Logarithmic Geometry and Moduli

Tropical Geometry and Moduli Spaces
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Moduli of curves

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> *Working with a non-complete moduli space is like keeping change in a pocket with holes*
>
> Angelo Vistoli

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- allow only nodes as singularities
- What's so great about nodes?
- One answer: from the point of view of logarithmic geometry, these are the *logarithmically smooth* curves.

Logarithmic structures

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Definition

It is a *logarithmic structure* if $\alpha : \alpha^{-1}\mathcal{O}^*_{\underline{X}} \to \mathcal{O}^*_{\underline{X}}$ is an isomorphism.

“Trivial” examples

Examples

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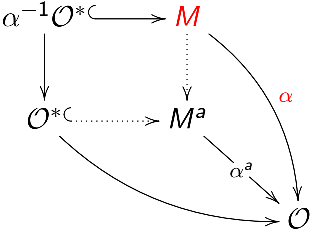
- $(\underline{X}, \mathbb{N} \stackrel{\alpha}{\to} \mathcal{O}_{\underline{X}})$, where α is determined by an arbitrary choice of $\alpha(1)$.
  This one is important but only pre-logarithmic.

[a] not according to Dhruv!

The associated logarithmic structure

You can always fix a pre-logarithmic structure:



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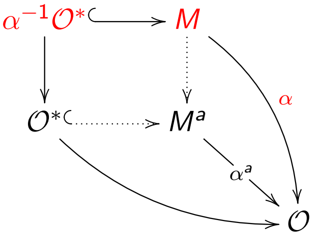

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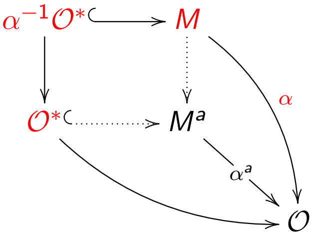

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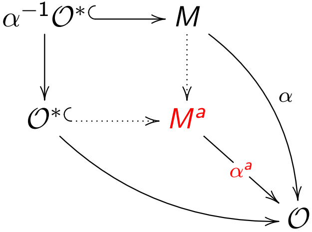

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This is particularly important for normal crossings divisors and toric divisors - these will be logarithmically smooth structures.

Example (Standard logarithmic point)

Let k be a field,

$$\begin{array}{ccc} \mathbb{N} \oplus k^\times & \to & k \\ (n, z) & \mapsto & z \cdot 0^n \end{array}$$

defined by sending $0 \mapsto 1$ and $n \mapsto 0$ otherwise.

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Works with P a monoid with $P^\times = 0$, giving the *P-logarithmic point*. This is what you get when you restrict the structure on an affine toric variety associated to P to the maximal ideal generated by $\{p \neq 0\}$.

Morphisms

A morphism of (pre)-logarithmic schemes $f : X \to Y$ consists of

- $\underline{f} : \underline{X} \to \underline{Y}$

$$\mathcal{O}_{\underline{X}} \xleftarrow{\underline{f}^\sharp} \underline{f}^{-1}\mathcal{O}_{\underline{Y}}$$

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A morphism of (pre)-logarithmic schemes $f : X \to Y$ consists of

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- A homomorphism $f^\flat$ making the following diagram commutative:

$$\begin{array}{ccc} M_X & \xleftarrow{f^\flat} & \underline{f}^{-1}M_Y \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ \mathcal{O}_{\underline{X}} & \xleftarrow{\underline{f}^\sharp} & \underline{f}^{-1}\mathcal{O}_{\underline{Y}} \end{array}$$

Definition (**Inverse image**)

Given $\underline{f} : \underline{X} \to \underline{Y}$ and $Y = (\underline{Y}, M_Y)$ define the *pre-logarithmic inverse image* by composing

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$X \to Y$ is **strict** if $M_X = \underline{f}^*M_Y$.

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- If N is the pushout of

$$\begin{array}{ccc} & \pi_Z^{-1} M_Z & \\ \swarrow & & \searrow \\ \pi_X^{-1} M_X & & \pi_Y^{-1} M_Y \end{array}$$

  then the log structure on $X \times_Z Y$ is defined by N^a.

