BOUNDEDNESS OF THE SPACE OF STABLE LOGARITHMIC MAPS

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1. Introduction

2. Resolution

The goal of this section is to describe a projective resolution of the universal target.

2.1. **Kato Fan.** Denote by $N = \mathbb{Z}^n$ the lattice of rank n, and $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$. A *fan* F in $N_{\mathbb{Q}}$ is a set of strong rationally convex cones in $N_{\mathbb{Q}}$ whose comment faces are still elements in F. Denote by $|F|$ the image of F in $N_{\mathbb{Q}}$. We say a fan F is *strong rationally convex* if $|F|$ is a strong rationally convex cones in $N_{\mathbb{Q}}$.

For a strong rationally convex fan F, denote by ∂F the boundary of $|F|$. We call ∂F the *convex hull* of F. For each face $\sigma \subset \partial F$, we obtain a fan $F_\sigma = F \cap \sigma$. We call F_σ the restriction of F to σ.

A fan F is said of *maximal rank*, if the set $|F|$ generates the vector space $N_{\mathbb{Q}}$.

`def:kato-fan` **Definition 2.2.** A *generalized Kato fan* (or simply Kato fan) is a pair $\Sigma = (\{\sigma_i\}_{i\in\Lambda}, <)$ consisting of a finite set of cones $\{\sigma_i\}$, and a partial ordering $<$ on Λ satisfying

(1) each cone σ_i is strong rationally convex, of maximal rank contained in a vector space $N_i \otimes_{\mathbb{Z}} \mathbb{Q}$ with the integral structure given by $N_i \cong \mathbb{Z}^{r_i}$;
(2) for any $i, j \in \Lambda$ such that $i < j$, we have
    (i) $r_i < r_j$;
    (ii) there is finitely many embeddings of lattices
$$\phi_{ij} : N_i \to N_j$$
    which identifies σ_i as a face $\sigma_{\phi_{ij}} \subset \sigma_j$;
(3) Every face of a cone σ_i is contained in Σ under the identifications in (2).

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The generalized Kato fan Σ is called *simplicial* if every cone is simplicial in the usual sense, and it is called *smooth* if every cone is smooth in the usual sense.

`def:subdivision` **Definition 2.3.** Consider a Kato fan $\Sigma = (\{\sigma_i\}_{i\in\Lambda}, <)$. A *subdivision* of Σ is another Kato fan $\Sigma' = (\{\sigma_i\}_{i\in\Lambda'}, <)$ obtained by subdivide cones in Σ in the usual sense.

Consider a Kato fan Σ as in the above definition. Let σ be the class of cones which are isomorphic under ϕ_{ij}. We call such σ a *cone* of Σ. As usual, we will denote by $\Sigma(n)$ the cones in Σ of dimension n. Cones in $\Sigma(1)$ are called rays.

Similarly, let v be the class of vectors in Σ, which are identified under $\phi_{ij} \otimes \mathbb{Q}$. If v can be represented by a vector $v_i \subset \Sigma_i$ for some $i \in \Lambda$, then we say that v_i is a *representative* of v, and Σ_i contains v.

Denote by $|\Sigma|$ the set of all such vectors. For any element $v \in |\Sigma|$, let $v_i \in N_i \times \mathbb{Q}$ be its representative in F_i. Note that if $v_i \in N_i$ for some i, then it is true for all i containing v_i by Definition 2.2(2). We call such v_i an *integral element* in $|\Sigma|$. Denote by $|\Sigma|^{int} \subset |\Sigma|$ the set of all integral elements. An element $v \in |\Sigma|^{int}$ is called *primitive* if one, hence all of its representatives are primitive.

The most important operation we will use in this note is the following:

`def:star-division` **Definition 2.4.** Given a Kato fan $\Sigma = (\{\sigma_i\}_{i\in\Lambda}, <)$, and a primitive element $v \in |\Sigma|^{int}$, we define a new Kato fan $\Sigma(v) = (\{\sigma_i\}_{i\in\Lambda'}, <)$ such that

1. if v is not contained in σ_i, then $i \in \Lambda'$, i.e. σ_i is cone in Σ';
2. if v is contained in σ_i, then take σ'_i to be the star subdivision of F_i at v as in [tv, Section 11.1], and add all faces of the fan σ'_i to the new Kato fan $\Sigma(v)$ with the obvious compatible integral structure.

It is easy to check that $\Sigma(v)$ obtained here still satisfies Definition 2.2(2), hence is a Kato fan. We call $\Sigma(v)$ the *star subdivision* of Σ at v.

`def:graph-res` **Definition 2.5.** Given a Kato fan $\Sigma = (\{\sigma_i\}_{i\in\Lambda}, <)$, a Kato fan $\Sigma' = (\{F'_i\}_{i\in\Lambda}, <)$ is called a *resolution* of Σ if Σ' is a subdivision of Σ such that each cone of Σ' is smooth.

