THE EVALUATION SPACE OF LOGARITHMIC STABLE MAPS

D. ABRAMOVICH, Q. CHEN, W. GILLAM, AND S. MARCUS

Abstract. For a fine log algebraic space X, we construct an algebraic stack $\wedge X$ parameterizing standard log points in X. The stack $\wedge X$ is the natural target for evaluation morphisms from the space of logarithmic stable maps, hence is a key ingredient in logarithmic Gromov-Witten theory. We also construct a variant $\wedge X$ parameterizing standard log nodes in X, which is the natural target for evaluations at nodes of log curves, hence it plays a crucial role in all gluing constructions in logarithmic Gromov-Witten theory.

1. Introduction

1.1. **Logarithmic stable maps.** A theory of *logarithmic stable maps* and *logarithmic Gromov-Witten (GW) theory* was first proposed during a 2001 workshop lecture by Bernd Siebert [Sie01]. It provides an approach to vastly generalize *relative* GW theory (see [LR01], [IP03, IP04], [Li01, Li02]) in which enumerative invariants of curves on varieties satisfying certain contact conditions are defined and used for computing usual GW invariants through degenerations.

The aim of this paper is to continue the development of logarithmic GW theory along the lines of [Kim09], [Che10a], and [AC10], where the category of logarithmic stable maps $\overline{M}_{g,m}(X)$ to a (fine, saturated) log scheme X is introduced and studied. The category $\overline{M}_{g,m}(X)$ is a stack over the category **LogSch** of (fine, saturated) log schemes. An object of $\overline{M}_{g,m}(X)$ over a base log scheme Y is a diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle\pi}\downarrow & & \\ Y & & \end{array} \tag{1}$$

of maps of log schemes, where $\pi : C \to Y$ is a log curve (in the sense of [FK]) with marking sections $p_1, \ldots, p_m$ labelling the locus where the relative characteristic of π is $\mathbb{N}$. The underlying diagram of schemes is required to be a stable map to X in the usual sense of Kontsevich. Under reasonable hypotheses on X, the category $\overline{M}_{g,m}(X, \Gamma)$ of log stable maps with specified discrete data Γ is representable by

---

Date: October 6, 2011.

Research of Abramovich and Chen supported in part by funds from NSF award 0603284. Gillam supported by an NSF Postdoctoral Fellowship. Marcus supported by an NSERC PGS-D Doctoral Fellowship.

a proper Deligne-Mumford (DM) stack over schemes with a (fine, saturated) log structure.

1.2. **Log points.** Given a log stable map (1) over a base Y, the log scheme $Y_i := (\underline{Y}, p_i^*\mathcal{M}_C)$ is naturally a log scheme over Y, locally isomorphic to

$$Y \times \operatorname{Spec}(0 : \mathbb{N} \to \mathbb{C}),$$

and is the motivating example of what we call a *standard log point*.

In Definition 6 we define the category $\wedge X$ of *(standard) log points* in a log scheme X. The category $\wedge X$ is a stack over **LogSch**. Our first main result is the following:

Theorem 1. *There is a logarithmic algebraic stack* $(\wedge X, \mathcal{M}_{\wedge X})$ *representing* $\wedge X$.

The definition of $\wedge X$ is arranged so that taking a diagram (1) to the pair $(Y_i \to Y, f|Y_i : Y_i \to X)$ defines an *evaluation morphism*

$$\mathrm{ev}_i : \overline{M}_{g,m}(X) \quad \to \quad \wedge X. \tag{2}$$

Theorem 1 is actually the special case $P = \mathbb{N}$ of a more general respresentability result, Theorem 12, for the stack $\wedge_P X$ of P *log points* in X.

1.3. **Log nodes.** The treatment of maps out of nodal curves in the logarithmic setting is more complex than in the usual theory of stable maps. For example, suppose $\pi : C \to Y$ is a log curve over a base log scheme Y whose underlying curve $\underline{C} = \underline{C}_1 \coprod_{\hat{p}} \underline{C}_2$ splits into two components $\underline{\pi}_i : \underline{C}_i \to \underline{Y}$ glued along a common section $\hat{p}$. Then we can consider the log scheme $C_1 := (\underline{C}_1, \mathcal{M}_C|\underline{C}_1)$ over Y. The map $C_1 \to Y$ is not a log curve: it is not even log smooth. Rather, $C_1 \to Y$ is what we call a *prenodal log curve*, with prenodal structure along the marking section $\hat{p}$ of the underlying nodal curve. For a typical target log scheme X, there is in general no reasonable way to describe $\mathrm{Hom}_{\mathbf{LogSch}}(C_1, X)$ in terms of maps out of an actual log curve over Y.[1] On the other hand, maps out of such prenodal log curves will arise frequently in the study of:

(1) the boundary of the space of log stable maps
(2) localization in the presence of a torus action on X
(3) degeneration formulas

We are led to introduce the category $\overline{M}_{g,m,n}(X)$ of log stable maps to X from prenodal log curves with n marked prenodal points $\hat{p}_1, \ldots, \hat{p}_n$. This $\overline{M}_{g,m,n}(X)$ is a stack over **LogSch** whose objects over Y are as in (1), except that $\pi : C \to Y$ is now a prenodal log curve with sections $\hat{p}_i$ marking the prenodal points, where the relative characteristic is $\mathbb{Z}$. In this situation, $\hat{Y}_i := (\underline{Y}, \hat{p}_i^*\mathcal{M}_C)$ is naturally a log scheme over Y, and is the motivating example of what we call a *standard log node*.

