Logarithmic geometry and moduli

Lectures at the Sophus Lie Center

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Brown University

June 16-17, 2014

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- Olsson
- Chen, Gillam, Huang, Satriano, Sun
- Gross - Siebert

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- $\overline{\mathcal{M}}_g$

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> *Working with a non-complete moduli space is like keeping change in a pocket with holes*
>
> Angelo Vistoli

Deligne–Mumford

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- What's so great about nodes?
- One answer: from the point of view of logarithmic geometry, these are the *logarithmically smooth* curves.

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Definition

It is a *logarithmic structure* if $\alpha : \alpha^{-1}\mathcal{O}^*_{\underline{X}} \to \mathcal{O}^*_{\underline{X}}$ is an isomorphism.

“Trivial” examples

Examples

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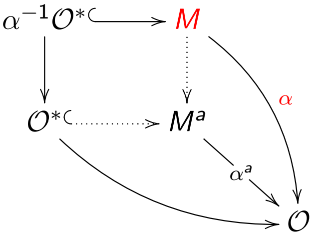
- $(\underline{X}, \mathbb{N} \stackrel{\alpha}{\to} \mathcal{O}_{\underline{X}})$, where α is determined by an arbitrary choice of $\alpha(1)$.
  This one is important but only pre-logarithmic.

The associated logarithmic structure

You can always fix a pre-logarithmic structure:



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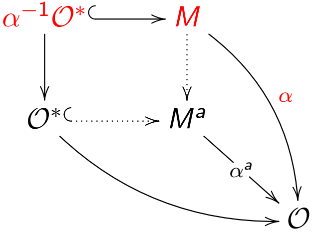

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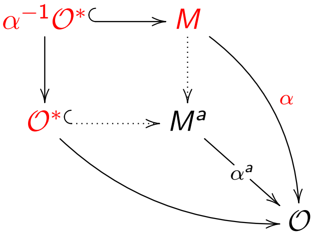

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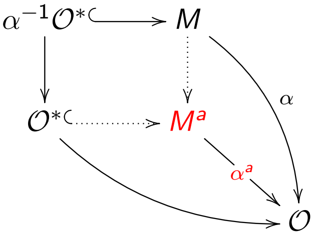

Key examples

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This is particularly important for normal crossings divisors and toric divisors - these will be logarithmically smooth structures.

Example (Standard logarithmic point)

Let k be a field,

$$\begin{array}{ccc} \mathbb{N} \oplus k^\times & \to & k \\ (n, z) & \mapsto & z \cdot 0^n \end{array}$$

defined by sending $0 \mapsto 1$ and $n \mapsto 0$ otherwise.

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Works with P a monoid with $P^\times = 0$, giving the *P-logarithmic point*. This is what you get when you restrict the structure on an affine toric variety associated to P to the maximal ideal generated by $\{p \neq 0\}$.

Morphisms

A morphism of (pre)-logarithmic schemes $f : X \to Y$ consists of

- $\underline{f} : \underline{X} \to \underline{Y}$

$$\mathcal{O}_{\underline{X}} \xleftarrow{\underline{f}^\sharp} \underline{f}^{-1}\mathcal{O}_{\underline{Y}}$$

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- $\underline{f} : \underline{X} \to \underline{Y}$
- A homomorphism $f^\flat$ making the following diagram commutative:

$$\begin{array}{ccc} M_X & \xleftarrow{f^\flat} & \underline{f}^{-1} M_Y \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ \mathcal{O}_{\underline{X}} & \xleftarrow{\underline{f}^\sharp} & \underline{f}^{-1} \mathcal{O}_{\underline{Y}} \end{array}$$

Definition (**Inverse image**)

Given $\underline{f} : \underline{X} \to \underline{Y}$ and $Y = (\underline{Y}, M_Y)$ define the *pre-logarithmic inverse image* by composing

$$\underline{f}^{-1} M_Y \to \underline{f}^{-1} \mathcal{O}_{\underline{Y}} \stackrel{\underline{f}^\sharp}{\to} \mathcal{O}_{\underline{X}}$$

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$X \to Y$ is **strict** if $M_X = \underline{f}^* M_Y$.

