LOGARITHMIC GEOMETRY AND MODULI LECTURES AT THE SOPHUS LIE CENTER JUNE 16-17, 2014

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Contents

1. Preamble

in which

we tell a story on how moduli spaces
lead to logarithmic geometry

1.1. **Moduli spaces.** At least since Riemann, we want to classify varieties. The first thing we do is fix some numerical invariants; for instance "dimension" and "genus", which lead to the classical case of moduli of algebraic curves. We expect varieties with fixed invariants, e.g. algebraic curves with fixed genus, to be parametrized by an algebraic variety - in the example, the moduli space M_g of curves of genus g is a quasi-projective algebraic variety (Ahlfors-Bers, Mumford).

But quasi-projective varieties are not necessarily projective.

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> *Working with a non-complete moduli space is like keeping change in a pocket with holes*
>
> Angelo Vistoli

1.2. **Compactification.** Deligne and Mumford provided a natural compactification $M_g \subset \overline{M}_g$ which is projective, and in fact "smooth", by allowing certain singular algebraic curves to be parametrized by the moduli space - these are stable curves.[1] The singularities are normal
1→ crossings: $xy = 0$.

What's so great about normal crossings?

1.3. **Differentials.** Consider a smooth family $\pi : X \to S$ of curves. The homomorphism $\pi^*\Omega^1_S \to \Omega^1_X$ has a locally free quotient $\Omega^1_{X/S}$. This is a sheaf theoretic manifestation of smoothness. Its rank, $rk(\Omega^1_{X/S}) = 1$ is a sheaf theoretic manifestation of the fact we are looking at curves.

If we now consider a family $\pi : X \to S$ of curves acquiring a node over $s \in S$, say with equation $xy = t$, then in $\Omega^1_{X/S}$ we have $0 = dt = d(xy) = x\,dy + y\,dx$. This implies that the nonzero section $x\,dy = -y\,dx$ of $\Omega^1_{X/S}$ is annihilated by both x, y, so it is torsion. This is a sheaf theoretic manifestation of non-smoothness. It focuses our attention on the divisor s in S and its inverse image $\{t = 0\} = Y_1 + Y_2 \subset X$.

Consider instead logarithmic differentials: $\Omega^1_S(\log(s))$ generated by $\frac{dt}{t}$; and $\Omega^1_X(\log(Y_1 + Y_2))$ generated by $\frac{dx}{x}$ and $\frac{dy}{y}$. Then $\frac{dt}{t} = \frac{dx}{x} + \frac{dy}{y}$, and the quotient $\Omega^1_{X/S,\log}$ is free, generated by $\frac{dx}{x} = -\frac{dy}{y}$.

It follows that from the point of view of logarithmic differentials, a node is as good as a smooth point.

More generally, for a normal crossings degeneration $t = x_1 \cdots x_k$, the sheaf $\pi^*\Omega^1_S(\log s)$, generated by

$$\frac{dt}{t} = \frac{dx_1}{x_1} + \cdots + \frac{dx_n}{x_n}$$

is a subbundle of $\Omega^1_X(\log(Y_1 + \ldots + Y_n))$, so the quotient is a vector bundle. Again:

> From the point of view of logarithmic differentials, a semistable degeneration is as good as a smooth family.

1.4. **The structure of a semistable variety.** Mouli spaces require looking at families, but also at the fibers of the families.

Consider the singular fiber of $X \to S$ above. Can one say that any such variety is "good"?

[1] (Dan) Add picture

Here is a problem: consier $X = E \times \mathbb{P}^1$, where E is an elliptic curve. Inside we have a section $D = \{0\} \times \mathbb{P}^1$. If we take the trivial family $X \times \mathbb{A}^1 \to \mathbb{A}^1$ and blow up $D \times 0$, the central fiber is $X \sqcup_D X$. One can deform one component to $X' = \mathbb{P}_E(\mathcal{O} \oplus L)$, where L is any line bundle L of degree 0 on E. The resulting variety cannot be smoothed. From the point of view of moduli spaces this is bad: near the moduli point of $X \sqcup_D X$ there is the point corresponding to X (good) but also points corresponding to $X' \sqcup_D X$, lying on a different and totally undesirable component (bad).

Logarithmic geometry, by some magic, knows not to deform the bad way! There is a structure on $X \to S$, called a logarithmic structure, which restricts nicely to the fiber, and the fiber *with this additional structure* can only deform in a good way.

2. Logarithmic structures

in which

our main characters are introduced

2.1. **Pre-logarithmic and logarithmic structures.** We define a *pre-logarithmic structure* to be

$$X = (\underline{X}, M \xrightarrow{\alpha} \mathcal{O}_{\underline{X}})$$

for which one usually uses the shorthand $X = (\underline{X}, M)$, where

- $\underline{X}$ is a scheme,
- M is a sheaf of monoids on X, and
- α is a monoid homomorphism, where the monoid structure on $\mathcal{O}_{\underline{X}}$ is the multiplicative structure.

Example 2.1.1.

- $(\underline{X}, \mathcal{O}^*_{\underline{X}} \hookrightarrow \mathcal{O}_{\underline{X}})$, the trivial structure. We sometimes write just $\underline{X}$ for this structure.
- $(\underline{X}, \mathcal{O}_{\underline{X}} \xrightarrow{\sim} \mathcal{O}_{\underline{X}})$, looks as easy but surprisingly not interesting, and
- $(\underline{X}, \mathbb{N} \xrightarrow{\alpha} \mathcal{O}_{\underline{X}})$, where α is determined by an arbitrary choice of $\alpha(1)$. This one is important but will need to be modified soon.

We define a *logarithmic structure* to be a pre-logarithmic structure in which the restriction $\alpha : \alpha^{-1}\mathcal{O}^*_{\underline{X}} \to \mathcal{O}^*_{\underline{X}}$ is an isomorphism. So the first two examples are logarithmic structures, and the last isn't.

A *logarithmic scheme* X is another name for a logarithmic structure on a scheme $\underline{X}$.

2.2. **Morphisms.** For a scheme $\underline{X}$ we define a morphism from a pre-logarithmic structure $M_1 \stackrel{\alpha_1}{\to} \mathcal{O}_{\underline{X}}$ to $M_2 \stackrel{\alpha_2}{\to} \mathcal{O}_{\underline{X}}$ to be

$$\begin{tikzcd} M_1 \arrow[rr, "\beta"] \arrow[dr, "\alpha_1"'] & & M_2 \arrow[dl, "\alpha_2"] \\ & \mathcal{O}_{\underline{X}} & \end{tikzcd}$$

This also defines a morphism of pre-logarithmic schemes $(\mathcal{X}, M_2) \stackrel{\beta}{\to} (\mathcal{X}, M_1)$ in a contravariant manner. The reason for doing it contravariantly is that a logarithmic structure is thought of as an extension of the structure sheaf, which behaves contravariantly in schemes.