[1]There is a log curve $C_1' \to Y'$ with $\underline{C}_1' = \underline{C}_1$ naturally associated to this situation, but $\mathrm{Hom}_{\mathbf{LogSch}}(C_1', X)$ generally has little to do $\mathrm{Hom}_{\mathbf{LogSch}}(C_1, X)$.

In Definition 5, we define the category $\wedge_\Delta X$ of *(standard) log nodes* in X. This $\wedge_\Delta X$ is a stack over **LogSch**, and we prove

Theorem 2. *There is a logarithmic algebraic stack $(\wedge_\Delta X, \mathcal{M}_{\wedge_\Delta X})$ representing $\wedge_\Delta X$.*

Again, the definition of $\wedge_\Delta X$ is arranged so that we obtain evaluation maps

$$\hat{\mathrm{ev}}_i : \overline{M}_{g,m,n}(X) \quad \to \quad \wedge_\Delta X \tag{3}$$

and, again, the log stack $\wedge_\Delta X$ is the special case of more general log stacks $\wedge_h X$ for monoid homomorphisms $h : Q \to P$, where $h = \Delta : \mathbb{N} \to \mathbb{N}^2$.

2. Constructions

In this section, we will give the definition and construction, under appropriate assumptions, of the log stack of log nodes in X, along with the special case of log points in X, to which we devote considerable attention. The basic idea of the construction is similar to the idea used to prove representability of many moduli problems: We first consider a rigid version of log nodes (Section 2.2) which is clearly a presheaf over log schemes, then we argue that the desired stack of log nodes can be recovered from this presheaf by taking a quotient by a group action (Section 2.11). It then remains to prove that the presheaf of rigid log nodes is representable by an algebraic space. This is done by constructing this presheaf explicitly in some simple cases (Section 2.6), then bootstrapping up (Section 2.7) to general X using formal properties of the presheaves of rigid log nodes (established in Section 2.5).

2.1. **Preliminaries.** In order to treat log algebraic spaces and log schemes on the same footing, it is natural for us to work frequently with the category **PShLogSch** of presheaves on the category of log schemes. All the presheaves of ultimate interest to us will be sheaves (in, say, the strict étale topology), but the sheaf property is rarely important to us, and when it is needed, we can easily spell out the specific assumption we use. We suppress all notation for the Yoneda functor

$$\mathbf{LogSch} \to \mathbf{PShLogSch},$$

regarding $Y \in$ **LogSch** as an object $Y \in$ **PShLogSch** using the same notation. The identification

$$\mathrm{Hom}_{\mathbf{PShLogSch}}(Y, X) \quad = \quad X(Y),$$

for $Y \in$ **LogSch**, $X \in$ **PShLogSch** coming from Yoneda's Lemma is used without further comment.

Let **SMon** be the category whose objects are sharp monoids and whose morphisms are monoid homomorphisms $h : Q \to P$ with trivial kernel: $h^{-1}(0) = \{0\}$. These are exactly the monoid homomorphisms that can arise as maps on characteristic monoids of log structures. We have a functor

$$\begin{array}{rcl} \mathbb{P} : \mathbf{SMon}^{\mathrm{op}} & \to & \mathbf{LogSch} \\ P & \mapsto & \mathbb{P}(P) := \operatorname{Spec}(0 : P \to \mathbb{Z}) \\ (h : Q \to P) & \mapsto & \mathbb{P}(h) := \operatorname{Spec}(h : (0 : Q \to \mathbb{Z}) \to (0 : P \to \mathbb{Z})). \end{array} \tag{4}$$

The log scheme $\mathbb{P}(P)$ represents the presheaf

$$\begin{array}{rcl} \mathbf{LogSch}^{\mathrm{op}} & \to & \mathbf{Sets} \\ Y & \mapsto & \operatorname{Hom}_{\mathbf{Mon}}(P, \operatorname{Ker}(\alpha_Y : \mathcal{M}_Y(Y) \to \mathcal{O}_Y(Y))). \end{array}$$

The **SMon** morphism $h : Q \to P$ is recovered from the map of log schemes $\mathbb{P}(h) : \mathbb{P}(P) \to \mathbb{P}(Q)$ as the map on characteristics.