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- If N is the pushout of

$$\begin{array}{ccccc} & & \pi_Z^{-1} M_Z & & \\ & \swarrow & & \searrow & \\ \pi_X^{-1} M_X & & & & \pi_Y^{-1} M_Y \end{array}$$

then the log structure on $X \times_Z Y$ is defined by N^a.

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The morphism $\underline{f} : \operatorname{Spec}(R[P]) \to \operatorname{Spec}(\mathbb{Z}[P])$ gives

$$X = \underline{X} \times_{\underline{X_0}} X_0.$$

Charts

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- the diagram

$$\begin{array}{ccc} Q_X & \longrightarrow & f^{-1}M_Y \\ \downarrow & & \downarrow \\ P_X & \longrightarrow & M_X \end{array}$$

  is commutative.

Types of logarithmic structures

- We say that $(\underline{X}, M_X)$ is *coherent* if *étale locally* at every point there is a *finitely generated* monoid P and a local chart $P_X \to \mathcal{O}_{\underline{X}}$ for X.

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- We say that a logarithmic structure is *fine* if it is *coherent* with local charts $P_X \to \mathcal{O}_X$ with P *integral*.
- We say that a logarithmic structure is *fine and saturated* (or fs) if it is coherent with local charts $P_X \to \mathcal{O}_X$ with P *integral and saturated*.

Definition (The characteristic sheaf)

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The characteristic sheaf records the combinatorics of a logarithmic structure, especially for fs logarithmic structures.

Differentials

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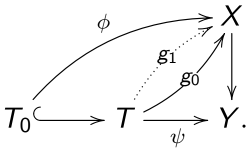
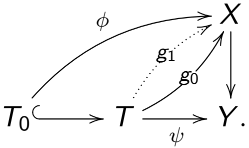
We contemplate the following diagram:



Differentials (continued)



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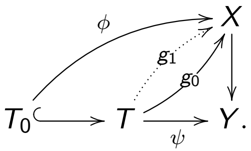
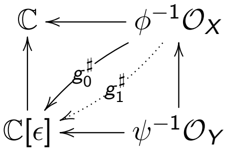


This translates to a diagram of groups



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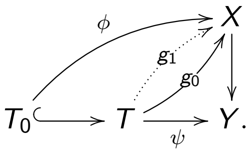
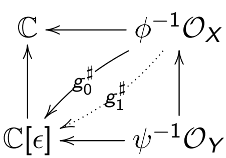
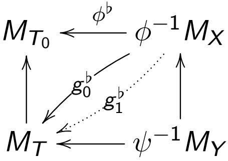


This translates to a diagram of groups and a diagram of monoids





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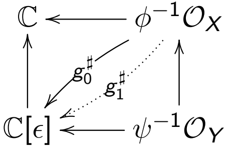

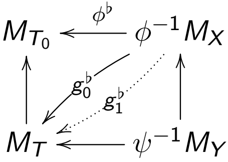

Differentials (continued)

$$
\begin{array}{ccc}
\mathbb{C} & \longleftarrow & \phi^{-1}\mathcal{O}_X \\
\uparrow & \swarrow^{g_0^\sharp} \ \swarrow^{g_1^\sharp} & \uparrow \\
\mathbb{C}[\epsilon] & \longleftarrow & \psi^{-1}\mathcal{O}_Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
M_{T_0} & \xleftarrow{\phi^\flat} & \phi^{-1}M_X \\
\uparrow & \swarrow^{g_0^\flat} \ \swarrow^{g_1^\flat} & \uparrow \\
M_T & \longleftarrow & \psi^{-1}M_Y
\end{array}
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The difference $g_1^\sharp - g_0^\sharp$ is a derivation $\phi^{-1}\mathcal{O}_X \xrightarrow{d} \epsilon\mathbb{C} \simeq \mathbb{C}$