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The morphism $\underline{f} : \operatorname{Spec}(R[P]) \to \operatorname{Spec}(\mathbb{Z}[P])$ gives

$$X = \underline{X} \times_{\underline{X_0}} X_0.$$

Charts

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Types of logarithmic structures

- We say that $(\underline{X}, M_X)$ is *coherent* if *étale locally* at every point there is a *finitely generated* monoid P and a local chart $P_X \to \mathcal{O}_{\underline{X}}$ for X.

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- We say that a logarithmic structure is *fine and saturated* (or fs) if it is coherent with local charts $P_X \to \mathcal{O}_X$ with P *integral and saturated*.

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The characteristic sheaf records the combinatorics of a logarithmic structure, especially for fs logarithmic structures.

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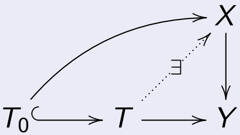

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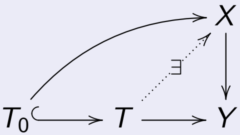

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The morphism is *logarithmically étale* if the lifting in (2) is unique.

Strict smooth morphisms

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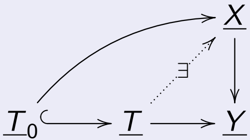

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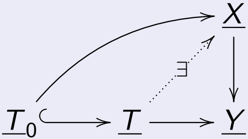

since $\underline{X} \to \underline{Y}$ smooth, and the lifting of morphism of monoids comes by the universal property of pullback. ♠

Combinatirially smooth morphisms

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Then $X \to Y$ *is logarithmically smooth.*

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Then $X \to Y$ *is logarithmically smooth.*
If also the cokernel is finite then $X \to Y$ *is logarithmically étale.*

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One direction:

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Deformations

Proposition (K. Kato)

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Sketch of proof: locally $X_0 \to X_0' \to Y_0$, where

$$X_0' = Y_0 \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P].$$

So $X_0' \to Y_0$ is combinatorially smooth, and automatically provided a deformation to

$$Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P],$$

and $X_0 \to X_0'$ is strict and smooth so deforms by the classical result.

Kodaira-Spencer theory

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Let Y_0 be artinian, $Y_0 \subset Y$ a strict square-0 extension with ideal J, and $f_0 : X_0 \to Y_0$ logarithmically smooth.

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Corollary

Logarithmically smooth curves are unobstructed.

Saturated morphisms

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[a] Has natural universal property

This guarantees that if $X \to Y$ is logarithmically smooth, then the fibers are reduced.

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- the fibers are curves i.e. pure dimension 1 schemes.

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Moduli of stable log curves

We define a category $\overline{\mathcal{M}}_{g,n}^{\log}$ of stable log curves: objects are log (g,n)-curves $X \to S$ and arrows are fiber diagrams compatible with sections

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There is a forgetful functor

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So $\overline{\mathcal{M}}_{g,n}^{\log}$ is a category fibered in groupoids over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Moduli of stable log curves (continued)

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Theorem (F. Kato)

$$\overline{\mathcal{M}}^{\log}_{g,n} \simeq (\overline{\mathcal{M}}_{g,n}, M_{\Delta_{g,n}}).$$

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- We claim that some stable log curves $X \to S$ are more fundamental than others.
- Not all logarithmic moduli problems are so lucky! There are issues with logarithmic $\mathbb{G}_m$ and logarithmic Picard (Molcho-Wise).

Minimality

Given a stable curve $\underline{X} \to \underline{S}$ we define

$$X^{\min} = \underline{X} \times_{\overline{\mathcal{M}}_{g,n+1}} \overline{\mathcal{M}}^{\log}_{g,n+1} \qquad \text{and} \qquad S^{\min} = \underline{S} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}^{\log}_{g,n}.$$

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We write

$$S^{\min} = (\underline{S}, M^{S}_{X/S}) \qquad \text{and} \qquad X^{\min} = (\underline{X}, M^{X}_{X/S}).$$