Given a monoid P, we let $\mathbb{G}(P) := \operatorname{Spec} \mathbb{Z}[P^{\mathrm{gp}}]$, regarded as a group object in log schemes with trivial log structure. The association $P \mapsto \mathbb{G}(P)$ is a functor

$$\mathbb{G} : \mathbf{Mon}^{\mathrm{op}} \to \mathbf{GpLogSch}.$$

The log group scheme $\mathbb{G}(P)$ represents the presheaf

$$\begin{array}{rcl} \mathbb{G}(P) : \mathbf{LogSch}^{\mathrm{op}} & \to & \mathbf{Sets} \\ Y & \mapsto & \operatorname{Hom}_{\mathbf{Ab}}(P^{\mathrm{gp}}, \mathcal{O}_Y^*(Y)). \end{array}$$

Definition 1. Let $h : Q \to P$ be a monoid homomorphism. A *splitting set* for h is a sub*set* $S \subseteq P$ such that $(s, q) \mapsto s + h(q)$ defines a bijection of sets $S \times Q \to P$.

Evidently the existence of a splitting set requires h to be monic. If $h : Q \hookrightarrow P$ is an integral monomorphism of fine monoids, then F. Kato's Integral Splitting Lemma ([FK], Section 1, [G], Lemma 7) implies that the set $S := P^{Q-\mathrm{prim}}$ of Q-primitive elements of P is a splitting set for h in bijective correspondence with $\operatorname{Cok} h$. For example, $S = (\{0\} \oplus \mathbb{N}) \cup (\mathbb{N} \oplus \{0\})$ is a splitting set for $\Delta : \mathbb{N} \hookrightarrow \mathbb{N}^2$ in bijective correspondence with $\operatorname{Cok} \Delta \cong \mathbb{Z}$. For reasonable monoid maps, splitting sets are stable under pushout:

Lemma 3. *Suppose $h : Q \hookrightarrow P$ is an integral monomorphism of integral monoids with splitting set S and $f : Q \to R$ is a monoid homomorphism to an integral monoid R. Then S is also a splitting set for $R \hookrightarrow P \oplus_Q R$, in the sense that $(s, r) \mapsto [s, r]$ defines a bijection $S \times R \to P \oplus_Q R$.*

Proof. The surjectivity of $(s, r) \mapsto [s, r]$ requires no assumptions at all about Q, P, R or the maps between them, and only uses surjectivity of the original addition map $(s, q) \mapsto s + q$. Given $[p, r] \in P \oplus_Q R$, just find $s \in S$ and $q \in Q$ so $p = s + q$. Then the computation

$$[p, r] = [s, r] + [q, 0] = [s, r] + [0, f(q)] = [s, r + f(q)]$$

shows $[p, r]$ is in the image of the map in question. For injectivity, suppose $[s, r] = [s', r']$ in $P \oplus_Q R$ for some $s, s' \in S$, $r, r' \in R$. We want to show that $s = s'$ and $r = r'$. Now, using all our hypotheses on Q, P, R and $Q \hookrightarrow P$, we know from the description of the pushout monoid in this situation that $[s, r] = [s', r']$ implies the existence of $q, q' \in Q$ such that $s + q = s' + q'$ in P and $r + f(q') = r' + f(q)$ in R. By the injectivity of the original addition map, the first of these equalities implies $s = s'$ and $q = q'$, so the second of the equalities says $r + f(q) = r' + f(q)$. But this implies $r = r'$ because R is integral. $\square$

2.2. **Rigid log nodes.** Consider a morphism $h : Q \to P$ in **SMon** and a presheaf X on the category **LogSch** of log schemes. We define a new presheaf $\wedge_h^{\mathrm{rig}} X$ on **LogSch** as follows: the set $(\wedge_h^{\mathrm{rig}} X)(Y)$ is the set of pairs (f, F) where $f \in \mathbb{P}(Q)(Y)$ and $F \in X(Y \times^f_{\mathbb{P}(Q)} \mathbb{P}(P))$. The latter fiber product is with respect to the maps f, $\mathbb{P}(h)$, as emphasized by the superscript f, which we will drop if f is clear from context. The set $(\wedge_h^{\mathrm{rig}} X)(Y)$ is clearly natural in Y and the formation of the presheaf $\wedge_h^{\mathrm{rig}} X$ is contravariantly functorial in h and covariantly functorial in X, so we have a functor

$$\begin{array}{rcl} \wedge^{\mathrm{rig}} : \mathbf{MapSMon}^{\mathrm{op}} \times \mathbf{PShLogSch} & \to & \mathbf{PShLogSch} \\ (h : Q \to P, X) & \mapsto & \wedge_h X. \end{array}$$