2.3. **Key examples.**

2.3.1. *Divisorial logarithmic structure.* Let $\underline{X}, D \subset \underline{X}$ be a variety with a divisor. Define

$$M_D(U) = \left\{ f \in \mathcal{O}_{\underline{X}}(U) \;\middle|\; f_{U \smallsetminus D} \in \mathcal{O}^{\times}_{\underline{X}}(U \smallsetminus D) \right\}.$$

This is particularly important for normal crossings divisors and toric varieties - these will be logarithmically smooth structures.

2.3.2. *Logarithmic points.* Let k be a field, P a monoid with unique invertible element 0. Consider the map $P \to k$ sending $0 \mapsto 1$ and $p \mapsto 0$ otherwise, and take the resulting $M := P \oplus k^\times \stackrel{\alpha}{\to} k$.

This is what you get when you restrict the structure on an affine toric variety associated to P (denoted $\operatorname{Spec}(P \to k[P])$ below) to the maximal ideal generated by $\{p \neq 0\}$. It is called the *P-logarithmic point.* When $P = \mathbb{N}$ it is known as the *standard logarithmic point.*

2.4. **The associated logarithmic structure.** A morphism of logarithmic structures is a morphism of pre-logarithmic structures which happen to be logarithmic structures. So we have a fully faithful embedding

$$\{\text{logarithmic structures on } \underline{X}\} \hookrightarrow \{\text{pre-logarithmic structures on } \underline{X}\}.$$

Theorem 1 (K. Kato)**.** *This has an adjoint - given $M \stackrel{\alpha}{\to} \mathcal{O}_{\underline{X}}$ there is a logarithmic structure $M^a \xrightarrow{\alpha^a} \mathcal{O}_{\underline{X}}$, the* associated logarithmic structure*,* *and a universal morphism*

$$\begin{tikzcd}
M \arrow[rr, "\beta"] \arrow[dr, "\alpha"'] & & M^a \arrow[dl, "\alpha^a"] \\
& \mathcal{O}_{\underline{X}} &
\end{tikzcd}$$

Proof sketch: this is a three step exercise.

STEP 1: define the pushout of a diagram of monoids:

$$\begin{tikzcd}
& M_1 \arrow[dr, dotted] & \\
M_0 \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & M_3 \\
& M_1 \arrow[ur, dotted] &
\end{tikzcd}$$

by $M_3 = M_1 \oplus M_2/\sim$, with the equivalence generated by $(m_1 + f_1(x), m_2) = (m_1, m_2 + f_2(x))$.

STEP 2: show that if f_1 is injective then $M_2 \to M_3$ is injective.

STEP 3: given a pre-logarithmic structure $\alpha : M \to \mathcal{O}$ use the diagram

$$\begin{tikzcd}
\alpha^{-1}\mathcal{O}^* \arrow[r, hook] \arrow[d] & M \arrow[d, dotted] \arrow[ddr, bend left, "\alpha"] & \\
\mathcal{O}^* \arrow[r, hook, dotted] \arrow[drr, bend right] & M^a \arrow[dr, "\alpha^a"] & \\
& & \mathcal{O}
\end{tikzcd}$$

Check that it is a logarithmic structure and is universal. ♣

Note that this gives $(\underline{X}, M^a) \to (\underline{X}, M)$.

2.5. **Inverse image.** Given $f : \underline{X} \to \underline{Y}$ and $Y = (\underline{Y}, M_Y)$ define the pre-logarithmic inverse image by composing

$$f^{-1}M_Y \to f^{-1}\mathcal{O}_{\underline{Y}} \xrightarrow{f^\sharp} \mathcal{O}_{\underline{X}}$$

and then the logarithmic inverse image is

$$f^*M_Y = (f^{-1}M_Y)^a.$$

This is the universal logarithmic structure on $\underline{X}$ with a diagram

$$\begin{tikzcd}
(\underline{X}, f^*M_Y) \arrow[r] \arrow[d] & Y \arrow[d] \\
\underline{X} \arrow[r] & \underline{Y}
\end{tikzcd}$$

2.6. **Morphisms of logarithmic schemes.** A morphism $X \to Y$ over a given $\underline{f} : \underline{X} \to \underline{Y}$ is finally defined to be a morphism of logarithmic structures $f^*M_Y \to M_X$.

2.7. **Fibered products.** The fibered product $X \times_Z Y$ exists, and is defined as follows:

- $\underline{X \times_Z Y} = \underline{X} \times_{\underline{Z}} \underline{Y}$
- If N is the pushout of

$$\begin{tikzcd}
& \pi_Z^{-1}M_Z \arrow[dl] \arrow[dr] & \\
\pi_X^{-1}M_X & & \pi_Y^{-1}M_Y
\end{tikzcd}$$

then the log structure on $X \times_Z Y$ is defined by N^a.

2.8. **Monoid algebras.** Let P be a monoid, R a ring. We obtain a monoid algebra $R[P]$ and a scheme $\underline{X} = \operatorname{Spec} R[P]$. There is an evident monoid homomorphism $P \to R[P]$ inducing sheaf homomorphism $P_X \to \mathcal{O}_{\underline{X}}$, a pre-logarithmic structure, giving rise to a logarithmic structure

$$(P_X)^a \to \mathcal{O}_{\underline{X}}.$$

This is a basic example. It deserves a notation:

$$X = \operatorname{Spec}(P \to R[P]).$$

The most basic example is $X_0 = \operatorname{Spec}(P \to \mathbb{Z}[P])$. The morphism $f : \operatorname{Spec}(R[P]) \to \operatorname{Spec}(\mathbb{Z}[P])$ gives

$$X = \underline{X} \times_{\underline{X_0}} X_0.$$

2.9. **Charts.** Logarithmic structures X are particularly manageable if one has a constant sheaf P_X and a homomorphism $P_X \to \mathcal{O}_{\underline{X}}$ such that X is the associated logarithmic structure:

A *chart* for M_X is a monoid homomorphism $P \to \Gamma(X, M_X)$ such that $P^a \to M_X$ is an isomorphism, where P^a is associated to $P_X \to \mathcal{O}_{\underline{X}}$ coming from $P \to \Gamma(M_X) \to \Gamma(\mathcal{O}_X)$.

A *local chart* is of course a chart on an open U of X.

Note that giving a chart is equivalent to giving a logarithmic morphism
$$X \to \mathrm{Spec}(P \to \mathbb{Z}[P])$$
such that $f^\flat$ is an isomorphism.

Also note:
$$Hom_{LogSch}(X, \mathrm{Spec}(P \to \mathbb{Z}[P])) \to Hom_{Mon}(P, \Gamma(X, M_X))$$
is canonically bijective.

Let us now look at charts for morphisms:

Given a morphism of logarithmic schemes $f : X \to Y$, a chart for f is a triple
$$(P_X \to M_X, Q_Y \to M_Y, Q \to P)$$
such that

(1) $P_X \to M_X$ and $Q_Y \to M_Y$ are charts for M_X and M_Y, and
(2) the diagram
$$\begin{tikzcd} Q_X \arrow[r] \arrow[d] & f^{-1}M_Y \arrow[d] \\ P_X \arrow[r] & M_X \end{tikzcd}$$
is commutative.