Fundamental diagram

$$\begin{array}{ccccc}
X & \longrightarrow & X^{\min} & \longrightarrow & \mathcal{X}_{g,n}^{\log} \\
\downarrow & & \downarrow & & \downarrow \\
S & \longrightarrow & S^{\min} & \longrightarrow & \overline{\mathcal{M}}_{g,n}^{\log} \\
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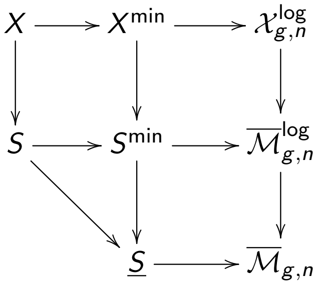
Fundamental diagram



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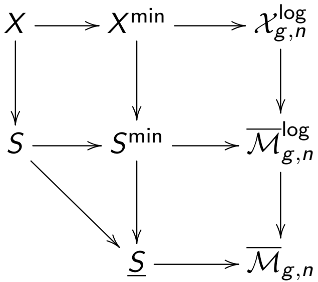

- $\overline{\mathcal{M}}_{g,n}^{\log}$ parametrizes stable log curves over $\mathfrak{LogSch}^{\text{fs}}$
- $\overline{\mathcal{M}}_{g,n}$ parametrizes minimal stable log curves over $\mathfrak{Sch}$.

Stable logarithmic maps

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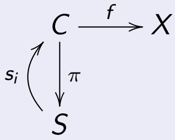

where

- $(C/S, s_i)$ is a prestable log curve, and
- in fibers $\mathrm{Aut}(\underline{C}_s \to \underline{X}, s_i)$ is finite.

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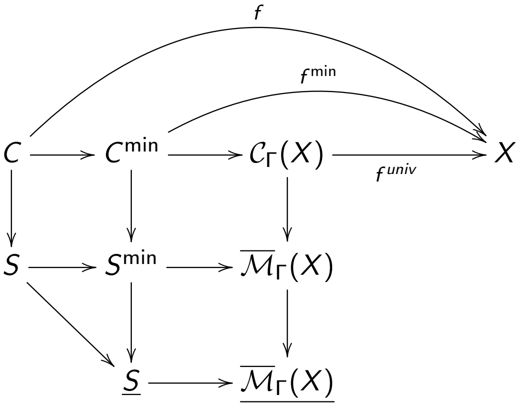
We collect the numerical data under the umbrella $\Gamma = (g, \beta, c_i)$.

Stable logarithmic maps (continued)

Theorem (Gross-Siebert, Chen, $\aleph$-Chen-Marcus-Wise)

Let X be projective logarithmically scheme. Stable logarithmic maps to X form a logarithmic Deligne–Mumford stack $\overline{\mathcal{M}}_\Gamma(X)$. It is finite and representable over $\overline{\mathcal{M}}_{\underline{\Gamma}}(\underline{X})$.

Fundamental diagram



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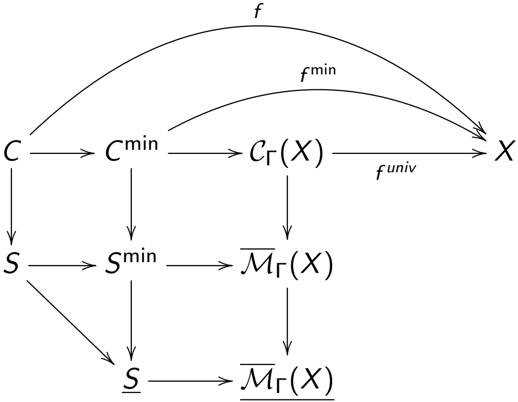

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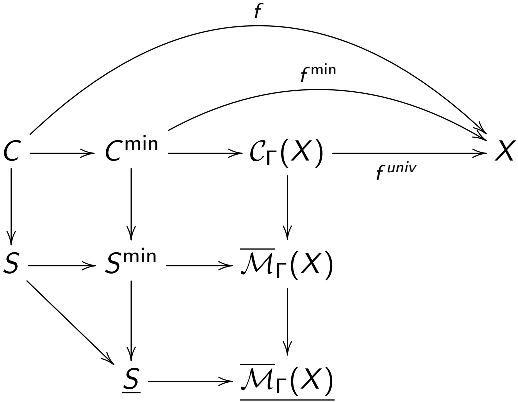