For a fixed h, the forgetful map $(f, F) \mapsto f$ makes $\wedge_h^{\mathrm{rig}} X$ a presheaf over $\mathbb{P}(Q)$, so we can view $\wedge_h^{\mathrm{rig}}$ as a functor

$$\begin{array}{rcl} \wedge_h^{\mathrm{rig}} : \mathbf{PShLogSch} & \to & \mathbf{PShLogSch}/\mathbb{P}(Q) \\ X & \mapsto & \wedge_h^{\mathrm{rig}} X. \end{array} \tag{5}$$

Definition 2. For $X \in \mathbf{PShLogSch}$, the presheaf $\wedge_h^{\mathrm{rig}} X$ is called the presheaf of *rigid h log nodes* in X. In case $P = \Delta : \mathbb{N} \to \mathbb{N}^2$, we call $\wedge_\Delta^{\mathrm{rig}} X$ the presheaf of *rigid standard log nodes in X.*

Remark 1. For $Y \in \mathbf{LogSch}$, if $f : Y \to \mathbb{P}(Q)$ is identified with a monoid homomorphism $f : Q \to \operatorname{Ker} \alpha_Y(Y) \subseteq \mathcal{M}_Y(Y)$ as discussed in (2.1), then $Y' := Y \times^f_{\mathbb{P}(Q)} \mathbb{P}(P) \in \mathbf{LogSch}$ is the log scheme with the same underlying scheme as Y, but with log structure $\mathcal{M}_{Y'} = \underline{P} \oplus^f_{\underline{Q}} \mathcal{M}_Y$ and structure map $\alpha_{Y'} : \mathcal{M}_{Y'} \to \mathcal{O}_Y$ given by $\alpha_{Y'} = (0, \alpha_Y)$ using the universal property of pushout. If $h : Q \to P$ is an integral morphism of integral monoids, the same discussion makes sense in the category of integral log schemes.

2.3. **Logarithmic Weil restriction.** In the special case where $h = 0 : 0 \hookrightarrow P$, we write $\wedge_P^{\mathrm{rig}}$ instead of $\wedge_h^{\mathrm{rig}}$. The functor

$$\begin{array}{rcl} \wedge_P^{\mathrm{rig}} : \mathbf{PShLogSch} & \to & \mathbf{PShLogSch} \\ X & \mapsto & \wedge_P^{\mathrm{rig}} X \end{array}$$

is then the formal right adjoint to the base change functor

$$\begin{array}{rcl} \_ \times \mathbb{P}(P) : \mathbf{PShLogSch} & \to & \mathbf{PShLogSch} \\ X & \mapsto & X \times \mathbb{P}(P) \end{array}$$

in the sense that, for any $Y \in \mathbf{LogSch}$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{PShLogSch}}(Y, \wedge_P^{\mathrm{rig}} X) &= (\wedge_P^{\mathrm{rig}} X)(Y) \\
&= X(Y \times \mathbb{P}(P)) \\
&= \mathrm{Hom}_{\mathbf{PShLogSch}}(Y \times \mathbb{P}(P), X).
\end{aligned}$$

The problem of respresenting the formal right adjoint to base change is the subject of *Weil restriction.* Much is known in the context of schemes.

Definition 3. For $X \in \mathbf{PShLogSch}$, the presheaf $\wedge_P^{\mathrm{rig}} X$ is called the presheaf of *rigid P log points* in X. In case $P = \mathbb{N}$, we drop P from the notation and call $\wedge_{\mathbb{N}}^{\mathrm{rig}} X$ the presheaf of *rigid standard log points in X.*

2.4. **Group actions.** For $P \in \mathbf{SMon}$, the log group scheme $\mathbb{G}(P)$ acts on $\mathbb{P}(P)$ as follows: Given a log scheme Y, a map

$$(f : P \to \operatorname{Ker} \alpha_Y) \in \mathbb{P}(P)(Y),$$

and a map $(g : P^{\mathrm{gp}} \to \mathcal{O}_Y^*(Y)) \in \mathbb{G}(P)(Y)$, we obtain a new map

$$(g \cdot f : P \to \operatorname{Ker} \alpha_Y) \in \mathbb{P}(P)(Y)$$

by setting $(g \cdot f)(p) := g(p)f(p)$. This action identifies $\mathbb{G}(P)$ with the subpresheaf of

$$\begin{aligned}
\mathrm{Aut}(\mathbb{P}(P)) : \mathbf{LogSch}^{\mathrm{op}} &\to \mathbf{Sets} \\
Y &\mapsto \mathrm{Aut}(Y \times \mathbb{P}(P)/Y)
\end{aligned}$$

consisting of those relative automorphisms acting trivially on the (absolute) characteristic monoid.

