. For simplicity, we first assume that B is a point with trivial log structures. Thus the target of the punctured map is a fs log scheme X.

Fix a punctured map $f : C/S \to X$ of genus g, with n marked points. We assume that the underlying base $\underline{S}$ is a geometric point. Away from the marked points, everything beharves the same as the case of log maps.

Consider a marking point $\Sigma \in C$. Choose a local coordinate σ in a neighborhood U of Σ, which vanishes at Σ. Denote by $\log\sigma \in \mathcal{M}_C$ a local section in U. By abuse of notations, we will identify $\log\sigma$ with its image in the characteristics, when there is no confusion. We are allowed to shrink the neighborhood U when it is necessary.

Write $U^\circ = U \smallsetminus \{\Sigma\}$. We may assume that U contains only one marked points. For any sections $\delta \in (f^*\mathcal{M}_X)_{U^\circ}$, its image under f is of the form

`equ:along-marking`

$$f^\flat(\delta) = e + c_\Sigma(\delta)\cdot\log\sigma + u \tag{2.2.1}$$

where $e \in \mathcal{M}_S$ is a section from the base log structure, $u \in \mathcal{O}_U^*$ is an invertible section, and $c(\delta) \in \mathbb{Z}$ is an integer determined uniquely by δ. Thus, we have

`lem:contact-order`

Lemma 2.2.1. *There is a map of monoids*

$$c_\Sigma : (f^*\overline{\mathcal{M}}_X)_\Sigma \to \mathbb{Z}$$

defined by (2.2.1). Here $\mathbb{Z}$ *is the group generated by* $\log\sigma$ *by the identification* $1 \mapsto \log\sigma$*.*

`def:contact-order`

Definition 2.2.2. The map of monoids c_Σ in Lemma 2.2.1 is called the *contact order* of the marked point Σ.

`lem:extend-to-markings`

Lemma 2.2.3. *Using the notations as above, if the image of* c_Σ *is in* $\mathbb{Z}_{\geq 0}$*, then the punctured map* f *can be extended to the marking* Σ *uniquely. In* *this case, the map c_Σ is called* positive. *It induces the contact order along the marked points in the sense of log maps.*

Proof. This follows directly from (2.2.1). ♠

Remark 2.2.4. Given a punctured map f with positive contact order maps along all marked points, is equivalent to given a log map with the induced contact orders long the marked points.

There is another easy but important observation:

`lem:force-degeneration`

Lemma 2.2.5. *With the notations as above, if $f(\Sigma) \in (\alpha(\delta) = 0)$, and $c_\Sigma(\delta) < 0$, then the section e in (2.2.1) is not invertible.*

Proof. Assume on the contrary that e is a unit. Then $\alpha(\delta)$ pulls back to a rational function on U with poles at Σ. This contradicts the condition that $f(\Sigma) \in (\alpha(\delta) = 0)$. ♠

`ss:contact-order`

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2.3. **Contact orders.** We next introduce some necessary notations and results for the use of contact order maps. Fix a connected fs log scheme X. For any geometric point $x \in X$ with the pull-back log structure[1], we are interested in the set of maps of monoids:

$$\mathcal{H}_x := \mathcal{H}om_{Mon}(\overline{\mathcal{M}}_x, \mathbb{Z}).$$

Definition 2.3.1. Elements in $\mathcal{H}_x$ are called *contact orders at x.*

Consider contact orders c_1 and c_2 at two different strict points x_1 and x_2 of X respectively. Further assume that there is a chart $U \to X$ containing both x_1 and x_2, with a chart of log structures

$$\beta : M \to \mathcal{M}_U := \mathcal{M}_X|_U.$$

This induces two maps of monoids:

$$g_1 : M \to \overline{\mathcal{M}}_{x_1} \to P$$

and

$$g_2 : M \to \overline{\mathcal{M}}_{x_2} \to P.$$

`tact-order-compatible`

Definition 2.3.2. Two contact orders c_1 and c_2 is called *compatible* if there is an étale chart (U, β) as above such that $g_1 = g_2$. Two contact orders c and c' is called *equivalent* if there exists a finite sequence of contact orders $c = c_1, \cdots, c' = c_k$ such that c_i is compatible with c_{i+1} for $i = 1, \cdots, k-1$.

Denote by $Con(X)$ the set of equivalence class of contact orders.

Remark 2.3.3. In fact, we can sheafify the family of contact orders as follows. For any $\underline{X}$-scheme $\underline{S}$, denote by S the log scheme over $\underline{S}$ with the pulled back log structure from X. For any $c \in Con(X)$, denote by $\mathcal{C}on_c(S) \subset Hom_S(\overline{\mathcal{M}}_S, \mathbb{Z})$ the subset of maps sheaf of monoids which is compatible with c. We should be able to check that $\mathcal{C}on_c$ is a sheaf over $\underline{X}$ with étale topology. Question is under what assumption, is this sheaf $\mathcal{C}on_c$ of finite type? This should equivalent to the boundedness problem.

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[1] (Qile) I call it strict point in X.

Definition 2.3.4. For any $c \in Con(X)$, denote by

$$Supp(c) := \{x \in X \mid c_x \neq 0\},$$

and call it the *support of* c. Clearly, $Supp(c)$ is a closed subset of X under the Zarisky topology. We call c *connected*, if its support is connected.

The following is obvious.

Lemma 2.3.5. *Any element $c \in Con(X)$ can be expressed as the unique linear combination of connected elements in $Con(X)$.*

Definition 2.3.6. An element $c \in Con(X)$ is called a *contact order along* X, or simply a *contact order* if it is connected.

Lemma 2.3.7. *Any contact order c is determined uniquely by its fiber at any point of $Supp(c)$.*

Proof. This follows from Definition 2.3.2. ♠

Consider a puctured map $f : C/S \to X$ over a geometric point $\underline{S}$. Then any marked point $\Sigma \in C$ induces a contact order $c_\Sigma \in Con(X)$.

`prop:contact-open`

Proposition 2.3.8. *Consider a puctured map $f : C/S \to X$ over a connected fs log scheme S. Then for any marking Σ, the contact order c_Σ as in Definition 2.2.2 at each geometric point of S induces the same contact order in $Con(X)$. By abuse of notation, we denote this contact order by c_Σ.*

Proof. It is enough to show that punctured maps with a fixed contact order along Σ forms an open family. Fix a strict point $s \in S$, and pick a neighborhood V of s. Choose another open neighborhood U of $\Sigma_s \in C$. Shrink U and V if necessary, we may assume that

1. $V \subset \phi(U)$;
2. There is a chart $\beta : M := f^*(\overline{\mathcal{M}}_X)_{\Sigma_s} \to f^*(\mathcal{M}_X)$ over V.