`prop:graph-res` **Proposition 2.6.** *For any Kato fan $\Sigma = (\{F_i\}_{i\in\Lambda}, <)$, there is a smooth Kato fan Σ' obtained by taking a sequence of star subdivision.*

Proof. We first take the barycentric subdivision of Σ as in [tv, Exercise 11.1.10]. Note that barycentric subdivision is given by a sequence of star subdivision, hence is well defined in our situation. This will result a simplicial Kato fan. So we may assume Σ is simplicial. Then the same construction as in [tv, Theorem 11.1.9] yields a smooth Kato fan Σ' obtained from a sequence of subdivision of Σ. This finishes the proof. ♠

In general, the integral structure of a Kato fan only exists locally on each cone. However, after some resolution the integral structure the integral structure can glued globally.

`prop:global-int`

Proposition 2.7. *For any Kato fan* $\Sigma = (\{\sigma_i\}_{i\in\Lambda}, <)$*, there is a resolution* Σ'' *of* Σ *obtained by taking a sequence of star subdivision such that there is an inclusion of integral vectors*

$$|\Sigma''|^{int} \to \mathbb{N}^n$$

and each cone of Σ'' *maps to a cone in* $\mathbb{N}^n$ *with compatible integral structures.*

Proof. Let Σ' be the Kato fan obtained in the proof of Proposition 2.6. Let Σ'' be the barycentric sub-division of Σ'. Take n to be the number of rays in $\Sigma''(1)$. Let $\mathbb{N}^n$ to be the free monoid generated by elements of $\Sigma''(1)$. Since $\Sigma''(1)$ is smooth, each integral vector in $|\Sigma''(1)|$ is given by the integral linear combinations of the rays in the minimal cone containing it. This yields the map

$$|\Sigma''|^{int} \to \mathbb{N}^n.$$

To prove the statement, it suffices to check the above map is injective. Take two integral vectors $v_1, v_2 \in \Sigma''$ contained in the two cones σ_1 and σ_2 respectively. First notice that if $\sigma_1 = \sigma_2 = \sigma$, then no two different rays of σ is identified by Definition 2.2(2) after the first barycentric subdivision.

Now assume that $\sigma_1 \neq \sigma_2$. We then notice that the two cones must have at least one different ray after the second barycentric subdivision. Since v_i is given by the linear combination of the rays in σ_i, the injectivity follows. ♠

2.8. **Associated Artin Fan.** For each strong rational convex cone σ, we introduce the log algebraic stack

$$\mathcal{A}_\sigma = [\operatorname{Spec} k[\sigma^\vee] / \operatorname{Spec} k[(\sigma^\vee)^{gp}]].$$

with its log structure descent from $\operatorname{Spec}(\sigma^\vee \to k[\sigma^\vee])$.

Consider two cones σ_1 and σ_2 with an embedding $\sigma_1 \to \sigma_2$, which identifies σ_1 as a face of σ_2. Then by [ACGS], we have a canonical strict open immersion

$$\mathcal{A}_{\sigma_1} \to \mathcal{A}_{\sigma_2}.$$

For each Kato fan Σ, we define

$$\mathcal{A}_\Sigma = \varinjlim \mathcal{A}_\sigma$$

with the limit taking over all cones of Σ with the partial ordering. We call $\mathcal{A}_\Sigma$ the *Artin fan* associated to Σ.

`em:stack-subdivision` **Lemma 2.9.** *Consider a sub-division Σ' of a given Kato fan Σ. Then there is an induced representable, proper, birational, and log étale morphism of log stacks*

$$\psi_{\Sigma'\to\Sigma} : \mathcal{A}_{\Sigma'} \to \mathcal{A}_\Sigma.$$

Furthermore, if Σ' is obtained from a sequence of star sub-division of Σ, then $\psi_{\Sigma'\to\Sigma}$ is projective.

Proof. By locally gluing the birational modification. ♠

`thm:artin-fan-res` **Theorem 2.10.** *Consider a Kato fan Σ. Then there is a subdivision Σ' of Σ such that*

1. *The associated Artin fan $\mathcal{A}_{\Sigma'}$ is smooth in the usual sense.*
2. *The birational morphism $\psi_{\Sigma'\to\Sigma} : \mathcal{A}_{\Sigma'} \to \mathcal{A}_\Sigma$ is projective.*

We may replace Σ' by a further sub-division such that there is a strict open immersion

$$\mathcal{A}_{\Sigma'} \to \mathcal{A}^n$$

for some positive integer n. In this case, we have the global quotient of log stacks:

$$\mathcal{A}_{\Sigma'} \cong [U/\mathbb{G}_m^n]$$

for a $\mathbb{G}_m^n$-invariant open subset $U \subset \mathbb{A}^n$ with its log structure given by the coordinate hyperplanes.