2.10. **Types of logarithmic structures.** We say that $(\underline{X}, M_X)$ is *quasi coherent* if *étale locally* at every point there is a monoid P and a local chart $P_X \to \mathcal{O}_{\underline{X}}$ for X.

We say that $(\underline{X}, M_X)$ is *coherent* if such a monoid P can be taken *finitely generated* - the image of $\mathbb{N}^k$.

A monoid P is *integral* if $P \to P^{\mathrm{gp}}$ is injective.

It is *saturated* if integral and whenever $p \in P^{\mathrm{gp}}$ and $m \cdot p \in P$ for some integrer $m > 0$ then $p \in P$. I.e., not like $\{0, 2, 3, \dots\}$.

We say that a logarithmic structure is *fine* if it is *coherent* with local charts $P_X \to \mathcal{O}_X$ with P *integral*.

We say that a logarithmic structure is *fine and saturated* (or fs) if it is coherent with local charts $P_X \to \mathcal{O}_X$ with P *integral and saturated*.

We say that a logarithmic structure is *locally free* if it is coherent with local charts $\mathbb{N}^k \to \mathcal{O}_X$ (where k depends on the chart).

Almost all work is done with fine logarithmic schemes, and much of that is restricted to fs logarithmic schemes. General logarithmic schemes are used as a tool.

For instance, if X, Y, Z are coherent then $X \times_Z Y$ is still coherent.[2] ←2

[2](Dan) exercises fine, saturated

2.11. **The characteristic sheaf.** Given a logarithmic structure $X = (\underline{X}, M)$, the quotient sheaf $\overline{M} := M/\mathcal{O}_{\underline{X}}^\times$ is called the *characteristic sheaf* of X.

3. Differentials

in which
Grothendieck's formalism is adapted
to logarithmic structures

3.1. **How to deform a point on a scheme (Grothendieck).**

3.1.1. Ingredients: $T_0 = \operatorname{Spec} \mathbb{C}$, $T = \operatorname{Spec} \mathbb{C}[\epsilon/\epsilon^2]$, $f : X \to Y$
Diagram:

$$\begin{tikzcd} & & X \arrow[d] \\ T_0 \arrow[r, hook] \arrow[rru, bend left] & T \arrow[r] \arrow[ru, dotted] & Y \end{tikzcd}$$

If Y is a point, or more generally if $T \to Y$ factors through the retraction $T \to T_0$, then there is a lift $g_0 : T \to X$. But anyway let us assume a lift g_0 exists and analyze possible other lifts g_1:

$$\begin{tikzcd} & & X \arrow[d] \\ T_0 \arrow[r, hook] \arrow[rru, bend left, "\phi"] & T \arrow[r, "\psi"'] \arrow[ru, dotted, "g_1"] \arrow[ru, bend right, "g_0"'] & Y. \end{tikzcd}$$

This translates to a diagram of groups:

$$\begin{tikzcd} \mathbb{C} \arrow[d, leftarrow] & \phi^{-1}\mathcal{O}_X \arrow[l] \arrow[dl, "g_0^\sharp"'] \arrow[dl, dotted, bend left, "g_1^\sharp"] \\ \mathbb{C}[\epsilon] & \psi^{-1}\mathcal{O}_Y \arrow[l] \arrow[u] \end{tikzcd}$$

The difference $g_1^\sharp - g_0^\sharp$ is a map $\phi^{-1}\mathcal{O}_X \xrightarrow{d} \epsilon\mathbb{C} \simeq \mathbb{C}$. Since $g_i^\sharp$ are ring homomorphisms d is linear. Moreover $d(f_1 f_2) = f_2 df_1 + f_1 df_2$: [3]

3→

[3] (Dan) exercise

$$\begin{aligned} d(f_1f_2) =& g_1(f_1f_2) - g_0(f_1f_2) \\ =& \;\; g_1(f_1)g_1(f_2) - \boxed{g_0(f_1)g_1(f_2)} \\ & + \boxed{g_0(f_1)g_1(f_2)} - g_0(f_1)g_0(f_2) \\ =& g_1(f_2)df_1 + g_0(f_1)df_2 \\ =& f_2df_1 + f_1df_2 \end{aligned}$$

This is a derivation $\phi^{-1}\mathcal{O}_X \to \mathbb{C}$.

3.1.2. Remove assumptions: T_0 any scheme, $T = T_0[J]$ a square-0 extension. Get instead a derivarion

$$\phi^{-1}\mathcal{O}_X \to J.$$

3.1.3. Can further allow $\mathrm{Ker}(\mathcal{O}_T \to \mathcal{O}_{T_0})$ to be an $\mathcal{O}_{T_0}$-module J. Get again a derivarion $\phi^{-1}\mathcal{O}_X \to J$.

3.1.4. The *universal derivation* $d : \mathcal{O}_X \to \Omega^1_{X/Y}$ occurs when $T_0 = X$, $T = X' := \mathrm{Spec}_{X\times_Y X}\, \mathcal{O}/(I_\Delta)^2$:

$$\begin{tikzcd} & & X \arrow[d] \\ X \arrow[r, hook] \arrow[urr, bend left, "Id"] & X' \arrow[r] \arrow[ur, dotted, "?"] & Y. \end{tikzcd}$$

Here $\Omega^1_{X/Y}$ is the $\mathcal{O}_X$-module generated by symbols df modulo all required relations.

3.2. **How to deform a point on a logarithmic scheme (Kato).**

3.2.1. We upgrade the data: a morphism $T_0 \to T$ is said to be *strict* if $T_0 = \underline{T}_0 \times_{\underline{T}} T$. A *strict closed embedding* is a strict morphism which is a closed embedding on underlying schemes. A *strict square-0 embedding* is a strict morphism which is a square-0 embedding on underlying schemes:

$$0 \to J \to \mathcal{O}_{\underline{T}} \to \mathcal{O}_{\underline{T}_0} \to 0.$$

Note that we obtain an exact sequence of multiplicative groups

$$1 \to (1+J) \to \mathcal{O}^\times_{\underline{T}} \to \mathcal{O}^\times_{\underline{T}_0} \to 1.$$

This induces an exact sequence of monoids

$$1 \to (1+J) \to M_T \to M_{T_0} \to 1,$$

in the sense that the group $1+J$ acts freely on M_T with quotient M_{T_0}.