Thus, for any $t \in V$, the fiber c_{Σ_t} is induced by the composition

$$M \to f^*(\overline{\mathcal{M}}_X) \to \mathbb{Z}.$$

Now the statement follows from Definition 2.3.2. ♠

`sec:combinatorics`
`ss:marked-graph`

3. Combinatorics preparation

3.1. Marked graph and the associated monoid.

We first introduce the following conception:

`def:generalization`

Definition 3.1.1. Consider a map of two sharp and fine monoids $h : P \to Q$. Let $F \subset P^{gp}$ be the subgroup generated by $h^{-1}(0)$, and $F^{-1}P \subset P^{gp}$ the submonoid generated by F and P. The map f is called is called *generalizing* if it induces an isomorphism $F^{-1}P/F \to Q$. We also say Q is a *generalization* of P.

`def:marked-graph`

Definition 3.1.2. A *marked graph* denoted by G, is a connected finite graph $\underline{G}$, consisting of a set of vertices $V(G)$, edges $E(G)$, and legs $L(G)$, along with the following data:

(1) For each $v \in V(G)$, we associate a toric monoid P_v. (This denotes the stratum of log structures that v sits in.)
(2) For each $l \in E(G)$ joining v_1 and v_2, we associate a toric monoid P_l, and an orientation on l paired with a nodal contact order map $c_l : P_l \to \mathbb{Z}$, and two generalizing maps of monoids $\psi_i : P_l \to P_{v_i}$ for $i = 1, 2$, such that for any nontrivial element $\delta \in P_l$, we have either $\psi_1(\delta) \neq 0$ or $\psi_2(\delta) \neq 0$.
(3) Legs are unbounded edges attached to vertices. For each $l \in L(G)$, we associate a contact order map $c_l : P_l \to \mathbb{Z}$.

We further require that loops of G have only trivial nodal contact orders.

Definition 3.1.3. Given a marked graph G as above, a *flop* of an edge $l \in E(G)$ is a modification of l in G by reversing the orientation and replacy c_l by $-c_l$. Two marked graph G_1 and G_2 is called *equivalent* if they are differ by a sequence of flops of their edges.

For a marked graph G, we consider the following monoid:

$$M := \Big(\sum_{v \in V(G)} P_v \oplus \sum_{l \in E(G)} \mathbb{N}_l \Big) / \sim$$

where $\mathbb{N}_l = < e_l > \cong \mathbb{N}$ is the rank one free monoid generated by the element e_l associated to the edge l. Here the equivalence relation $\sim$ is given by

equ:degeneracy-jump1
$$\psi_1(\delta) + c_l(\delta) \cdot e_l = \psi_2(\delta) \qquad \text{if } c_l(\delta) \geq 0 \tag{3.1.1}$$

equ:degeneracy-jump2
$$\psi_1(\delta) = \psi_2(\delta) - c_l(\delta) \cdot e_l \qquad \text{if } c_l(\delta) \leq 0 \tag{3.1.2}$$

where l is an edge oriented from v_1 to v_2, and $\delta \in P_l$ is an arbitrary section. Denote by $\overline{\mathcal{M}}(G)$ the monoid obtained by taking the saturation of M, and quotient its torsion.

def:associated-monoid **Definition 3.1.4.** The fs monoid $\overline{\mathcal{M}}(G)$ constructed above is called *the associated monoid* of the marked graph G.

Remark 3.1.5. The formation of $\overline{\mathcal{M}}(G)$ does not depend on legs.

The following is a direct result of (3.1.1) and (3.1.2):

Lemma 3.1.6. *The associated monoid $\overline{\mathcal{M}}(G)$ only depends on the equivalent class of G.*

For a marked graph G, we have the following canonical maps

$$\epsilon_l : \mathbb{N}_l \to \overline{\mathcal{M}}(G),$$

and

$$\chi_v : P_v \to \overline{\mathcal{M}}(G)$$

for any $v \in V(G)$ and $l \in E(G)$.

def:admissible **Definition 3.1.7.** A marked graph G is called *admissible* if $\ker \epsilon_l$ and $\ker \chi_v$ are all trivial for any $v \in V(G)$ and $l \in E(G)$. In the rest of this notes, a marked graph will be automatically admissible unless otherwise specified.

def:deg-graph

Definition 3.1.8. A *degenerate graph* G is a marked graph with the extra data:

(1) For each $l \in E(G)$, we associate an injective map of monoids $\mathbb{N} \to P_l$ such that the composition map $\mathbb{N} \to P_l \to \mathbb{Z}$ is zero. Note that this does change under flops of marked graphs.
(2) For each $v \in V(G)$, we associate an injective map of monoids $\mathbb{N} \to P_v$.

Furthermore, the above maps of monoids shoud be compatible with the generalizing map $P_l \to P_v$ if v is an end of l. We denote by e_G the generator of the copy of $\mathbb{N}$, and identify it with its image in $\overline{\mathcal{M}}(G)$.

prop:deg-graph

Proposition 3.1.9. *Consider a degenerate graph G, there is a canonical map $\pi_G : \mathbb{N} \to \overline{\mathcal{M}}(G)$, which is compatible with $\mathbb{N} \to P_v$ and $\mathbb{N} \to P_l$ for any vertex v and edge l. Furthermore, this map does not depend on flops of the degenerated graph.*

Proof. This follows from the fact that the composition $\mathbb{N} \to P_l \to \mathbb{Z}$ is zero. ♠

From now on, we will use the terminology marked graph for the equivalence classes of marked graph, and a representative for a choice of orientations of all the edges in a fixed equivalence. This will be the same for degenerated graphs.

ss:contraction

3.2. **Contraction of graphs.**

We now focus on operation of different marked or degenerated graphs.

def:graph-arrow

Definition 3.2.1. Consider two marked graphs G_1 and G_2. A *contraction* $\phi : G_1 \to G_2$ is given by the following data:

(1) An injection of edges $\phi_E : E(G_2) \hookrightarrow E(G_1)$, such that for any $l \in E(G_2)$, we associate a generalizing map of monoids
$$\phi(l) : P_{\phi_E(l)} \to P_l,$$
which fits in a commutative diagram possibly after a flop of the edge
$$\begin{tikzcd} P_{\phi_E(l)} \arrow[rr] \arrow[dr, "c_{\phi_E(l)}"'] & & P_l \arrow[dl, "c_l"] \\ & \mathbb{Z} & \end{tikzcd}$$
(2) A surjection of vertices $\phi_V : V(G_1) \to V(G_2)$, such that $\phi_V(v_1) = \phi_V(v_2)$ if and only if there is a chain of (not necessarily oriented) edges $l_1 \cdots l_m$ joining v_1 and v_2 with $l_i \notin \phi_E(E(G_2))$, for $i = 1, \cdots, m$, and for any $v \in V(G_1)$, we associate a generalizing map of monoids
$$\phi(v) : P_v \to P_{\phi_V(v)}.$$
(3) For each $v \in V(G_2)$, the legs attached to v are one-to-one corresponds to legs attached to $\phi_V^{-1}(v)$ with the same contact orders.