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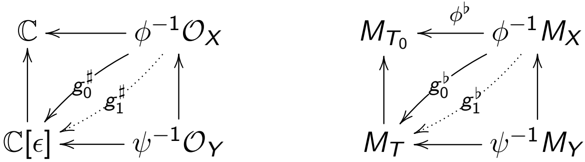

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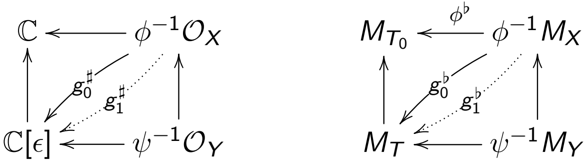

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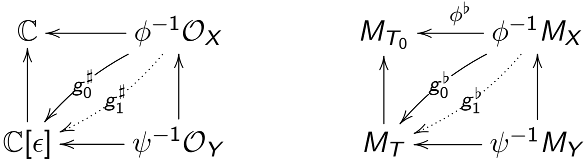

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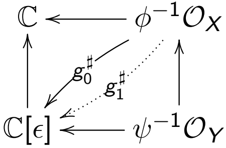

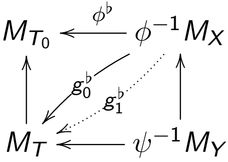

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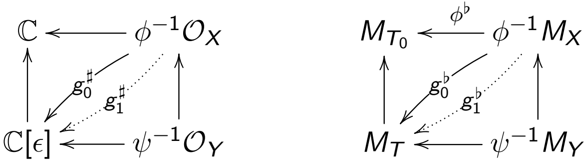

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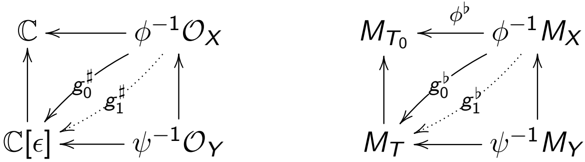

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Namely $D(m) =$ "$g_1^\flat(m) - g_0^\flat(m)$" $\in J$.

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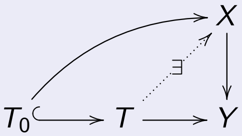

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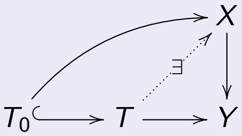

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The morphism is *logarithmically étale* if the lifting in (2) is unique.

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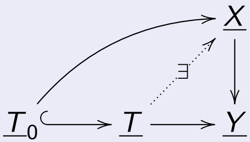

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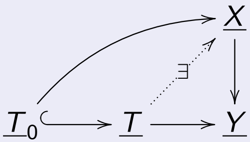

since $\underline{X} \to \underline{Y}$ smooth, and the lifting of morphism of monoids comes by the universal property of pullback. ♠

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(proof on board!)

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One direction:

$$\begin{array}{ccc}
\underline{X} \longrightarrow \underline{Y} \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P] & \longrightarrow & \operatorname{Spec} \mathbb{Z}[P] \\
\downarrow & & \downarrow \\
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Sketch of proof: locally $X_0 \to X_0' \to Y_0$, where

$$X_0' = Y_0 \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P].$$

So $X_0' \to Y_0$ is combinatorially smooth, and automatically provided a deformation to

$$Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P],$$

and $X_0 \to X_0'$ is strict and smooth so deforms by the classical result.

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Corollary

Logarithmically smooth curves are unobstructed.

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This guarantees that if $X \to Y$ is logarithmically smooth, then the fibers are reduced.

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      $$\mathrm{Spec}(\mathbb{N}^2 \to \mathbb{Z}[\mathbb{N}^2]) \to Spec(\mathbb{N} \to \mathbb{Z}[\mathbb{N}])$$

Theorem (F. Kato)

Assume $\pi : X \to S$ is a log curve. Then

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Stable log curves

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Moduli of stable log curves

We define a category $\overline{\mathcal{M}}_{g,n}^{\log}$ of stable log curves: objects are log (g,n)-curves $X \to S$ and arrows are fiber diagrams compatible with sections

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There is a forgetful functor

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}^{\log} & \longrightarrow & \mathfrak{LogSch}^{\text{fs}} \\ (X \to S) & \mapsto & S. \end{array}$$

So $\overline{\mathcal{M}}_{g,n}^{\log}$ is a category fibered in groupoids over $\mathfrak{LogSch}^{\text{fs}}$.