3.2.2. The diagram

$$\begin{tikzcd}
 & & X \arrow[d] \\
T_0 \arrow[r, hook] \arrow[urr, bend left, "\phi"] & T \arrow[r, "\psi"'] \arrow[ur, bend right, "g_0"'] \arrow[ur, dotted, "g_1"] & Y
\end{tikzcd}$$

induces two diagrams. First the familiar

$$\begin{tikzcd}
\mathcal{O}_{\underline{T}_0} & \phi^{-1}\mathcal{O}_{\underline{X}} \arrow[l] \arrow[dl, bend right, "g_0^\sharp"'] \arrow[dl, dotted, "g_1^\sharp"] \\
\mathcal{O}_{\underline{T}} \arrow[u] & \psi^{-1}\mathcal{O}_{\underline{Y}} \arrow[l] \arrow[u]
\end{tikzcd}$$

which induces the familiar derivation $\phi^*\mathcal{O}_{\underline{X}} \to J$. In addition we obtain:

$$\begin{tikzcd}
M_{T_0} & \phi^{-1}M_X \arrow[l, "\phi^\flat"'] \arrow[dl, bend right, "g_0^\flat"'] \arrow[dl, dotted, "g_1^\flat"] \\
M_T \arrow[u] & \psi^{-1}M_Y \arrow[l] \arrow[u]
\end{tikzcd}$$

We define $D : \phi^{-1}M_X \to J$ by the equation

$$g_1^\flat(m) = (1 + D(m)) + g_2^\flat(m).$$

It is well defined. Also it is a monoid homomorphism: $D(m_1+m_2) = D(m_1) + D(m_2)$, since it measures the difference, in a group, between two monoid homomorphisms.

Key properties:

(1) $D|_{\psi^{-1}M_Y} = 0$
(2) $\alpha(m)D(m) = d(\alpha(m))$,

in other words,

$$D(m) = d\log(\alpha(m)),$$

which justifies the name of the theory.

I have checked the second property. I refuse to reproduce it here.

3.3. **Logarithmic derivations.** The discussion above justifies defining a *logarithmic derivation in* J to be a pair (d, D) where $d : \mathcal{O}_X \to J$ is a derivation and $D : M_X \to J$ a monoid homomorphism satisfying (1) and (2). The collection of all logarithmic derivations form the logarithmic tangent sheaf $T_{X/Y}$. Better still, there exists a module $\Omega^1_{X/Y}$ and a universal logarithmic derivation (d, D) in $\Omega^1_{X/Y}$. It is the module generated by symbols df and $D(m)$ subject to the conditions above:

$$\Omega^1_{X/Y} = (\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O}_{\underline{X}} \otimes M_X^{\mathrm{gp}}))/\mathcal{K},$$

where $\mathcal{K}$ is the collection of necessary relations coming from (1) and (2).

Examples: compute the cases of DNC, toric, log point.

4. Logarithmic smoothness

in which

the plot thickens

4.1. Smoothness and logarithmic smoothness. Recall from Grothendieck that a morphism of schemes $X \to Y$ is *smooth* if

(1) it is locally of finite presentation, and
(2) Whenever T_0 is affine and $T_0 \subset T$ a square-0 embedding and any diagram

$$\begin{tikzcd} & & X \arrow[d] \\ T_0 \arrow[r, hook] \arrow[urr, bend left] & T \arrow[r] \arrow[ur, dotted, "\exists"] & Y \end{tikzcd}$$

there exists a lifting as indicated.

The morphism is *étale* if the lifting in (2) is unique.

We define a morphism $X \to Y$ of *fine logarithmic schemes* to be *logarithmically smooth* if

(1) $\underline{X} \to \underline{Y}$ is locally of finite presentation, and
(2) Whenever a fine logarithmic scheme T_0 is affine and $T_0 \subset T$ a strict square-0 embedding and given a diagram of *logarithmic* schemes

$$\begin{tikzcd} & & X \arrow[d] \\ T_0 \arrow[r, hook] \arrow[urr, bend left] & T \arrow[r] \arrow[ur, dotted, "\exists"] & Y \end{tikzcd}$$

there exists a lifting as indicated.

The morphism is *logarithmically étale* if the lifting in (2) is unique.

The value of the assumption that everything be fine will remain a bit mysterious. It is an indication that logarithmic schemes which are not fine are a bit of a problem. The characterization below will simply be false otherwise.

Remark: the assumption that T_0 is affine is a way to say that for whatever T_0, a lifting exists *locally on* T. One direction is evident. For the other: if T is affine and there are local liftings, the differences are sections of sheaves of derivations $LogDer(\phi^{-1}\mathcal{O}_X, J)$ which are quasi coherent on an affine T so cohomology vanishes.

Remark: For étale morphisms uniqueness means that "existence locally" implies "existence": unique liftings must coincide on overlaps. Maps which coincide on overlaps glue together.

4.2. **Characterization of logarithmic smoothness.**

Principle: logarithmic smoothness is characterized by (1) a collection of basic cases and (2) pullbacks and covers.

4.2.1. *Strict smooth morphisms:* if $X \to Y$ is strict and $\underline{X} \to \underline{Y}$ smooth then $X \to Y$ is logarithmically smooth. Indeed there is a lifting

$$\begin{tikzcd}
 & & \underline{X} \arrow[d] \\
\underline{T}_0 \arrow[r, hook] \arrow[urr, bend left] & \underline{T} \arrow[r] \arrow[ur, dotted, "\exists"] & \underline{Y}
\end{tikzcd}$$

since $\underline{X} \to \underline{Y}$ smooth, and the lifting of morphism of monoids comes by the universal property of pullback.

4.2.2. *Combinatorial morphisms.*

Proposition 1. *Say P, Q are finitely generated* integral *monoids, R a ring, $Q \to P$ a monoid homomorphism.*

Write $X = \mathrm{Spec}(P \to R[P])$ and $Y = \mathrm{Spec}(Q \to R[Q])$.

Assume

(1) $\mathrm{Ker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ *is finite and with order invertible in R,*
(2) $\mathrm{TorCoker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ *has order invertible in R.*

4→ *Then $X \to Y$ is logarithmically smooth.*[4]

If also the cokernel is finite then $X \to Y$ is logarithmically étale.

Idea in proof: first, it is enough to lift the maps of monoids in the lifting diagram.

We use the sequence

$$1 \to (1 + J) \to M_T \to M_{T_0} \to 1.$$

[4] (Dan) indicate proof in notes?

We note that by integrality $M_{T_0}^{\mathrm{gp}} = M_T^{\mathrm{gp}}/(1+J)$, and there is a cartesian diagram

$$\begin{tikzcd} M_T \arrow[r] \arrow[d] & M_T/(1+J) = M_{T_0} \arrow[d] \\ M_T^{\mathrm{gp}} \arrow[r] & M_T^{\mathrm{gp}}/(1+J) = M_{T_0}^{\mathrm{gp}}. \end{tikzcd} \tag{1}$$

A lifting on the group level

$$\begin{tikzcd} M_{T_0}^{\mathrm{gp}} & P^{\mathrm{gp}} \arrow[l] \arrow[dl, dotted, "\exists"] \\ M_T^{\mathrm{gp}} \arrow[u] & Q^{\mathrm{gp}} \arrow[l] \arrow[u] \end{tikzcd}$$

exists by kernel/cokernel assumptions and diagram chasing. The cartesian diagram (1) shows that the lifting exists on the monoid level.