These data shoud be compatible in the following sense. For any $l \in E(G_2)$ oriented from v_1 to v_2 in G_2, we have the image edge $\phi_E(l) \in E(G_1)$ oriented from v_1' to v_2' in G_1. By (2), we have $\phi_V(v_i') = v_i$ for $i = 1, 2$. The we further require the following diagram is commutative:

$$\begin{tikzcd}
& P_{v_1'} \arrow[rr] & & P_{v_1} \\
P_{\phi_E(l)} \arrow[ur] \arrow[dr] \arrow[rr] & & P_l \arrow[ur] \arrow[dr] & \\
& P_{v_2'} \arrow[rr] & & P_{v_2}.
\end{tikzcd}$$

Clearly the identity of a marked graph is a contraction.

Definition 3.2.2. A contraction $\phi : G_1 \to G_2$ is called *admissible*, if both G_1 and G_2 are admissible. Unless otherwise specified, a contraction will automatically mean an admissible contraction.

`cons:contract-edge`

Construction 3.2.3. Consider a marked graph G and an edge $l \in E(G)$. We next construct a contraction $\phi : G \to G_l$ as follows. Denote by $F =< e_l >\subset \overline{\mathcal{M}}(G)^{gp}$ the subgroup generated by e_l. Write $\overline{\mathcal{M}} = F^{-1}\overline{\mathcal{M}}(G)/F$. (Verify that $\overline{\mathcal{M}}$ is a sharp fs monoid.) We thus have the following composition

$$c\epsilon_{l'} : \mathbb{N}_{l'} \to \overline{\mathcal{M}}(G) \to \overline{\mathcal{M}},$$

and

$$c\chi_v : P_v \to \overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}.$$

for any $l' \in E(G)$ and $v \in V(G)$.

1. We first contract edges $l' \in E(G)$ such that $c\epsilon = 0$, and identify the end points. This yields a new graph $\underline{G_l}$ with orientation of the edges defined by orientation of G. Clearly we have $c\epsilon_l = 0$.
2. For any $v' \in \underline{G_l}$, we pick a vertex $v \in V(G)$ such that $\phi(v) = v'$. Denote by $F_v =< \ker c\chi_v >\subset P_v^{gp}$ the subgroup generated by $\ker c\chi_v$. We associate the monoid $P_{v'} = F_v^{-1}P_v/F_v$. We check that this does not depend on the choice of v over v'.
3. For any $l'' \in E(G_l)$ joining v_1 and v_2, write $l' = phi(l') \in E(G)$, with $\psi_1' : P_{l'} \to P_{v_1}$ and $\psi_2' : P_{l'} \to P_{v_2}$ by (2). Denote by $F_{l''} =< \ker\psi_1' \cap \ker\psi_2' >\subset P_{l'}^{gp}$ the subgroup. We now associate $P_{l''} = F_{l''}^{-1}P_{l'}/F_{l''}$ with the natural maps $P_{l''} \to P_{v_i}$ for $i = 1, 2$.
4. Legs with their contact orders attached to each $v \in V(G_l)$ is defined in an obvious way.

The above construction yields a marked graph G_l with a contraction ϕ. One check that $\overline{\mathcal{M}}(G_l) = \overline{\mathcal{M}}$.

Definition 3.2.4. An edge of a marked graph is called *distinguished* if its contact order is non-trivial. A marked graph is called *distinguished* if it has no non-distinguished edge. In particular, a distinguished marked graph has no loops.

We first observe that

Lemma 3.2.5. *Consider an admissible marked graph G. For an nondistinguished edge l joining v_1 and v_2, we have the two associated map $\psi_1 : P_l \to P_{v_1}$ and $\psi_2 : P_l \to P_{v_2}$ to be isomorphic.*

Proof. Since the contact order of l is trivial, the generalizing maps ψ_i, (3.1.1) and (3.1.2) induces an isomorphism $P_{v_1} \to P_{v_2}$. Therefore $\ker \psi_i = 0$. This proves the statement. ♠

`prop:split-monoid`

Proposition 3.2.6. *For any marked graph G, there is a unique contraction $\phi_d : G \to G_d$ such that G_d is obtained by contracting non-distinguished edges in G with out changing orientations and those associated monoids. Furthermore, we have*

$$\overline{\mathcal{M}}(G) \cong \overline{\mathcal{M}}(G_d) \oplus \mathbb{N}^n$$

where n is the number of non-distinguished edges in G.

Proof. This follows from (3.1.1) and (3.1.2), and the previous lemma. ♠

`raction-generalization`

Proposition 3.2.7. *For any contraction $\psi : G \to G'$, the monoid $\overline{\mathcal{M}}(G')$ is a generalization of $\overline{\mathcal{M}}(G)$.*

Conversely, we have

`ralization-contraction`

Proposition 3.2.8. *Given a marked graph G, for any generalization $\overline{\mathcal{M}}$ of $\overline{\mathcal{M}}(G)$, there is a unique contraction $\psi : G \to G'$ with a canonical isomorphism $\overline{\mathcal{M}}(G') \cong \overline{\mathcal{M}}$.*

`p:contraction-monoids`

Proposition 3.2.9. *Consider a contraction of two degenerate graph $\psi : G \to G'$. Then we have a commutative diagram of monoids:*

$$\begin{tikzcd}
\overline{\mathcal{M}}(G) \arrow[rr] & & \overline{\mathcal{M}}(G') \\
& \mathbb{N} \arrow[ul, "\pi_G"] \arrow[ur, "\pi_{G'}"'] &
\end{tikzcd}$$

Proof. Notice that the contraction preserves (3.1.1) and (3.1.2). ♠

`ss:graph-splitting`

3.3. **Graph splitting.**

Definition 3.3.1. A degenerate graph G is called a *splitting graph* if $\overline{\mathcal{M}}(G) \cong \mathbb{N}$. A *splitting* of a degenerate graph G' is a contraction $G' \to G$ such that G is a splitting graph.

`cons:splitting`

Construction 3.3.2. Given a splitting $\phi : G \to G_0$, we construct a set of degenerate graphs $\{G_v\}_{v \in V(G_0)}$ as follows. We cut $\underline{G}$ along the edges in the subset $E(G_0) \subset E(G)$. After cutting the edges, we obtain a disconnected graph with its connected components one to one corresponds to the set $V(G_0)$. We denote the connected component associated to $v \in V(G_0)$ by $\underline{G}_v$. We then construct the degenerate graph G_v as follows:

(1) The underlying graph of G_v is $\underline{G}_v$.
(2) The data of monoids, generalizing maps of monoids and contact order of edges on G_v is obtained directly from those on G.
(3) The legs $L(G_v) = L(v) \cup R(v)$ is the union of two parts: the set L_v consists of legs in G attached to vertices in $V(G_v) \subset V(G)$ with the same contact orders; the set $R(v)$ consists of legs corresponds to the cut edges $E(G_0)$ attached to vertices in $V(G_v)$ with the compatible contact orders in G defined by taking the edges with the orientation starting from vertice in $V(G_v)$. We call $R(v)$ the set of *roots* of the splitting ϕ.