Moduli of stable log curves (continued)

We also have a forgetful functor

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Theorem (F. Kato)

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(Proof sketch on board)

Minimality

Given a stable curve $\underline{X} \to \underline{S}$ we define

$$X^{\min} = \underline{X} \times_{\overline{\mathcal{M}}_{g,n+1}} \overline{\mathcal{M}}_{g,n+1}^{\log} \qquad \text{and} \qquad S^{\min} = \underline{S} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}^{\log}.$$

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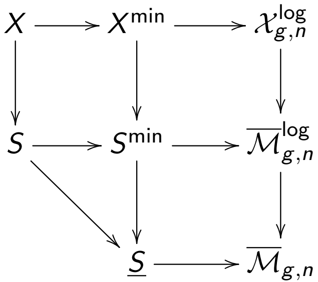
We write

$$S^{\min} = (\underline{S}, M^S_{X/S}) \qquad \text{and} \qquad X^{\min} = (\underline{X}, M^X_{X/S}).$$

Fundamental diagram



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- $\overline{\mathcal{M}}_{g,n}^{\text{log}}$ parametrizes stable log curves over $\mathfrak{LogSch}^{\text{fs}}$
- $\overline{\mathcal{M}}_{g,n}$ parametrizes minimal stable log curves over $\mathfrak{Sch}$.

Stable logarithmic maps

Definition

A **stable logarithmic map** is a diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle s_i} \uparrow\downarrow {\scriptstyle \pi} & & \\ S & & \end{array}$$

where

- $(C/S, s_i)$ is a prestable log curve, and
- in fibers $\mathrm{Aut}(\underline{C}_s \to \underline{X}, s_i)$ is finite.

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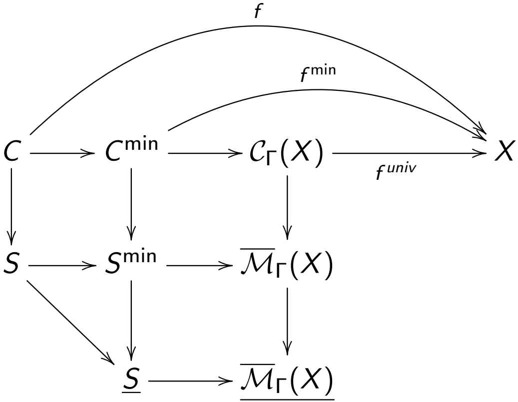
We collect the numerical data under the umbrella $\Gamma = (g, \beta, c_i)$.

Stable logarithmic maps (continued)

Theorem (Gross-Siebert, Chen, ℵ-Chen-Marcus-Wise)

Let X be projective logarithmically scheme. Stable logarithmic maps to X form a logarithmic Deligne–Mumford stack $\overline{\mathcal{M}}_\Gamma(X)$. It is finite and representable over $\overline{\mathcal{M}}_{\underline{\Gamma}}(\underline{X})$.

Fundamental diagram



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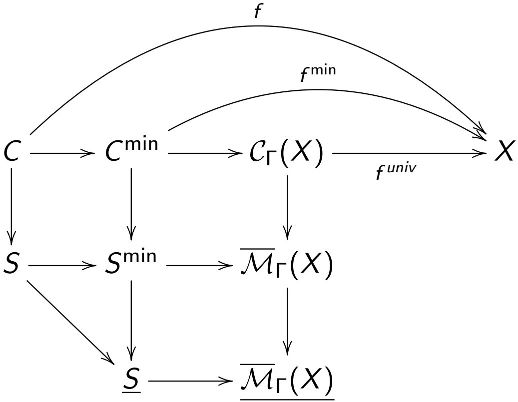