4.2.3. *Characterization.*

Theorem 2 (K. Kato). *Let X, Y be fine logarithmic schemes, and let $Q_Y \to M_Y$ be a chart. Then a morphism $X \to Y$ is logarithmically smooth if and only if étale locally on X there is a chart $P_X \to M_X$ and a monoid homomorphism $Q \to P$, which, together with $Q_Y \to M_Y$ makes a chart for $X \to Y$ such that*

1. *$Q \to P$ satisfies the conditions of the proposition for smoothness, and*
2. *$X \to Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]$ is smooth*

For the proof, see Kato. It uses the fact that $\Omega^1_{X/Y}$ is coherent locally free, but more.

4.3. **Key examples.**

1. Nodal curves.
2. Marked nodal curves.
3. Toric varieties
4. $\operatorname{Spec}\mathbb{C}[t] \to \operatorname{Spec}\mathbb{C}[s]$ given by $s = t^2$
5. $\operatorname{Spec}\mathbb{C}[x,y] \to \operatorname{Spec}\mathbb{C}[t]$ given by $t = x^m y^n$
6. $\operatorname{Spec}\mathbb{C}[x,y] \to \operatorname{Spec}\mathbb{C}[x,z]$ given by $z = xy$.
7. $\operatorname{Spec}(\mathbb{N} \to \mathbb{C}[\mathbb{N}]) \to \operatorname{Spec}(\mathbb{N}_{\smallsetminus 1} \to \mathbb{C}[\mathbb{N}_{\smallsetminus 1}])$.

Exercise. Show by explicit calculation that these are logarithmically smooth.

4.4. **Integral morphisms.** We note a disturbing feature: the last two examples are not flat. It is of interest to delineate those logarithmically smooth morphisms which are flat.

We define a monoid homomorphism $Q \to P$ to be *integral* if

$$\mathbb{Z}[Q] \to \mathbb{Z}[P]$$

is flat.

A morphism $f : X \to Y$ of logarithmic schemes is *integral* if for every geometric point x of X the homomorphism

$$(f^{-1}\overline{M}_Y)_x \to (\overline{M}_X)_x$$

of characteristic sheaves is integral.

Equivalently, there is a chart with $Q \to P$ integral.

Remark 4.4.1. This implies that $X \to Y$ is universally integral. $h : Q \to P$ is integral if for every integral Q' and homomorphism $Q \to Q'$ the pushout $P' = P \oplus^Q Q'$ is integral.

There is an explicit criterion for integrality: say $a_1, a_2 \in Q$ and $b_1, b_2 \in P$ satisfy $h(a_1) + b_1 = h(a_2) + b_2$. Then in fact there are $a_3, a_4 \in Q$ and $b \in P$ so that $b_1 = h(a_3) + b$ and $b_2 = h(a_4) + b$, and $a_1 + a_3 = a_2 + a_4$.

Exercise 4.4.2. $P = \mathbb{N}$, $Q = \mathbb{N} \smallsetminus 1$, show that $P \oplus^Q P$ is not integral. Show that the criterion fails for $a_1 = 2, a_2 = 3, b_1 = 1, b_2 = 0$. Show that the blowup example of $z = xy$ is not integral.

Remark 4.4.3. Later we'll worry about saturated morphisms

5. Logarithmically smooth deformations

in which

the thickening is plotted

5.1. **Local deformations.** Recall that if $\underline{X}_0 \to \underline{Y}_0$ is smooth, $\underline{Y}_0 \subset \underline{Y}$ a square-0 (or artinian) extension, then locally $\underline{X}_0$ can be lifted to a smooth $\underline{X} \to \underline{Y}$.

Proposition 2 (K. Kato)**.** *The same is true for logarithmically smooth deformations.*

Sketch of proof: locally $X_0 \to X_0' \to Y_0$, where

$$X_0' = Y_0 \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P].$$

So $X_0' \to Y_0$ is combinatorially smooth, and automatically provided a deformation to

$$Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P],$$

and $X_0 \to X_0'$ is strict and smooth so deforms by the classical result.

5.2. Kodaira-Spencer theory.

Theorem 3 (K. Kato)**.** *Let Y_0 be artinian, $Y_0 \subset Y$ a strict square-0 extension with ideal J, and $f_0 : X_0 \to Y_0$ logarithmically smooth. Then*

(1) *There is a canonical element $\omega \in \mathrm{Ext}^2(\Omega^1_{X_0/Y_0}, f_0^* J)$ such that a logarithmically smooth deformation $X \to Y$ exists if and only if $\omega = 0$.*
(2) *If $\omega = 0$, then isomorphism classes of such $X \to Y$ correspond to elements of a torsor under $\mathrm{Ext}^1(\Omega^1_{X_0/Y_0}, f_0^* J)$.*
(3) *Given such deformation $X \to Y$, its automorphism group is $\mathrm{Hom}(\Omega^1_{X_0/Y_0}, f_0^* J)$.*

Sketch of proof: cover X_0 by affines $U_0^{(i)}$, and on each $U_0^{(i)}$ fix a deformation $U^{(i)} \to Y$. As we have seen, all such lifts are isomorphic.

To glue, we need to choose isomorphisms $U^{(i,j)} \to U^{(j,i)}$ - these lie in a torsor under $\mathrm{Hom}(\Omega^1_{U_0^{(j,i)}/Y_0}, f_0^* J)$. Such isomorphisms patch if they are in agreement on $U^{(j,i,k)}$.

Corollary 5.2.1. *Logarithmically smooth curves are unobstructed.*

5.3. **Saturated morphisms.** Recall that the monoid homomorphism $\mathbb{N} \xrightarrow{\cdot 2} \mathbb{N}$ gives an integral logarithmically étale map with non-reduced fibers. An integral $Q \to P$ of saturated monoids is said to be *saturated* if $\mathrm{Spec}(P \to \mathbb{Z}[P]) \to \mathrm{Spec}(Q \to \mathbb{Z}[Q])$ has reduced fibers. An integral morphism $X \to Y$ of fs logarithmic schemes is *saturated* if it has a saturated chart. This guarantees that if $X \to Y$ is logarithmically smooth, then he fibers are reduced. It also guarantees that the morphism is universally saturated.

6. Interlude: moduli of curves

in which

a crash course on the moduli of stable
curves is provided

We describe the Deligne–Mumford–Knudsen theory. No logarithmic structures here.

6.1. **Prestable curves.** A *prestable n-marked curve C/S* is a flat, proper morphism with connected reduced fibers of dimension 1, along with disjoint sections $s_i : S \to C$ for $i = 1, \dots, n$ in the smooth locus of C/S. We require all fibers have at most nodes as singularities.

We denote by p_i the images of s_i.

6.2. Stable curves.

6.2.1. A prestable curve C/S is *stable* if for every geometric fiber the automorphism group $\mathrm{Aut}(C_0, p_1, \ldots, p_n)$ is finite.

6.2.2. A prestable curve C/S is *stable* if for every irreducible component C' of the normalization C^ν of a geometric fiber

1. If $C' \simeq \mathbb{P}^1$ then C' contains at least 3 special points: marked points or points mapping to nodes.
2. If $g(C') = 1$ then C' contains at least 1 special point.