We call $(G_v, R(v))_{v\in V(G0)}$ the *gluing data* of the splitting $G \to G_0$.

Consider a splitting graph G. Then we have the natural map of monoids $\pi_G : \mathbb{N} \to \overline{\mathcal{M}}(G) \cong \mathbb{N}$. Thus, the map π_G is given by the mutiplication with a positive integer n_G.

`def:twisting-index`

Definition 3.3.3. We call n_G the *twisting index* of the the splitting graph G.

Consider a degenerate graph G with a splitting $\phi : G \to G_0$. By Construction 3.3.2, we have the gluing data $(G_v, R(v))_{v\in V(G0)}$.

Lemma 3.3.4. *There is a commutative diagram of monoids*

$$\begin{tikzcd}
\sum_{v\in V(G)} \mathbb{N} \arrow[r] \arrow[d, "\sum \pi_{G_v}"'] & \mathbb{N} \arrow[d] \\
\sum_{v\in V(G)} \overline{\mathcal{M}}(G_v) \arrow[r] & \overline{\mathcal{M}}(G)
\end{tikzcd}$$

where the top arrow is given by $e_{G_v} \mapsto e_G$.

Proof. This can be checked directly using the construction of the associated monoids of marked graphs. ♠

`def:arrow-splitting`

Definition 3.3.5. Consider two splittings $\phi_1 : G_1 \to G_0$ and $\phi_2 : G_2 \to G_0$. An arrow $\phi_1 \to \phi_2$ over G_0 is a contraction $G_1 \to G_2$ fits in a commutative diagram

$$\begin{tikzcd}
G_1 \arrow[rr] \arrow[dr] & & G_2 \arrow[dl] \\
& G_0 &
\end{tikzcd}$$

This means that the induced diagram of sets of edges, vertices, and monoids are commutative.

Lemma 3.3.6. *Given an arrow of splittings $\phi_1 \to \phi_2$ as above, we have a commutative diagram of monoids*

$$\begin{tikzcd}
\overline{\mathcal{M}}(G_1) \arrow[rr] \arrow[dr] & & \overline{\mathcal{M}}(G_2) \arrow[dl] \\
& \overline{\mathcal{M}}(G_0) &
\end{tikzcd}$$

which is compatible with π_{G_i} for $i = 1, 2, 3$.

For later use, we introduce the following notations. Denote by $\mathfrak{G}_{\mathbf{c}}$ ($\mathfrak{G}^d_{\mathbf{c}}$ respectively) the category of marked graphs (degenerate graphs respectively) with legs having contact orders given by $\mathbf{c} = \{c_i\}_{i=1}^n$, and the arrows are given by contractions. For a fixed splitting graph G, denote by $\mathfrak{G}(G)$ the category of splittings $\phi : G' \to G$ with arrows given by Definition 3.3.5. Note that the contact orders of the legs of G' is given by those of G.

`ss:discrepant-gluing`

3.4. Discrepant gluing of the splitting data.

`ss:map-graph`

3.5. **Graphs associated to punctured maps.** Consider a punctured map $f : C/S \to X$ where $\underline{S}$ is a geometric point. We now associate a marked graph G to the geometric fiber f as follows.

(1) Let $\underline{G}$ be the dual graph of the nodal curve $\underline{C}$, where vertices correspond irreducible components, edges correspond to nodes, and legs correspond to marked points.
(2) For each vertex v, we associate the fs monoid $P_v = (f^*\overline{\mathcal{M}}_X)_x$ where x is a general point of the component corresponding to v.
(3) For each edge l of $\underline{G}$ joining v_1 and v_2, we associate the monoid $P_l = (f^*\overline{\mathcal{M}}_X)_l$ with the two natural generalizing map $P_l \to v_1$ and $P_l \to v_2$.
(4) The contact orders of each edge or legs are given by the contact orders of each node or marked point respectively.

Definition 3.5.1. The marked graph G as above is called the *associated graph of f.*

The following result implies that the associated graph is admissible.

Proposition 3.5.2. *There is a canonical map of monoids $\overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}_S$, whose kernel is trivial.*

Proof. Need details. $\spadesuit$

`def:minimality`

Definition 3.5.3. The punctured map $f : C/S \to X$ over a geometric point $\underline{S}$ is called *minimal (or basic)* if the canonical map $\overline{\mathcal{M}}(G) \to \overline{\mathcal{M}}_S$ is an isomorphism of monoids. A family of punctured maps is called *minimal* if every geometric fiber is minimal.

`prop:min-open` **Proposition 3.5.4.** *Given a family of punctured maps $f : C/S \to X$ over an fs log scheme $\underline{S}$, the fiber with minimal punctured maps form an open family.*

Proof. Need details ♠

Consider the case of a proper, flat, and integral family of fs targets $X \to B$ where B is a smooth (not necessarily proper) curve with only one marked point $0 \in B$. The log structure on B is given by the divisor $0 \in B$. Consider again a geometric fiber of punctured maps $f : C/S \to X/B$. If the image of S in B is not 0, then this reduce to the case as above.

Proposition 3.5.5. *Consider a punctured map $f : C/S \to X_0/0$. Denote by G the marked graph associated to $f : C/S \to X_0$. Then*

1. *For every $v \in V(G)$ and $l \in E(G)$, there is a natural map $\mathbb{N} \cong \overline{\mathcal{M}}_0 \to P_v$ and $\mathbb{N} \cong \overline{\mathcal{M}}_0 \to P_l$ which is compatible with the generalizing map $P_l \to P_v$ if l is attached to v.*
2. *For each $l \in E(G)$, the image of the composition $\mathbb{N} \to P_l \to \mathbb{Z}$ is zero, where the section arrow is the contact order map.*

Thus, G becomes a degenerate graph.

Proof. Verify directly. ♠

`def:min-degenerate` **Definition 3.5.6.** A punctured map $f : C/S \to X/B$ is called *minimal*, if the punctured map $f : C/S \to X$ by forgetting the base B is minimal.

Minimality for the family case is again an open condition by the above definition and Proposition 3.5.4.

`prop:min-universal` **Proposition 3.5.7.** *Consider a punctured map $f : C/S \to X/B$, where B could be a point with the trivial log structure, or the marked curve $(B, 0)$ as above. There is a minimal punctured map $f_m : C_m/S_m \to X/B$ and a morphism of fs log schemes $\theta : S \to S_m$ which fits in a commutative diagram:*

$$\begin{tikzcd} S \arrow[rr, "\theta"] \arrow[dr] & & S_m \arrow[dl] \\ & B & \end{tikzcd}$$

such that

1. *The underlying map $\underline{\theta}$ is the identity of $\underline{S}$.*
2. *The f is the pull-back of f_m via θ.*
3. *The pair (f_m, θ) is unique up to a unique isomorphism.*

Proof. Similar to the log case. ♠

3.6. **Open immersion of log stack induced by generalizing maps.** For a sharp fine monoid P, denote by

$$\mathcal{T}_P := \Big[\operatorname{Spec} k[P] / \operatorname{Spec} k[P^{gp}]\Big]$$

where the quotient is the stack quotient. Note that the stack $\mathcal{T}_P$ has a natural fs log structure $\mathcal{M}_{\mathcal{T}_P}$ given by the log scheme T_P, see [Ols03a]. For simplicity, we sometimes view $\mathcal{T}_P$ as the log stack with its natural log structures, if no confusion could arise.