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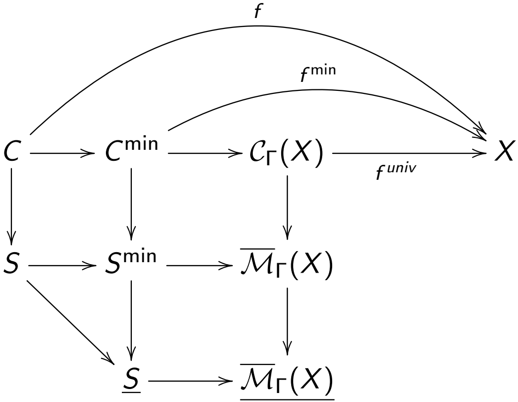

We are in search of a moduli stack $\underline{\overline{\mathcal{M}}_\Gamma(X)}$ parametrizing *minimal* stable logarithmic maps over $\mathfrak{Sch}$.
As such it comes with a logarithmic structure $\overline{\mathcal{M}}_\Gamma(X)$ which parametrizes *all* stable logarithmic maps over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Stable logarithmic maps (continued)

This requires two steps:

- first find a morphism from $(C \to S, f : C \to X)$ to a *minimal* object $(C^{\min} \to S^{\min}, f^{\min} : C^{\min} \to X)$.
- then show that the object $(C^{\min} \to S^{\min}, f^{\min} : C^{\min} \to X)$ has a versal deformation space, whose fibers are also minimal.

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But the nodes impose crucial conditions.

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$$P \oplus^{\mathbb{N}} \mathbb{N}^2 = P\langle \log x, \log y \rangle / (\log x + \log y = \rho_q)$$

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Recall that the stalk of a sheaf at a point q maps, via a "generization map", to the stalk at any point specializing to q, such as η_q^1, η_q^2.

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The map to the stalk at η_q^2 where $y = 0$ sends $\log x \mapsto 0$, and so $\log y \mapsto \rho_q$.
This means that we have a monoid homomorphism, which is clearly injective,

$$P \oplus^{\mathbb{N}} \mathbb{N}^2 \to P \times P.$$

Its image is precisely the set of pairs

$$\{(p_1, p_2) | p_2 - p_1 \in \mathbb{Z}\rho_q\}$$

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Or better: we have $u_q : M_q \to \mathbb{Z}$ such that

$$(p_2 - p_1) \circ f_q^\flat(m) = u_q(m) \cdot \rho_q. \qquad (0.0.1)$$

Putting nodes and generic points together

The maps $p_1 \circ f_q^\flat : M_q \to P$ and $p_2 \circ f_q^\flat : M_q \to P$, since they come from maps of sheaves, are compatible with generization maps.

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the data of $p_1 \circ f_q^\flat$ and $p_2 \circ f_q^\flat$ is already determined by the data at the generic points η_i of the curve.

Putting nodes and generic points together (continued)

The *only* data the node provides is the element $\rho_q \in P$ and homomorphism $u_q : M_q \to \mathbb{Z}$, in such a way that equation

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holds.

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where R is generated by all the relations implied by equation (0.0.2)

Putting nodes and generic points together (continued)

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It is quite a bit more natural to describe the dual lattice

$$Q_f^\vee = \left\{ ((v_\eta), (e_\eta)) \in \prod_\eta N_{\sigma_\eta} \times \prod_q \mathbb{N} \;\middle|\; \begin{array}{c} \forall\ \eta_q^1 \xrightarrow{\quad q \quad} \eta_q^2 \\ v_q^1 - v_q^2 = e_j u_q \end{array} \right\}.$$

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- The equations $v_q^1 - v_q^2 = e_j u_q$ define the cone of all such graphs
- $Q_f^\vee$ is the integer lattice in that cone.

Theorem (Gross-Siebert)

The minimal object exists, with characteristic sheaf Q_f, dual to the lattice in the corresponding space of tropical curves.