For a map $h : Q \to P$ in $\mathbf{SMon}$, and $X \in \mathbf{PShLogSch}$, we will now describe an action of the log group scheme $\mathbb{G}(P)$ on the presheaf $\wedge_h^{\mathrm{rig}} X$. The action will be natural in X, but it will *not* be an action commuting with the structure morphism $\wedge_h^{\mathrm{rig}} X \to \mathbb{P}(Q)$. Rather, the action of $\mathbb{G}(P)$ on $\wedge_h^{\mathrm{rig}} X$ will intertwine with the action of $\mathbb{G}(Q)$ on $\mathbb{P}(Q)$ defined in the previous paragraph via the map

$$\mathbb{G}(h)^{-1} : \mathbb{G}(P) \to \mathbb{G}(Q)$$

of log group schemes appearing in the commutative diagram below.

$$\begin{tikzcd}
\mathbb{G}(P) \arrow[r, "\mathbb{G}(h)"] \arrow[d, "_^{-1}"'] \arrow[dr, "\mathbb{G}(h)^{-1}"] & \mathbb{G}(Q) \arrow[d, "_^{-1}"] \\
\mathbb{G}(P) \arrow[r, "\mathbb{G}(h)"] & \mathbb{G}(Q)
\end{tikzcd}$$

Note that inverse is a map of group objects since our group objects are commutative.

Given $Y \in \mathbf{LogSch}$ and $(g : P^{\mathrm{gp}} \to \mathcal{O}_Y^*(Y)) \in \mathbb{G}(P)(Y)$, note

$$(\mathbb{G}(h)^{-1}(g) : Q^{\mathrm{gp}} \to \mathcal{O}_Y^*(Y)) \in \mathbb{G}(Q)(Y)$$

is given by $q \mapsto g(h(q))^{-1}$. Given $f \in \mathbb{P}(Q)(Y)$, we construct an isomorphism

$$a(g) : Y \times_{\mathbb{P}(Q)}^{\mathbb{G}(h)^{-1}(g)\cdot f} \mathbb{P}(P) \;\; \to \;\; Y \times_{\mathbb{P}(Q)}^{f} \mathbb{P}(P)$$

making the diagram

$$\begin{tikzcd}
Y \times_{\mathbb{P}(Q)}^{\mathbb{G}(h)^{-1}(g)\cdot f} \mathbb{P}(P) \arrow[r, "a(g)", "\cong"'] \arrow[d] \arrow[rr, bend left] & Y \times_{\mathbb{P}(Q)}^{f} \mathbb{P}(P) \arrow[r] \arrow[d] & \mathbb{P}(P) \arrow[d, "\mathbb{P}(h)"] \\
Y \arrow[rr, bend right, "\mathbb{G}(h)^{-1}(g)\cdot f"'] & Y \arrow[r, "f"] & \mathbb{P}(Q)
\end{tikzcd} \tag{6}$$

commute (the two "squares" here are cartesian). It is easier to think in terms of log structures, where $a(g)$ is the isomorphism

$$\begin{array}{rcl} a(g) : \underline{P} \oplus_{\underline{Q}}^{f} \mathcal{M}_Y & \to & \underline{P} \oplus_{\underline{Q}}^{\mathbb{G}(h)^{-1}\cdot f} \mathcal{M}_Y \\ {[p, m]} & \mapsto & [p, g(p)m] \end{array}$$

of log structures on $\underline{Y}$. Here we view $\mathcal{O}_Y^*$ as sitting inside $\mathcal{M}_Y$, and the juxtaposition of $g(p)$ and m is the monoid operation on $\mathcal{M}_Y$. This is well-defined by the computation

$$\begin{array}{rcl} a(g)[h(q), 0] & = & [h(q), g(h(q))] \\ & = & [0, g(h(q))(\mathbb{G}(h)^{-1} \cdot f)(q)] \\ & = & [0, g(h(q))g(h(q))^{-1} f(q)] \\ & = & [0, f(q)] \\ & = & a(g)[0, f(q)]. \end{array}$$

The data consisting of the action of $\mathbb{G}(Q)$ on $\mathbb{P}(Q)$ and the assignment $g \mapsto a(g)$ defines an action of $\mathbb{G}(h) : \mathbb{G}(P) \to \mathbb{G}(Q)$ on $\mathbb{P}(h) : \mathbb{P}(P) \to \mathbb{P}(Q)$ in the sense of Definition 4 in Section 2.8.

This action on h log nodes induces an action on $\wedge_h^{\mathrm{rig}} X$ by setting

$$g \cdot (f, F) \;\; := \;\; (\mathbb{G}(h)(g)^{-1} \cdot f, X(a(g))(F)),$$

where

$$\begin{array}{rcl} (f, F) & \in & (\wedge_h^{\mathrm{rig}} X)(Y) \\ g & \in & \mathbb{G}(P)(Y) \\ X(a(g)) : X(Y \times_{\mathbb{P}(Q)}^{f} \mathbb{P}(P)) & \to & X(Y \times_{\mathbb{P}(Q)}^{\mathbb{G}(h)^{-1}(g)\cdot f} \mathbb{P}(P)) \\ (\mathbb{G}(h)(g)^{-1} \cdot f, X(a(g))) & \in & (\wedge_h^{\mathrm{rig}} X)(Y). \end{array}$$

2.5. **Properties of $\wedge_h^{\mathrm{rig}}$.** The basic properties of the functor (5) are summed up in the following

Lemma 4. *The functor $\wedge_h^{\mathrm{rig}}$: **PShLogSch** $\to$ **PShLogSch**$/\mathbb{P}(Q)$...*

(1) preserves inverse limits.