Consider a generalizing map of monoids $h : P \to Q$. Denote by $K = \ker(h : P \to Q$. Write $N = K^{-1}P \subset P^{gp}$ the submonoid generated by inverting K in P. We thus have an exact sequence of monoids:

`equ:monoid-contraction`

$$0 \longrightarrow K^{gp} \longrightarrow N \xrightarrow{h'} Q \longrightarrow 0. \tag{3.6.1}$$

Namely, for any $a, b \in N$ such that $h'(a) = h'(b)$, then $(a - b) \in K^{gp}$.

`lem:mon-cri` **Lemma 3.6.1.** *Consider an element $e \in P^{gp}$. Then $e \in N$ if and only if $h^{gp}(e) \in Q$.*

Proof. This follows from the exact sequence 3.6.1. ♠

Since h is generalizing, we have an exact sequence of free abelian groups of finite rank

$$0 \to K^{gp} \to P^{gp} \to Q^{gp} \to 0.$$

We fix a (non-canonical) splitting

$$P^{gp} \cong K^{gp} \oplus Q^{gp}. \tag{3.6.2}$$

We then identify Q as a submonoid of P^{gp} via the second factor.

`lem:mon-factor` **Lemma 3.6.2.** *With the fixed splitting as above, the inclusion $q : Q \hookrightarrow P^{gp}$ factors through N.*

Proof. This follows directly from Lemma 3.6.1. ♠

By abuse of notation, denote by $q : Q \to N$ the inclusion. Thus q becomes a section of h'. Now consider the injection of monoids

$$P \to N.$$

This induces a strict open embedding of log schemes:

$$\operatorname{Spec}(N \to k[N]) \to \operatorname{Spec}(P \to k[P]),$$

which is $\operatorname{Spec} k[P^{gp}]$-equivalent. Consequently, we have a strict open immersion of log stacks:

`equ:log-stack-embd`

$$\mathcal{T}_N \to \mathcal{T}_P. \tag{3.6.3}$$

`lem:log-stack-iso` **Lemma 3.6.3.** *There is a canonical isomorphism of log stacks*

$$\mathcal{T}_N \to \mathcal{T}_Q.$$

Proof. Note that for any scheme X, the groupoid $\mathcal{T}_P(\underline{X})$ consists of log structures $\mathcal{M}$ on $\underline{X}$, which admit maps (not necessarily unique) $P \to \overline{\mathcal{M}}$ that locally lifts to a chart. We refer to [Ols03a] for the detail. We construct the map $\mathcal{T}_N \to \mathcal{T}_Q$ as follows.

Consider a log structure $\mathcal{M}$ on $\underline{X}$ with the map $\beta : N \to \overline{\mathcal{M}}$ which locally lifts to a chart. Since $\overline{\mathcal{M}}$ is a sheaf of fs sharp monoid, we have the following commutative diagram

`equ:mon-charts`

$$\begin{tikzcd} N \arrow[rr, "h'"] \arrow[dr, "\beta"'] & & Q \arrow[dl, "\beta'"] \\ & \overline{\mathcal{M}} & \end{tikzcd} \tag{3.6.4}$$

Note that if locally the map β lifts to a chart $N \to \mathcal{M}$, then we obtain another chart $Q \to N \to \mathcal{M}$ by composing with q, which lifts β'. Thus, we obtain an object $(\mathcal{N}, \beta') \in \mathcal{T}_Q(\underline{X})$.

To see that the map $\mathcal{T}_N \to \mathcal{T}_Q$ is essentially surjective, consider an object $(\mathcal{N}, \beta') \in \mathcal{T}_Q(\underline{X})$. Then the image of the pair $(\mathcal{N}, \beta = \beta' \circ h') \in \mathcal{T}_N(\underline{X})$ is isomorphic to $(\mathcal{N}, \beta')$, since q is a section of h'.

Now the full faithfulness follows from the fact that the map $\mathcal{T}_N \to \mathcal{T}_Q$ does not change the log structure. This finishes the proof of the statement. ♠

`prop:log-stack-embd`

Proposition 3.6.4. *For any generalizing map $P \to Q$, we have a canonical open strict embedding of the associated log stacks*

$$h_{\mathcal{T}} : \mathcal{T}_Q \to \mathcal{T}_P.$$

Proof. This follows from (3.6.3) and Lemma 3.6.3. ♠

For simplicity, we use the notation $\mathcal{T}_G := \mathcal{T}_{\overline{\mathcal{M}}(G)}$ for a marked graph G. Consider a contraction of marked graphs $\phi : G_1 \to G_2$. Thus we have a generalizing map $\phi_{\overline{\mathcal{M}}} : \overline{\mathcal{M}}(G_1) \to \overline{\mathcal{M}}_{G_2}$. Combine this with Proposition 3.6.4, we have

`cor:asso-stack-embd`

Corollary 3.6.5. *For a contraction of marked graphs $\phi : G_1 \to G_2$, we have a natural open immersion of log stacks*

$$\phi_{\mathcal{T}} : \mathcal{T}_{G_2} \to \mathcal{T}_{G_1}.$$

3.7. **Log stacks charted by graphs.** Let $\mathfrak{G}$ be the category $\mathfrak{G}_{\mathbf{c}}$, $\mathfrak{G}^d_{\mathbf{c}}$, or $\mathfrak{G}(G_0)$ where $\mathbf{c}$ is the set of contact orders of the legs, and G_0 is a fixed splitting graph. Denote by $\mathfrak{G}^\circ$ the dual category of $\mathfrak{G}$. Consider the log stack given by the following limit:

$$\mathcal{T}_{\mathfrak{G}} := \varinjlim_{\mathfrak{G}^\circ} \mathcal{T}_G.$$

`prop:rel-graph-stack`

Proposition 3.7.1. *(1) The log stack $\mathcal{T}_{\mathfrak{G}}$ is irreducible, connected, and log smooth of pure dimension* 0.
(2) The canonical log map $\mathcal{T}_G \to \mathcal{T}_{\mathfrak{G}}$ is a strict open embedding for any $G \in \mathfrak{G}$.
(3) The set $\{\mathcal{T}_G \to \mathcal{T}_{\mathfrak{G}}\}_{G \in \mathfrak{G}}$ forms an open covering of $\mathcal{T}_{\mathfrak{G}}$.
(4) For any family of minimal punctured maps $f : C/S \to X$, there is a canonical strict log map $S \to \mathcal{T}_{\mathfrak{G}}$.