Proof. This follows from Lemma 2.9, and Proposition 2.6 and 2.7. ♠

2.11. **Resolution of a log scheme.** Let X be a log DM stack. Then by [ACGS] or some other references, we have a Kato fan Σ_X associated to X such that there is a strict log map

$$X \to \mathcal{A}_{\Sigma_X}.$$

Consider the cartesian product

$$\begin{tikzcd}
Y \arrow[r, "\psi"] \arrow[d] & X \arrow[d] \\
\mathcal{A}_{\Sigma_Y} \arrow[r, "\psi_{\mathcal{A}}"] & \mathcal{A}_{\Sigma_X}
\end{tikzcd}$$

with bottom map given by Theorem 2.10. The exceptional divisor $E_{\mathcal{A}}$ of $\psi_{\mathcal{A}}$ is a simple normal crossings divisor given by the union

$$\mathcal{E}_{\mathcal{A}} = \cup_i \mathcal{E}_i$$

with $\mathcal{E}_i$ irreducible. Denote by E_* be the pull-back of $\mathcal{E}_*$ over Y. Denote by $-L_*$ be line bundle over Y corresponding to the exceptional divisor E_*. Now we summarize:

`cor:target-modify` **Corollary 2.12.** *With the notations as above, We have:*

1. *The morphism $\psi : Y \to X$ is projective, log étale and surjective.*
2. *ψ is an isomorphism away from the exceptional locus $\cup_i E_i$.*
3. *The line bundle $L = \sum_i L_i$ is ψ-ample, and each L_i is ψ-nef.*
4. *Y has Deligne-Faltings log structure.*
5. *If X is log smooth, then the underlying structure $\underline{Y}$ is smooth in the usual sense.*

3. Boundedness of numerical data

In this section, we fix a log scheme or stack X and a resolution $\psi : Y \to X$ as in Corollary 2.12. We write Σ_X and Σ_Y for the two Kato fans associated to X and Y. We may assume that Σ_X is a subdivision of Σ_Y satisfying conditions in Proposition 2.7. In particular, the two sets of integral vectors $|\Sigma_X|$ and $|\Sigma_Y|$ are identical. We use $|\Sigma|$ denoting the two identical sets.

3.1. **Boundedness of contact orders.** Let X be a log scheme with the Kato fan Σ_X. Denote by $\wedge(X)$ the stack of log points in X.

`attice-contact-order` **Proposition 3.2.** *There is a one-to-one correspondence between the connected components of $\wedge(X)$ and the set $|\Sigma_X|$.*

Proof. By the construction in [Logpt], it suffices to treat the case $X = \mathcal{A}_{\Sigma_X}$. In the local case, the contact orders correspond to the lattice points by definition. Now for each $v \in |\Sigma_X|$, we obtain the contact order by viewing v locally on each cones. Then we use [Logpt] to deduce connectedness. ♠

Since $|\Sigma_X| = |\Sigma_Y| = |\Sigma|$, the connected components of $\wedge(X)$ are one-to-one corresponding to the connected components of $\wedge(Y)$. Fix an integral vector $v \in |\Sigma|$, and denote by $\wedge(Y, v)$ and $\wedge(X, v)$ the corresponding connected component.

`uation-change-target` **Lemma 3.3.** *The log map $Y \to X$ induces a natural morphism of the evaluation stacks*

$$\psi_v : \wedge(Y, v) \to \wedge(X, v)$$

Proof. This can be checked for example on each local family. ♠

`uation-birationality` **Lemma 3.4.** *The natural map*

$$\psi_v : \wedge(\mathcal{A}_Y, v) \to \wedge(\mathcal{A}_X, v)$$

is surjective, proper and of DM type. Furthermore, it is an isomorphism over the loci with the trivial log structure.

Proof. Surjectivity and the isomorphism can be checked using the construction in [Logpt]. Properness can be checked using valuative criterion. ♠

Combining the above lemma, we have:

`tangency-boundedness` **Proposition 3.5.** *The map $\psi_v : \wedge(Y, v) \to \wedge(X, v)$ as in Lemma 3.3 is surjective, and proper of DM type. In particular, $\wedge(X, v)$ is of finite type.*

3.6. **Boundedness of the curve classes.** We then fix the numerical data $\Gamma_X = (g, n, \beta, \{c_j\}_j^n)$ for stable log maps to X. By Proposition 3.2, we may identify $\{c_j\}_j^n$ with the set of integral vectors in $|\Sigma|$, hence contact orders to Y. Denote by $c_j(E_i)$ the contact order of the i-th marking with the exceptional divisor E_i as in Corollary 2.12.

`em:curve-boundedness` **Lemma 3.7.** *The choice of curve class $\beta_Y \in H_2(Y)$ satisfying $\psi_* \beta_Y = \beta$, and*

`qu:exceptional-curve`

$$\beta_Y \cap c_1(L_i) = \sum_j c_j(E_i) \tag{3.1}$$

for all j, is finite.

Proof. By Corollary 2.12(2), it suffices to show that curve classes $\beta \in H_2(Y)$ in the exceptional loci of ψ satisfying (3.1) is finite. But this follows from the positivity of Corollary 2.12(3). ♠

It follows that

`op:numerical-lifting` **Proposition 3.8.** *The choice of the numerical data Γ_Y for stable log maps to Y lifting Γ_X is finite.*

4. CONCLUSION

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