(2) preserves formally log étale maps.
(3) satisfies $\wedge_h^{\mathrm{rig}} \underline{X} = \mathbb{P}(Q) \times \underline{X}$ *for any presheaf* $\underline{X}$ *on schemes (regarded as a presheaf on log schemes by setting* $\underline{X}(Y) := \underline{X}(\underline{Y})$*).*
(4) preserves strict maps having a given property P *of morphisms of schemes stable under base change (e.g. being an étale cover, being of locally finite presentation, being proper, etc.).*
(5) preserves the property of having the sheaf property for a given subcanonical topology τ *on* **LogSch**.

Proof. For (1), we consider an inverse limit system $i \mapsto X_i$ in **PShLogSch** with inverse limit X. Note $X(Y) = \varprojlim X_i(Y)$ for $Y \in$ **LogSch**. An element of $(\varprojlim (\wedge_h^{\mathrm{rig}} X_i))(Y)$ is a set of pairs $(f_i, F_i) \in \prod_i (\wedge_h^{\mathrm{rig}} X_i)(Y)$ satisfying the usual compatibility for maps $i \to j$ in the indexing category. But actually, this describes the inverse limit of the $\wedge_h^{\mathrm{rig}} X_i$ taken in **PShLogSch**, whereas we actually take it in **PShLogSch**/$\mathbb{P}(Q)$, which amounts to requiring that all the f_i are given by some common $f \in \mathbb{P}(Q)(Y)$, so all the F_i are maps

$$F_i : Y \times^f_{\mathbb{P}(Q)} \mathbb{P}(P) \to X_i,$$

and the compatibility in i says these F_i come from a unique map $F : Y \times_{\mathbb{P}(Q)} \mathbb{P}(P) \to X$. Evidently then, $(f, F_i)_i \mapsto (f, F)$ defines a bijection

$$(\varprojlim (\wedge_h^{\mathrm{rig}} X_i))(Y) \quad = \quad (\wedge_h^{\mathrm{rig}} X)(Y)$$

natural in X, so (1) is proved.

Recall that a map $X \to X'$ in **PShLogSch** is formally log étale iff any commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] \arrow[ur, dotted] & X' \end{tikzcd}$$

in **PShLogSch** where $Y \to Y'$ is a strict square zero closed embedding of log schemes (representable presheaves) has a unique completion as indicated. Given such a map $X \to X'$, we must show that $\wedge_h^{\mathrm{rig}} X \to \wedge_h^{\mathrm{rig}} X'$ is formally log étale, so consider a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "{(f,F)}"] \arrow[d] & \wedge_h^{\mathrm{rig}} X \arrow[d] \\ Y' \arrow[r, "{(f',F')}"] \arrow[ur, dotted] & \wedge_h^{\mathrm{rig}} X' \end{tikzcd} \tag{7}$$

with $Y \hookrightarrow Y'$ a strict square zero closed embedding of log schemes. In particular,

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d] & \mathbb{P}(Q) \arrow[d, equal] \\ Y' \arrow[r, "f'"] & \mathbb{P}(Q) \end{tikzcd}$$

commutes, so the natural identification

$$Y \times^{f}_{\mathbb{P}(Q)} \mathbb{P}(P) \;=\; Y \times_{Y'} (Y' \times^{f'}_{\mathbb{P}(Q)} \mathbb{P}(P))$$

shows that the left vertical arrow in the diagram

$$\begin{tikzcd} Y \times^{f}_{\mathbb{P}(Q)} \mathbb{P}(P) \arrow[r, "F"] \arrow[d] & X \arrow[d] \\ Y' \times^{f'}_{\mathbb{P}(Q)} \mathbb{P}(P) \arrow[r, "F'"] \arrow[ur, dotted, "G"] & X' \end{tikzcd}$$

is a base change of $Y \to Y'$, hence is itself a square zero closed embedding of log schemes, hence a completion G exists as indicated and then (f', G) completes (7).

The point of (3) is that $\underline{Y \times^{f}_{\mathbb{P}(Q)} \mathbb{P}(P)} = \underline{Y}$ independent of the $f \in \mathbb{P}(Q)(Y)$ used to form the fibered product, so the datum of $F \in \underline{X}(Y \times_{\mathbb{P}(Q)} \mathbb{P}(P))$ is just the datum of an element of $X(\underline{Y}) = \underline{X}(Y)$.