Proof. This follows from the limit construction. ♠

Definition 3.7.2. The stack $\mathcal{T}_{\mathfrak{G}}$ is called the *log stack charted by* $\mathfrak{G}$. Denote by $\mathcal{M}_{\mathcal{T}_{\mathfrak{G}}}$ its natural log structure.

Denote by $\mathfrak{D}$ the category of distinguished marked graphs associated to graphs in $\mathfrak{G}$. We can similary take the limit:

$$\mathcal{T}_{\mathfrak{D}} := \varinjlim_{\mathfrak{D}^{\circ}} \mathcal{T}_{G}.$$

Note that for each marked graph G, we can associate a distinguished graph G^d and a canonical contraction $G \to G^d$. The splitting given in Proposition 3.2.6 induces a natural isomorphism

`equ:nd-monoid-split`

$$\mathcal{T}_{G} \cong \mathcal{T}_{G^d} \times \mathcal{A}^n, \tag{3.7.1}$$

where l is the number of non-distinguished edges in G, and $\mathcal{A} = [\mathbb{A}^1/\mathbb{G}_m]$. This induces a natural smooth projection

`equ:contract-logstack`

$$\mathcal{T}_{G} \to \mathcal{T}_{G^d}. \tag{3.7.2}$$

Gluing above projection, we obtain

`cor:stack-contract-nd`

Corollary 3.7.3. *There is a natural log morphism*

$$\pi_{\mathfrak{D}} : \mathcal{T}_{\mathfrak{G}} \to \mathcal{T}_{\mathfrak{D}}$$

which is induced by (3.7.2). Furthermore the underlying map $\underline{\pi_{\mathfrak{D}}}$ *and is a smooth morphism of the underlying stacks.*

We fix a connected smooth curve $\underline{B}$ with a special point $0 \in \underline{B}$. Then the pair $(\underline{B}, 0)$ can be viewed as a log scheme B with underlying structure given by $\underline{B}$, and the log structure given by the divisor 0. Thus, we have an induced strict map

`equ:log-point-curve`

$$B \to \mathcal{A}. \tag{3.7.3}$$

Given a degenerate graph G, the composition

$$\mathbb{N} \xrightarrow{\pi_G} \overline{\mathcal{M}}(G) \longrightarrow \overline{\mathcal{M}}_{\mathcal{T}_G}$$

induces a rank 1 sub-log structure of $\mathcal{M}_{\mathcal{T}_G}$, hence a log map

$$\pi_G : \mathcal{T}_G \to \mathcal{A}.$$

We still use π_G to denote this map.

Lemma 3.7.4. *Consider a contraction of degenerate graphs* $\phi : G_1 \to G_2$. *We have a commutative diagram of log stacks:*

$$\begin{tikzcd}
\mathcal{T}_{G_2} \arrow[rr, "\phi_{\mathcal{T}}"] \arrow[dr, "\pi_{G_2}"'] & & \mathcal{T}_{G_1} \arrow[dl, "\pi_{G_1}"] \\
& \mathcal{A} &
\end{tikzcd}$$

Proof. This follows from Propostion 3.2.9. ♠

Denote by $\mathfrak{G}$ the category of degenerate graphs $\mathfrak{G}^d_{\mathbf{c}}$ or $\mathfrak{G}(G_0)$. We consider the fibered product of log stacks:

$$\mathfrak{T}_{\mathfrak{G}} = \mathcal{T}_{\mathfrak{G}} \times_{\mathcal{A}} B.$$

Since the map $B \to \mathcal{A}$ is strict, the log structure on $\mathfrak{T}_{\mathfrak{G}}$ is the pull-back log structure from $\mathcal{T}_G$.

Definition 3.7.5. The log stack $\mathfrak{T}_{\mathfrak{G}}$ is called *the log stack charted by degenerate graphs in* $\mathfrak{G}$.

Similarly, we have

Proposition 3.7.6. *(1) The stack $\mathfrak{T}_{\mathfrak{G}}$ is irreducible, and log smooth of pure dimension* 1.
(2) There is a set of strict étale charts $\{\mathcal{T}_G \times_{\mathcal{A}} B \to \mathfrak{T}_{\mathfrak{G}}\}$.
(3) For any minimal punctured maps $f : C/S \to X/B$, we have a canonical strict log map $S \to \mathfrak{T}_{\mathfrak{G}}$.

We now consider the category $\mathfrak{D}$ consists of distinguished graphs associated to graphs in the category $\mathfrak{G}$ of degenerate graphs. Combining the above with Corollary 3.7.3, we have

ck-contract-degenerate

Corollary 3.7.7. *There is a natural log morphism*

$$\pi_{\mathfrak{D}} : \mathfrak{T}_{\mathfrak{G}} \to \mathfrak{T}_{\mathfrak{D}}$$

which is induced by the contractions of non-distinguished edges. Furthermore the underlying map $\underline{\pi_{\mathfrak{D}}}$ and is a smooth morphism of the underlying stacks.

Denote by $\Omega_{\mathbf{c}}$ the set of splitting graphs with fixed contact orders on legs. Consider the stack

$$\mathfrak{Q} = \mathfrak{T}_{\mathfrak{G}^d_{\mathbf{c}}} \times_B 0.$$

This stack is reducible and non-reduced in general. We consider the similar log stack

$$\mathfrak{Q}_{G_0} = \mathfrak{T}_{\mathfrak{G}(G_0)} \times_B 0.$$

This stack is again non-reduced and irreducible. In fact, over each chart $\mathcal{T}_G \times_{\mathcal{A}} B \to \mathfrak{T}_{\mathfrak{G}}$, the substack $\mathfrak{Q}_{G_0}$ is defined by $\alpha(e_0) = 0$, where e_0 is the local generator of $\mathcal{M}_B$. We have

Proposition 3.7.8. *The log stack $\mathfrak{Q}$ is of pure dimension zero, and has irreducible components given by $\{\mathfrak{Q}_{G_0}\}_{G_0 \in \Omega_{\mathbf{c}}}$.*

Denote by e_{G_0} the local generator of $\mathcal{T}_{G_0}$. Denote by $\mathfrak{T}_{G_0} \subset \mathfrak{Q}_{G_0}$ the substack locally defined by $\alpha(e_{G_0}) = 0$.

Proposition 3.7.9. *The stack $\mathfrak{T}_{G_0}$ is reduced, and of pure dimension zero. The map $\mathfrak{T}_{G_0} \to \mathfrak{Q}_{G_0}$ is a closed embedding of degree $n_{G_0}^{-1}$.*

Proof. Note that $e_0 = n_{G_0} \cdot e_{G_0}$. ♠

We can similarly formulate the statements for the category $\mathfrak{D}$.

4. The stack of punctured maps

Notation 4.0.10. Consider a projective, integral fs log map $\pi : X \to B$. We use Γ to denote the set of data $(g, n, \{c_i\}_{i=1}^n)$, where $c_i \in Con(X)$ is the contact order of the i-th marked points. Denote by $\mathfrak{M}_\Gamma(X/B)$ the category of stable punctured map to $X \to B$ with genus g, n marked points, and the prescribed contact orders c_i for the i-th marking, fibered over the category of fs log schemes. If B is a point with trivial log structure, then we write $\mathfrak{M}_\Gamma(X)$ for this fibered categroy.