6.2.3. A prestable curve C/S is *stable* if $\omega_{C/S}(\sum p_i)$ is π-ample.

Proposition 3. *All three definitions coincide*

6.3. Moduli of stable curves.

Theorem 4 (Deligne–Mumford–Knudsen)**.** *Stable curves form a proper, smooth Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}$ over $\mathbb{Z}$ with projective coarse moduli space. The universal curve is $\overline{\mathcal{M}}_{g,n+1}$.*

Brief sketch of a proof:

1. All stable n-pointed curves of genus g are uniformly embeddable in $\mathbb{P}^N$ by the 3-log-canonical system $(\omega_C(\sum p_i))^{\otimes 3}$.
2. The corresponding Hilbert scheme $Hilb_{g,6g-6+3n,N}$ contains point scorresponding to all such embeddings.
3. The locus of 3-log-canonically embedded curves is a locally closed subscheme H^0 of $Hilb_{g,6g-6+3n,N}$.
4. The ambiguity in choosing the embedding is accounted precisely by the action of PGL_{N+1} on H^0.
5. The quotient $\overline{M}_{g,n} = H^0/PGL_{N+1}$ exists as a projective scheme (GIT).
6. The quotient $\overline{\mathcal{M}}_{g,n} = [H^0/PGL_{N+1}]$ exists as a Deligne–Mumford stack.
7. It is a beautiful argument of Knudsen that the universal curve is indeed $\overline{\mathcal{M}}_{g,n+1}$

7. Log curves

in which

log curves are defined and characterized, and their moduli stack is identified

Definition 7.0.1. A *log curve* is a morphism $f : X \to S$ of fs logarithmic schemes satisfying:

(1) f is logarithmically smooth
(2) f is integral i.e. flat
(3) f is saturated i.e. reduced fibers
(4) the fibers are curves i.e. pure dimension 1 schemes.

Theorem 5 (F. Kato). *Assume $\pi : X \to S$ is a log curve. Then*

(1) Fibers have at most nodes as singularities
(2) étale locally on S we can choose disjoint sections $s_i : \underline{S} \to \underline{X}$ in the nonsingular locus $\underline{X}_0$ of $\underline{X}/\underline{S}$ such that
  (a) Away from s_i we have that $X^0 = \underline{X}_0 \times_{\underline{S}} S$, so π is strict away from s_i
  (b) Near each s_i we have a strict étale

$$X^0 \to S \times \mathbb{A}^1$$

  with the standard divisorial logarithmic structure on $\mathbb{A}^1$.
  (c) étale locally at a node $xy = f$ the log curve X is the pullback of

$$\mathrm{Spec}(\mathbb{N}^2 \to \mathbb{Z}[\mathbb{N}^2]) \to Spec(\mathbb{N} \to \mathbb{Z}[\mathbb{N}])$$

  where $\mathbb{N} \to \mathbb{N}^2$ is the diagonal. Here the image of $1 \in \mathbb{N}$ in $\mathcal{O}_S$ is f and the generators of $\mathbb{N}^2$ map to x and y.

Sketch of a proof. It is easy to see that the situations described above satisfy K. Kato's criteria for logarithmic smoothness. We need to show the converse.

We consider the local factorization $X \to X' \to S$, where $X \to X'$ is strict and smooth and $X' \to S$ is of combinatorial type. If $X \to X'$ is of relative dimension 1 there isn't much space for anything, so we get case (2a).

Otherwise we may replace X by X' and use the fact that everything is saturated and combinatorial to reduce to the case of toric varieties X and S where $\dim X = \dim S + 1$.

The fact that the map is integral means in particular that any cone in the fan of X maps onto a cone in the fan of S.

The fact that the map is saturated means that the map of lattices of co-characters in each cone is surjective.

If there is a ray in a cone of X which maps to 0, we are in case (2b).

Otherwise one pulls back to a curve S_1 in S and obtain a tori surface $X_1 = X \times_S S_1$. The case of toric surfaces is well understood.

7.1. **Stable log curves.** A stable log curve $X \to S$ is:

(1) a log curve $X \to S$,

(2) sections $s_i : \underline{S} \to \underline{X}$ for $i = 1, \ldots, n$,

such that

(1) $(\underline{X} \to \underline{S}, s_i)$ is stable,
(2) the log structure is strict away from sections and singularities of fibers, and "divisorial along the sections".

7.2. **Moduli.** We define a category $\overline{\mathcal{M}}_{g,n}^{\log}$ of stable log curves: objects are log curves $X \to S$ and arrows are fiber diagrams

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S_2 & \longrightarrow & S_2 \end{array}.$$

There is a forgetful functor

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}^{\log} & \longrightarrow & \mathfrak{LogSch}^{\mathrm{fs}} \\ (X \to S) & \mapsto & S. \end{array}$$

So $\overline{\mathcal{M}}_{g,n}^{\log}$ is a category fibered in groupoids over $\mathfrak{LogSch}^{\mathrm{fs}}$.

and also

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}^{\log} & \longrightarrow & \overline{\mathcal{M}}_{g,n} \\ (X \to S) & \mapsto & (\underline{X} \to \underline{S}) \end{array}$$

Note that the moduli stack $\overline{\mathcal{M}}_{g,n}$ has a natural logarithmic smooth structure $M_{\Delta_{g,n}}$ given by the boundary divisor - this is a bit of a furunate situation which does not repeat in other cases. As such it represents a category fibered in groupoids $(\overline{\mathcal{M}}_{g,n}, M_{\Delta_{g,n}})$ over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Theorem 6 (F. Kato)**.** $\overline{\mathcal{M}}_{g,n}^{\log} \simeq (\overline{\mathcal{M}}_{g,n}, M_{\Delta_{g,n}})$.

Sketch of a proof. First, it is not hard to show that the universal family $\mathcal{C}_{g,n} = \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ with its boundary logarithmic structure is a log curve, so indeed we have a morphism $\overline{\mathcal{M}}_{g,n}^{\log} \leftarrow (\overline{\mathcal{M}}_{g,n}, M_{\Delta_{g,n}})$.

Also the forgetful morphism $\overline{\mathcal{M}}_{g,n}^{\log} \longrightarrow \overline{\mathcal{M}}_{g,n}$ gives, for each log curve $X \to S$, a canonical morphism $\underline{S} \to \overline{\mathcal{M}}_{g,n}$ and $\underline{X} \to \overline{\mathcal{M}}_{g,n+1}$. We define

$$X^{\min} = \underline{X} \times_{\overline{\mathcal{M}}_{g,n+1}} \overline{\mathcal{M}}_{g,n+1}^{\log} \qquad \text{and} \qquad S^{\min} = \underline{S} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}^{\log}.$$

Lemma 7.2.1. *The morphism $S \to \underline{S}$ lifts uniquely to $S \to S^{\min}$, and similarly $X \to X^{\min}$. Moreover $X = S \times_{S^{\min}} X^{\min}$.*

Sketch of proof. It is enough to work étale locally, so we may replace $\overline{\mathcal{M}}_{g,n}$ by a versal deformation space V which is étale over

$\operatorname{Spec} k[t_1, \ldots, t_{3g-3+n}]$, where $t_1, \ldots, t_m$ are equations of boundary divisors defining nodes $q_1, \ldots, q_m$, and we may assume a given point $s \in S$ maps to the origin. The logarithmic structure on V is associated to $\mathbb{N}^m \to \mathcal{O}$ where generator $e_i \mapsto t_i$.