The statement (4) is mostly a matter of definitions. For $X \in$ **PShLogSch**, we obtain a new object $\underline{X} \in$ **PShLogSch** by setting $\underline{X}(Y) := X(\underline{Y})$. We may also regard $\underline{X}$ as an object of **PShSch**. To say that a map $X' \to X$ in **PShLogSch** is *strict with property* P means two things: First, the diagram

(8)
$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \underline{X'} \arrow[d] \\ X \arrow[r] & \underline{X} \end{tikzcd}$$

must be cartesian. Second, the map $\underline{X'} \to \underline{X}$ should have property P in the sense that for any map $\underline{Y} \to \underline{X}$ from a *scheme* $\underline{Y}$, the base change $\underline{Y'} := \underline{Y} \times_{\underline{X}} \underline{X'}$ must "be" a scheme (be representable), and the structure map $\underline{Y'} \to \underline{Y}$ should "be" a map of schemes with property P. The important point is that being strict with property P is stable under base change. Now suppose $X' \to X$ is strict with property P and we want to show that the same is true of $\wedge^{\mathrm{rig}}_h X' \to \wedge^{\mathrm{rig}}_h X$. Since (8) is cartesian, we obtain a cartesian diagram

$$\begin{tikzcd} \wedge^{\mathrm{rig}}_h X' \arrow[r] \arrow[d] & \mathbb{P}(Q) \times \underline{X'} \arrow[d] \\ \wedge^{\mathrm{rig}}_h X \arrow[r] & \mathbb{P}(Q) \times \underline{X} \end{tikzcd}$$

by applying the results (1) and (3) proved above. But the right vertical arrow is just the product of $\underline{X}' \to \underline{X}$ and the identity of $\mathbb{P}(Q)$ (the identification of (3) is natural), so in fact

$$\begin{tikzcd} \wedge_h^{\mathrm{rig}} X' \arrow[r] \arrow[d] & \underline{X}' \arrow[d] \\ \wedge_h^{\mathrm{rig}} X \arrow[r] & \underline{X} \end{tikzcd}$$

is also cartesian. This in turn implies that

$$\begin{tikzcd} \wedge_h^{\mathrm{rig}} X' \arrow[r] \arrow[d] & \underline{\wedge_h^{\mathrm{rig}} X'} \arrow[r] \arrow[d] & \underline{X}' \arrow[d] \\ \wedge_h^{\mathrm{rig}} X \arrow[r] & \underline{\wedge_h^{\mathrm{rig}} X} \arrow[r] & \underline{X} \end{tikzcd}$$

is cartesian, and the part about preservation of property P is now obvious.

The natural maps $Y \to \underline{Y}$ yield a natural map $X \to \underline{X}$ in **PShLogSch**. We know this new presheaf $\underline{X}$ satisfies $\wedge_h \underline{X} = \mathbb{P}(Q) \times \underline{X}$ (naturally in X) by the previous argument. By definition, a morphism $X \to X'$ in **PShLogSch** is strict (of (locally) finite presentation) iff

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \underline{X} \arrow[d] \\ X' \arrow[r] & \underline{X}' \end{tikzcd}$$

is cartesian (and $\underline{X} \to \underline{X}'$ is representable by maps of schemes of (locally) finite presentation). When this is holds, applying (1) and the "We know..." observation to this cartesian square yields (4).

For (5), we don't actually need that τ is subcanonical; we merely need that $\mathbb{P}(Q)$ is a sheaf in the topology τ. Consider a τ cover $t : Y \to Z$. We must prove that

$$(\wedge_h^{\mathrm{rig}} X)(Z) \to (\wedge_h^{\mathrm{rig}} X)(Y) \rightrightarrows (\wedge_h^{\mathrm{rig}} X)(Y \times_Z Y) \tag{9}$$

is an equalizer diagram of sets assuming X is a sheaf for τ. Consider an element (f, F) in the middle set with the same images

$$(f\pi_1, F_1) = (f\pi_2, F_2)$$

under the parallel arrows. The fact that $f\pi_1 = f\pi_2 \in \mathbb{P}(Q)(Y \times_Z Y)$ and the fact that $\mathbb{P}(Q)$ has the sheaf property for τ together imply that there is a unique $g \in \mathbb{P}(Q)(Z)$ with $f = gt = \mathbb{P}(Q)(Y)$. The identification

$$Y \times^f_{\mathbb{P}(Q)} \mathbb{P}(Q) \quad = \quad Y \times_Z (Z \times^g_{\mathbb{P}(Q)} \mathbb{P}(P))$$

proves that the natural map

$$Y \times^f_{\mathbb{P}(Q)} \mathbb{P}(P) \quad \to \quad Z \times^g_{\mathbb{P}(Q)} \mathbb{P}(P) \tag{10}$$