The goal of the section is to prove that $\mathfrak{M}_\Gamma(X)$ is represented by the proper DM stack parametrizing minimal punctured maps with its natural log structure. Repeat the argument in [AC], we obtain the construction for $\mathfrak{M}_\Gamma(X/B)$.

4.1. **Algebricity and boundedness.** Algebricity should follow from the construction of Hom of log structures. But for the purpose of boundedness, we need to modify the proof to add the data of prescribed poles.

4.2. **Weak valuative criterion.** The proof of this is the same as log maps.

`sec:deformation`

5. The deformation theory of punctured maps

In this section, we fix a projective family of integral and log smooth target with fs log structures $X \to B$. We first investegate the deformation theory of punctured maps with target $X \to B$. This construction naturally generalizes the deformation theory of log maps. We then use the result to construct a perfect obstruction theory of punctured maps. For any fs log scheme T over B, write $X_T := X \times_B T$.

`ss:small-extension`

5.1. **Deformation over small extensions.** Consider a punctured map $f_0 : C_0/S_0 \to X/B$ over an affine fs log scheme S_0 relative to B. Choose another fs log scheme S over B with a exact closed immersion $j : S_0 \to S$. Denote by J the ideal sheaf of S_0 in S. Assume further that $J^2 = 0$, i.e. S is a small extension of S_0. We would like to classify the deformations of f_0, i.e. for the commutative diagram of solid arrows:

`diag:total-deformation`

(5.1.1)

$$\begin{tikzcd}
C_0 \arrow[rr, "f_0"] \arrow[dr] \arrow[dd, dashed] & & X_{S_0} \arrow[dl] \arrow[dd] \\
& S_0 \arrow[dd] & \\
C \arrow[rr, dashed, "f"] \arrow[dr, dashed] & & X_S \arrow[dl] \\
& S &
\end{tikzcd}$$

we would like to classify punctured maps $f: C/S \to X/B$ making the above diagram commutes, such that the two side squares are cartesian. Since $j: S_0 \to S$ is a small extension, they have the same topological space and étale topology. Write $\Omega := f_0^*\Omega_{X/B}$ the pull-back of the vector bundle of log differentials, and $I := \mathcal{O}_C \otimes_{\mathcal{O}_S} J \cong \mathcal{O}_{C_0} \otimes J$ the sheaf of ideal defining $C_0 \subset C$.

We first look at an easier situation. Assume that we have an extension $C \to S$ of the log curve $C_0 \to S_0$. Then we get a diagram of solid arrows:

`iag:arrow-deformation` (5.1.2)

$$\begin{tikzcd} C_0 \arrow[r, "f_0"] \arrow[d] & X_{S_0} \arrow[d] \\ C \arrow[r, dashed, "f"] & X_S. \end{tikzcd}$$

`lem:arrow-deformation` **Lemma 5.1.1.** *(1) There is a canonical obstruction $o \in Ext^1(\Omega, I)$, whose vanishing is sufficient and necessary for the extension of the extension f of the punctured map f_0.*

(2) If $o = 0$, then the set of isomorphism classes form a torsor under the group $Ext^0(\Omega, I)$.

Proof. Note that away from the marked points, the punctured map f_0 is a well-defined log map. Hence the statement follows from [Ols05, Theorem 5.9]. Thus, we may assume that C_0 is a smooth affine curve with a unique marking $\Sigma \subset C_0$.

Now assume that we have two extensions $f_1 : C \to X_S$ and $f_2 : C \to X_S$ of f_0. These yield a commutative diagram:

$$\begin{tikzcd} C_0 \arrow[r] \arrow[d] & X \arrow[d] \\ C \arrow[r, "g"'] \arrow[ur, dotted, "f_1", shift left] \arrow[ur, dotted, "f_2"', shift right] & B. \end{tikzcd}$$

Away from Σ, we thus have a commutative diagram of log structures over $C_0 \setminus \Sigma$:

$$\begin{tikzcd} \mathcal{M}_{C_0} & f_0^*\mathcal{M}_X \arrow[l] \arrow[dl, dotted, "f_2^\flat"', shift right] \arrow[dl, dotted, "f_1^\flat", shift left] \\ \mathcal{M}_C \arrow[u] & g^*\mathcal{M}_B \arrow[l] \arrow[u] \end{tikzcd}$$

As in [Ogu06, Chapter IV Section 2] or [ACG$^+$, Section 3], we can define a morphism of sheaf of groups $D_{f_1-f_2} : f_0^{-1}\mathcal{M}_X^{gp} \to I$ such that for every $m \in f_0^*\mathcal{M}_X$ we have

$$(f_1 - f_2)(m) = 1 + D_{f_1-f_2}(m).$$

This defines a derivation $D_{f_1-f_2} \in Hom_{C_0 \setminus \Sigma}(\Omega, I)$. We now check that $D_{f_1-f_2}$ naturally extends to a section in $Hom_{C_0}(\Omega, I)$.

Choose a point $p \in \Sigma$. Locally around p, we fix a chart

$$\beta : P := f_0^{-1}\overline{\mathcal{M}}_{X,\Sigma} \to f_0^*\mathcal{M}_{X,\Sigma}.$$

Denote by $c : P \to \mathbb{Z}$ the contact order at p, and $\sigma \in \mathcal{M}_{C_0}$ the section locally at p corresponding to the marking Σ. Thus, for any $m \in f_0^*\overline{\mathcal{M}}_X$ we have

$$(f_1-f_2)(\beta(m)) = (\log u_1+e+c(m)\cdot\sigma)-(\log u_1+e+c(m)\cdot\sigma) = \log u_1-\log u_2,$$

where u_1 and u_2 are locally invertible sections around p, and e is a section of $\mathcal{M}_S$. This proves that $D_{f_1-f_2} \in Hom_{C_0}(\Omega, I)$.

Also notice that we have a commutative diagram of the structure sheaves:

$$\begin{tikzcd}
\mathcal{O}_{C_0} & f_0^{-1}\mathcal{M}_X \arrow[l] \arrow[dl, dotted, shift right, "f_2^\sharp"'] \arrow[dl, dotted, shift left, "f_1^\sharp"] \\
\mathcal{O}_C \arrow[u] & g^{-1}\mathcal{O}_B \arrow[l] \arrow[u]
\end{tikzcd}$$

which induce a derivation $\partial_{f_1-f_2} : f_0^{-1}\mathcal{O}_X \to I$ as usual. It is well-known that $D_{f_1-f_2}$ uniquely determines $\partial_{f_1-f_2}$.

On the other hand, given one extension f_1 and a section $D \in Hom_{C_0}(\Omega, I)$ we obtain a different extension f_2 by reversing the above construction.