To lift $S \to \underline{S}$ to $S \to S^{\min}$ we need to map $\mathbb{N}^m \to M_S$. Consider the i-th node q_i in X_s for $s \in S$. It is given étale locally by $xy = g_i$ where g_i is the pullback of t_i. The logarithmic structure on $X \to S$ near q_i was described as $M_S \oplus^{\mathbb{N}} \mathbb{N}^2$ where the generator e of $\mathbb{N}$ maps to g_i. This allows us to map $\mathbb{N}^m \to \mathcal{O}_{\underline{S}}$ by sending e_i to the generator corresponding to g_i.

The fact that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^{\min} \arrow[d] \\ S \arrow[r] & S^{\min} \end{tikzcd}$$

is commutative is easy diagram chasing. The fact that it is cartesian follows again from F. Kato's characterization of log curves. ♣

The theorem now follows: we have now defined a morphism $\overline{\mathcal{M}}^{\log}_{g,n} \to (\overline{\mathcal{M}}_{g,n}, M_{\Delta_{g,n}})$ which is inverse to the one described in the beginning of the proof. ♣

7.3. **Minimality.** The logarithmic structures $X^{\min} \to S^{\min}$ are called the *minimal* or *basic* logarithmic structures on a log curve. We write

$$S^{\min} = (\underline{S}, M^S_{X/S}) \qquad \text{and} \qquad X^{\min} = (\underline{X}, M^X_{X/S}).$$

The following cartesian diagram is fundamental for understanding what is going on:

(2)
$$\begin{tikzcd} X \arrow[r] \arrow[d] & X^{\min} \arrow[r] \arrow[d] & \mathcal{X}^{\log}_{g,n} \arrow[d] \\ S \arrow[r] \arrow[dr] & S^{\min} \arrow[r] \arrow[d] & \overline{\mathcal{M}}^{\log}_{g,n} \arrow[d] \\ & \underline{S} \arrow[r] & \overline{\mathcal{M}}_{g,n} \end{tikzcd}$$

The big rectangle in the top two rows exemplifies the fact that $\overline{\mathcal{M}}^{\log}_{g,n} = (\overline{\mathcal{M}}_{g,n}, M_{\Delta_{g,n}})$ is the stack parametrizing log curves. The two right columns exemplify something new and I contend quite surprising about $\overline{\mathcal{M}}_{g,n}$: it is a stack over the category of schemes parametrizing *minimal* log curves.

It is a remarkable coincidence that the same moduli space $\overline{\mathcal{M}}_{g,n}$ serves both purposes. This is because a family of curves canonically admits a log curve structure - the minimal structure. In other moduli space life is not so easy, but the picture here is worth keeping in mind. The typical effect will be that objects admitting a logarithmic structure are better, and representability of logarithmic objects is tantamount to the existence of minimal structures.

Indeed, F. Kato proceeds to show how to construct $X^{\min} \to S^{\min}$ directly from $\underline{X} \to \underline{S}$ without prior knowledge of $\overline{\mathcal{M}}_{g,n}$. If $\underline{S} = \operatorname{Spec} k$ is the spectrum of an algebraically closed field then indeed the minimal logarithmic structure on $\underline{S}$ is associated with $\mathbb{N}^m$. The local description of a node $xy = f$ and its logarithmic structure necessarily requires going to étale charts, so the logarithmic structure on X requires the étale topology and descent. Then one shows that these minimal logarithmic structures are stable under small deformations.

8. Interlude: stable maps and Gromov–Witten theory

in which

we introduce stable maps and Gromov–Witten theory in the classical sense.

Once again, no logarithmic structures here.

8.1. Stable maps. A **stable map** is a diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & X \\ S \arrow[u, bend left, "s_i"] & \end{tikzcd}$$

where

(1) $(C/S, s_i)$ is a prestable curve, and
(2) in fibers $\operatorname{Aut}(C_s \to X, s_i)$ is finite.

8.2. Gromov–Witten theory. We want to count curves on X of class $\beta \in H_2(X, \mathbb{Z})$ meeting cycles $\Gamma_1, \ldots, \Gamma_n$ corresponding to cohomology classes γ_i. For instance: lines through p_1, p_2.

Kontsevich's method: the moduli of stable maps $M := \overline{\mathcal{M}}_{g,n,\beta}(X)$ is a Deligne–Mumford stack with projective coarse moduli space. There are evaluation maps

$$\begin{array}{rcl} M & \xrightarrow{e_i} & X \\ (C/S, p_i) & \mapsto & f(p_i) \end{array}$$

and one defines the *Gromov–Witten invariants*

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,\beta} = \int_{[M]^{\mathrm{vir}}} e_1^* \gamma_1 \cdots e_n^* \gamma_n.$$

The mysterious part is $[M]^{\mathrm{vir}}$. This is there to make this a homological and deformation invariant. This is akin to the fact that the number of lines through p_1, p_2, namely the intersection number of the locus of lines through p_1 with the locus of lines through p_2, is 1, whether or not $p_1 = p_2$.

In order to define this one uses a *perfect obstruction theory.* In this case it is given by $R^\bullet \pi_* f^* T_X$, represented by a 2-term complex on S.

9. Stable logarithmic maps

in which

our main characters arrive at an enchanted place, and we leave them there.

9.1. **Definition of stable logarithmic maps.** A stable logarithmic map is the same diagram

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"] & X \\ S \end{tikzcd}$$

where $C \to S$ is a prestable log curve (with appropriate sections of the underlying curve etc.).

A stable logarithmic map has additional deformation-invariant numerical data - the contact orders of C with X at the marked points. At each such point the logarithmic structure at C has a factor $\mathbb{N}$ corresponding to the marked point, and the contact order is the homomorphism $f^* M_X \xrightarrow{c_i} \mathbb{N}$ at that marked point. We collect the numerical data under the umbrella $\Gamma = (g, \beta, c_i)$. The underlying numerical data are $\underline{\Gamma} = (g, \beta, n)$.

9.2. **Moduli.** The following theorem was proven by Gross-Siebert and Chen and ℵ-Chen under some assumptions; the assumptions removed by ℵ-Chen-Marcus-Wise:

Theorem 7. *Let X be projective logarithmically smooth scheme. Stable logarithmic maps to X form a logarithmic Deligne–Mumford stack $\overline{\mathcal{M}}_\Gamma(X)$. It is finite and representable over $\overline{\mathcal{M}}_{\underline{\Gamma}}(\underline{X})$.*

The fact that $\overline{\mathcal{M}}_\Gamma(X)$ is a logarithmic Deligne–Mumford stack is now a consequence of work of Wise, who constructs logarithmic Hom-spaces of logarithmic schemes.