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As such it comes with a logarithmic structure $\overline{\mathcal{M}}_\Gamma(X)$ which parametrizes *all* stable logarithmic maps over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Stable logarithmic maps (continued)

This requires two steps:

- first find a morphism from $(C \to S, f : C \to X)$ to a *minimal* object $(C^{\min} \to S^{\min}, f^{\min} : C^{\min} \to X)$.
- then show that the object $(C^{\min} \to S^{\min}, f^{\min} : C^{\min} \to X)$ has a versal deformation space, $\underline{S} \to \underline{V}$, with universal family

$$(C_V^{\min} \to V^{\min}, f_V^{\min} : C_V^{\min} \to X),$$

whose fibers over any $\underline{T} \to \underline{V}$ are also minimal.

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$$\begin{array}{ccc} S_P & \longrightarrow & S \\ & \searrow & \vdots \\ & & S_Q \end{array}$$

Indeed if $S_P \to S$ is such a point and $S_P \to S_Q$ is its map to the minimal structure, then minimality says there is a unique $S \to S_Q$ factoring $S_P \to S_Q$

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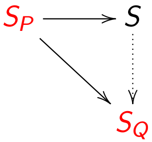

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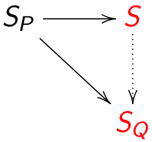

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At the generic points

The curve C has components C_i with generic points η_i corresponding to vertices in the dual graph, and nodes q_j with local equations $xy = g_j$ corresponding to edges in the dual graph.

- The map f sends η_i to some stratum X_i of X with cone σ_i having lattice $N_i \subset \sigma_i$.

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But the nodes impose crucial conditions.

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- At a node q with branches η_q^1, η_q^2 we similarly have a map

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$$P \oplus^{\mathbb{N}} \mathbb{N}^2 = P\langle \log x, \log y\rangle / (\log x + \log y = \rho_q)$$

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- The stalk at either η_q^1, η_q^2 is P.

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The map to the stalk P at η_q^2 where $y = 0$ sends $\log x \mapsto 0$, and so $\log y \mapsto \rho_q$.
This means that we have a monoid homomorphism, which is clearly injective,

$$P \oplus^{\mathbb{N}} \mathbb{N}^2 \to P \times P.$$

Its image is precisely the set of pairs

$$\{(p_1, p_2) | p_2 - p_1 \in \mathbb{Z}\rho_q\}$$

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Or better: we have $u_q : M_q \to \mathbb{Z}$ such that

$$(p_2 - p_1) \circ f_q^\flat(m) = u_q(m) \cdot \rho_q. \qquad (0.0.1)$$

Putting nodes and generic points together

The maps $p_1 \circ f_q^\flat : M_q \to P$ and $p_2 \circ f_q^\flat : M_q \to P$, since they come from maps of sheaves, are compatible with generization maps.

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the data of $p_1 \circ f_q^\flat$ and $p_2 \circ f_q^\flat$ is already determined by the data at the generic points η_i of the curve.

Putting nodes and generic points together (continued)

The *additional* data the node provides is the element $\rho_q \in P$ and homomorphism $u_q : M_q \to \mathbb{Z}$, in such a way that equation

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$$Q_f = \left(\left(\prod_\eta M_{\sigma_\eta} \times \prod_q \mathbb{N} \right) \Big/ R \right)^{sat}$$

where the saturated submonoid R is generated by all the relations implied by equation (0.0.2)

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It is quite a bit more natural to describe the dual lattice

$$Q_f^\vee = \left\{ ((v_\eta),(e_q)) \in \prod_\eta N_{\sigma_\eta} \times \prod_q \mathbb{N} \;\middle|\; \begin{array}{c} \forall \; \overset{\eta_q^1}{\bullet} \xrightarrow{\quad q \quad} \overset{\eta_q^2}{\bullet} \\ v_q^1 - v_q^2 = e_q u_q \end{array} \right\}.$$

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Theorem (Gross-Siebert)

The minimal object exists, with characteristic sheaf Q_f, dual to the lattice in the corresponding space of tropical curves.