Now the statement follows from the standard cocycle argument of the deformation theory. ♠

Now we turn to the situation of (5.1.1). It is not hard to check that the punctured map f_0 induces a morphism of the sheaves of logarithmic differentials:

$$(df_0)^\vee : \Omega \to \Omega_{C_0/S_0}.$$

Denote by $\mathbb{L}$ the cone of $(df_0)^\vee$, i.e. we have a distinguished triangle:

`cplx:deformation`

$$\Omega \to \Omega_{C_0/S_0} \to \mathbb{L}. \tag{5.1.3}$$

Recall that the deformations of log curves is controlled by the $Ext^i(\Omega_{C_0/S_0}, I)$ for $i = 0, 1, 2$. We refer to [Ols05, Theorem 5.6] for the precise statement. Combining the deformation of the source curve and the result of Lemma 5.1.1, we have

`rop:total-deformation`

Proposition 5.1.2. *(1) There is a canonical obstruction $o \in Ext^2(\mathbb{L}, I)$, whose vanishing is sufficient and necessary for the existence of the extensions as in (5.1.1).*

(2) If $o = 0$, then the set of isomorphism classes of such extension forms a torsor under the group $Ext^1(\mathbb{L}, I)$.

(3) The group of automorphisms of a such extension is a torsor under $Ext^0(\mathbb{L}, I)$.

Proof. Denote by Obs and Def the obstruction and deformation space for (5.1.1). Taking the long exact sequence of (5.1.3), we have

`cplx:deformation-seq`

$$
\begin{aligned}
0 \longrightarrow Ext^0(\mathbb{L}, I) &\longrightarrow Ext^0(\Omega_{C_0/S_0}, I) \longrightarrow Ext^0(\Omega, I) \\
\longrightarrow Ext^1(\mathbb{L}, I) &\longrightarrow Ext^1(\Omega_{C_0/S_0}, I) \longrightarrow Ext^1(\Omega, I) \\
\longrightarrow Ext^2(\mathbb{L}, I) &\longrightarrow Ext^2(\Omega_{C_0/S_0}, I) = 0
\end{aligned}
\tag{5.1.4}
$$

We fix the trivial deformation $C \to S$ of the source log curve $C_0 \to S_0$. Then any obstruction of the arrow f_0 as in (5.1.2) induces an obstruction in Obs. We thus obtain an arrow $Ext^1(\Omega, I) \to Obs$. The kernel of this map is $Ext^1(\Omega_{C_0/S_0}, I)$. In fact, we can take the trivial deformation of f_0 associated to each deformation of the source log curves. By forgetting the obstruction of the arrow, we have a natural map $Obs \to Ext^2(\Omega_{C_0/S_0}, I)$. But since the deformation of log curves are unobstructed. This means that the map $Ext^1(\Omega, I) \to Obs$ is surjective. Comparing with (5.1.4), we conclude that $Obs \cong Ext^2(\mathbb{L}, I)$.

When the obstruction vanishes, we have a natural map

$$Def \to Ext^1(\Omega_{C_0/S_0}, I)$$

by forgetting the deformation of the maps. The sequence

$$Def \to Ext^1(\Omega_{C_0/S_0}, I) \to Ext^1(\Omega, I)$$

is certainly exact for the vanishing of the obstruction.

On the other hand, fixing the trivial deformation of the source curve, there is an induced map

$$Ext^0(\Omega, I) \to Def.$$

We claim that the sequence

$$Ext^0(\Omega_{C_0/S_0}, I) \to Ext^0(\Omega, I) \to Def \to Ext^1(\Omega_{C_0/S_0}, O)$$

is exact. In fact, the image $\operatorname{im}(Ext^0(\Omega, I) \to Def)$ is precisely the deformations with trivial source curve extensions by definition, and the image $\operatorname{im}(Ext^0(\Omega_{C_0/S_0}, I) \to Ext^0(\Omega, I))$ is the extension of the map determined by the automorphism of the source curves. Again, comparing with (5.1.4), we conclude that $Def \cong Ext^1(\mathbb{L}, I)$.

Finally, elements in $Ext^0(\mathbb{L}, I)$ correspond to automorphisms of the extension of $C_0 \to S_0$, which induce the trivial deformation of f_0. This proves the last statement. $\spadesuit$

5.2. **A perfect obstruction theory of punctured maps.** This can be constructed relative to the stack of base log structures similarly as the case of expanded degeneration. The formation will commutes with arbitrary base change.

5.3. **The Splitting formula.** Follows from the push forward formula of virtual cycles and the formation of the perfect obstruction theory.

6. GLUING FORMULA

APPENDIX A. STACK OF NODES

A.1. **P-log points.** Fix a fs sharp monoid P. In the paper, we only need $P = \mathbb{N}^2$. Denote by

$$\mathbb{B}_P = [\operatorname{Spec} k / \operatorname{Spec} k[P^{gp}]]$$

and

$$\mathcal{A}_P = [\operatorname{Spec} k[P] / \operatorname{Spec} k[P^{gp}]]$$

where the quotients is given by the stack quotient. Thus, we have a natural closed embedding

equ:quotient-embedding

$$\mathbb{B}_P \to \mathcal{A}_P. \tag{A.1.1}$$

def:p-point

Definition A.1.1. A family of *standard P-log points over $\underline{S}$* is given by a log scheme $(\underline{S}, \mathcal{N}_S)$ such that for each geometric point $\bar{s} \in \underline{S}$, we have $\mathcal{N}_{S,\bar{s}} \cong P$. A family of *$P$-log points over a (fs) log scheme S* is given by a family $S' = (\underline{S}, \mathcal{N}_S \oplus \mathcal{M}_S) \to S$, where the arrow is given by the canonical injection of log structures $\mathcal{M}_S \to \mathcal{N}_S \oplus \mathcal{M}_S$.

cor:p-point

Corollary A.1.2. *For any P-log points $S' \to S$ as in the definition, we have a unique (up to a unique isomorphism) standard P-log points $(\underline{S}, \mathcal{M}'_S)$ with the following log cartesian diagram:*

$$\begin{tikzcd}
S' \arrow[r] \arrow[d] & (\underline{S}, \mathcal{M}'_S) \arrow[d] \\
S \arrow[r] & \underline{S}.
\end{tikzcd}$$

Note that we have a natural embedding (A.1.1). Denote by $\mathcal{N}_P$ the natural log structure on $\mathbb{B}_P$ obtained by pulling-back the canonical log structure of $\mathcal{A}_P$. We denote by $\mathbb{B}_P$ and $\mathbb{B}'_P$ the log stack with trivial log structure and the natural log structure $\mathcal{N}_P$ respectively. The following result is obvious:

prop:p-point

Proposition A.1.3. *(1) The stack $\mathbb{B}_P$ with its trivial log structure parametrizes family of standard P-log points over schemes.*
(2) The log stack $\mathbb{B}_P$ parametrizes family of P-log points over log schemes. Furthermore, the universal family is given by $\mathbb{B}'_P$.

Proof. This follows directly from Corollary A.1.2. ♠

A.2. **Construction of the stacks.**

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