The first observation is that what we are looking for is precisely the analogue of diagram 2:

(3)
$$\begin{tikzcd}
C \arrow[r] \arrow[d] \arrow[rrr, bend left=40, "f"] & C^{\min} \arrow[r] \arrow[d] \arrow[rr, bend left, "f^{\min}"] & \mathcal{C}_\Gamma(X) \arrow[r, "f^{univ}"'] \arrow[d] & X \\
S \arrow[r] \arrow[dr] & S^{\min} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_\Gamma(X) \arrow[d] & \\
& \underline{S} \arrow[r] & \underline{\overline{\mathcal{M}}_\Gamma(X)} &
\end{tikzcd}$$

We are in search of a moduli stack $\underline{\overline{\mathcal{M}}_\Gamma(X)}$ parametrizing *minimal* stable logarithmic maps over $\mathfrak{Sch}$. As such it comes with a logarithmic structure $\overline{\mathcal{M}}_\Gamma(X)$ which parametrizes *all* stable logarithmic maps over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Michael Artin devised a way to verify that moduli problems are algebraic stacks using deformation theory. There are some general properties one needs to verify. After this one takes an arbitrary geometric object of the moduli space and needs to show that it has an algebraic versal deformation space.

In our situation the general properties do hold for our moduli space. But given a stable logarithmic map $(C \to S, f : C \to X)$ with $\mathcal{S} = \operatorname{Spec} k$ a geometric point, diagram (3) requires two steps:

(1) first find a morphism from $(C \to S, f : C \to X)$ to a *minimal* object $(C^{\min} \to S^{\min}, f^{\min} : C^{\min} \to X)$.
(2) then show that the object $(C^{\min} \to S^{\min}, f^{\min} : C^{\min} \to X)$ has a versal deformation space, whose fibers are also minimal.

A precise study of minimality in the abstract is given in a paper of William Daniel Gillam. It was further tied to moduli problems in work of Junchao Shentu. In short, an object is minimal if it is essentially a final object in a connected component of the category of objects on the same underlying $\underline{S}$. Here "essentially final" means "final up to automorphisms of the minimal object".

Fundamental work of Olsson guarantees that deformation spaces exist, so point (2) is under control. The key remaining promlem is (1).

9.3. **Minimal stable logarithmic maps.** I will describe minimal objects in case X is a toric variety. In this case one can use fans and refer to Payne's lectures. In general one uses the polyhedral cone complex of a logarithmic scheme. I am focusing on the characteristic monoid - in a sense this suffices, but for the full argument see the original papers.

9.3.1. Consider a stable logarithmic map $(C/S, f : C \to X)$ over a P-logarithmic point S. We wish to find some Q-logarithmic point and a logarithmic map over it through which our object factors. The curve C has components C_i with generic points η_i corresponding to vertices v_i in the dual graph, and nodes q_j with local equations $xy = g_j$ corresponding to edges q_j in the dual graph.

9.3.2. The map f sends η_i to some stratum X_i of X with cone σ_i having lattice $N_i \subset \sigma_i$. Departing from toric conventions we denote $M_i = N_i^\vee = \mathrm{Hom}(N_i, \mathbb{N})$: the lattice M_i is the characteristic monoid of X at a general point of X_i. It is the quotient of the lattice of characters non-negative on σ by those vanishing on σ. Since the logarithmic structure of C at η_i is the pullback of the structure on S, we have a map $f_i^\flat : M_i \to P$. It can dually be viewed as a map $P^\vee \to N_i$.

If that were all we had, our final object would be $Q^\vee = \prod N_i$, and dually the initial monoid $Q = \oplus M_i$. But the nodes impose crucial conditions.

9.3.3. At a node q with branches η_q^1, η_q^2 we similarly have a map $f_q^\flat : M_q \to P \oplus^{\mathbb{N}} \mathbb{N}^2$. Unfortunately it is unnatural to consider maps into a coproduct, and we give an alterante description of

$$P \oplus^{\mathbb{N}} \mathbb{N}^2 = P\langle \log x, \log y\rangle / (\log x + \log y = \rho_q)$$

where $\rho_q = \log g_q \in P$.

Recall that the stalk of a sheaf at a point q maps, via a "generization map", to the stalk at any point specializing to q, such as η_q^1, η_q^2. The map to the stalk at η_q^1 where $x = 0$ sends $\log y \mapsto 0$, and so $\log x \mapsto \rho_q$.

The map to the stalk at η_q^2 where $y = 0$ sends $\log x \mapsto 0$, and so $\log y \mapsto \rho_q$.

This means that we have a monoid homomorphism, which is clearly injective,

$$P \oplus^{\mathbb{N}} \mathbb{N}^2 \to P \times P.$$

Its image is precisely the set of pairs

$$\{(p_1, p_2) | p_2 - p_1 \in \mathbb{Z}\rho_q\}$$

From this we obtain that the homomorphism $f_q^\flat$ can be viewed as mapping $M_q \to P \times P$, and the difference homomorphism $(p_2 - p_1) \circ f_q^\flat : M_q \to P^{\mathrm{gp}}$ maps M_q to $\mathbb{Z}\rho_q$. We record this difference through the homomorphism $u_q : M_q \to \mathbb{Z}$ such that

$$(p_2 - p_1) \circ f_q^\flat(m) = u_q(m) \cdot \rho_q. \tag{4}$$

9.3.4. Let us now analyze the two components. The maps $p_1 \circ f_q^\flat : M_q \to P$ and $p_2 \circ f_q^\flat : M_q \to P$, since they come from maps of sheaves, are compatible with generization maps. In other words $p_1 \circ f_q^\flat : M_q \to P$ is the composition of $M_q \to M_{\eta_q^1} \to P$ and similarly $p_2 \circ f_q^\flat : M_q \to P$ is the composition of $M_q \to M_{\eta_q^2} \to P$

In other words *the data of $p_1 \circ f_q^\flat$ and $p_2 \circ f_q^\flat$ is already determined by the data at the generic points η_i of the curve.*

The *only* data the node provides is the element $\rho_q \in P$ and homomorphism $u_q : M_q \to \mathbb{Z}$, in such a way that equation (4) holds. Therefore the initial object is

$$Q_f = \left(\left(\prod_\eta M_{\sigma_\eta} \times \prod_q \mathbb{N} \right) \Big/ R \right)^{sat}$$

where R is generated by all the relations implied by equation (4).

It is quite a bit more natural to describe the dual lattice

$$Q_f^\vee = \left\{ \big((v_\eta), (e_\eta)\big) \in \prod_\eta N_{\sigma_\eta} \times \prod_q \mathbb{N} \;\middle|\; \begin{array}{c} \forall\ \eta_q^1 \xrightarrow{\quad q \quad} \eta_q^2 \\ v_q^1 - v_q^2 = e_j u_q \end{array} \right\}.$$

This has a beautiful interpretation in terms of tropical curves in the fan of X, which I will describe if time permits. I am not sure I'll have time to put this in any revision of these notes. (These are remarks 1.18 and 1.21 in Gross and Siebert's paper.)

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