is a base change of t, hence a τ cover. Furthermore, the fibered product of this map with itself is identified with

$$(Y \times_Z Y) \times^{f\pi_1 = f\pi_2}_{\mathbb{P}(Q)} \mathbb{P}(P)$$

and F_1 and F_2 are the images of F under the parallel arrows in

$$X(Z \times^g_{\mathbb{P}(Q)} \mathbb{P}(P)) \to X(Y \times^f_{\mathbb{P}(Q)} \mathbb{P}(P)) \rightrightarrows X((Y \times_Z Y) \times^{f\pi_1=f\pi_2}_{\mathbb{P}(Q)} \mathbb{P}(P)).$$

But $F_1 = F_2$, so by the sheaf property of X for the cover (10), we conclude the existence of a unique $G \in X(Z \times^g_{\mathbb{P}(Q)} \mathbb{P}(P))$ pulling back to F via X of the covering map (10). The pair $(g, G) \in (\wedge^{\mathrm{rig}}_h X)(Z)$ is then the unique element mapping to (f, F) via the left arrow in (9). $\square$

2.6. **Explicit constructions.** The purpose of this section is to explicitly construct the presheaf $\wedge^{\mathrm{rig}}_h X$ for some simple choices of h, X. In this section, we say that $X \in$ **PShLogSch** is a *log algebraic space* if it is a sheaf in the strict étale topology and there is a log scheme Y and a strict étale cover $Y \to X$. A log algebraic space X is *fine* iff one can take Y fine.

In the following lemma, **LogSch** denotes the category of integral log schemes.

Lemma 5. *Suppose $h : Q \hookrightarrow P$ is an integral monomorphism of integral monoids with a splitting set S, and $X = \operatorname{Spec}(\mathbb{N} \to \mathbb{Z}[\mathbb{N}])$. Then we have a natural isomorphism*

$$\wedge^{\mathrm{rig}}_h X = \coprod_S (\mathbb{P}(Q) \times X)$$

in **PShLogSch**$/\mathbb{P}(Q)$. *In particular, $\wedge^{\mathrm{rig}}_h X$ is representable.*

Proof. Note that X "is" ("represents") the presheaf

$$\begin{aligned} \mathbf{LogSch}^{\mathrm{op}} &\to \mathbf{Sets} \\ Y &\mapsto \mathcal{M}_Y(Y). \end{aligned}$$

Consider an element $(f, F) \in (\wedge^{\mathrm{rig}}_h X)(Y)$. Then $F \in X(Y \times^f_{\mathbb{P}(Q)} \mathbb{P}(P))$ is a global section of the log structure $\underline{P} \oplus_{\underline{Q}} \mathcal{M}_Y$ on $\underline{Y}$ (c.f. Remark 1). By Lemma 3, there is an isomorphism

$$\underline{S} \times \mathcal{M}_Y \to \underline{P} \oplus_{\underline{Q}} \mathcal{M}_Y$$

in the étale topos of $\underline{Y}$ (we use the fact that Y is integral here), so, assuming $\underline{Y}$ is connected, this F is a pair (s, F') consisting of an element $s \in S$ and a global section $F' \in \mathcal{M}_Y(Y)$. The pair (f, F') is the same thing as a map $Y \to \mathbb{P}(Q) \times X$. Taking the possibility of disconnected $\underline{Y}$ into account yields the result. $\square$

The lemma above is essentially the only situation in which we have a direct construction of $\wedge^{\mathrm{rig}}_h X$. For rigid P log points $\wedge^{\mathrm{rig}}_P$, we can directly construct $\wedge^{\mathrm{rig}}_P X$ for slightly more general X.

As a matter of notation, we will always write $[q] \in \mathbb{Z}[Q]$ for the image of $q \in Q$ in the corresponding monoid algebra. This is because $[0] = 1$ is the unit of $\mathbb{Z}[Q]$, and we certainly do not want to confuse this with $0 \in \mathbb{Z}[Q]$.

Lemma 6. *Let $P \in$ **SMon**, $Q \in$ **Mon**, $X := \operatorname{Spec}(Q \to \mathbb{Z}[Q])$. Then we have an isomorphism*

$$\wedge_P^{\mathrm{rig}} X \;\;=\;\; \coprod_{P^Q} X$$

*in **PShLogSch**, where P^Q is the monoid of monoid homomorphisms from Q to P.*

Remark 2. If P, Q are finitely generated (resp. fine), then one can easily check that P^Q is finitely generated (resp. fine) by using the fact that Q is finitely presented and an equalizer of maps from a finitely generated monoid is finitely generated. This fact is not particularly relevant since P^Q arises only as a set indexing a disjoint union.

Proof. We may assume $Y = (\underline{Y}, \mathcal{M}_Y)$ is connected, so, as in the previous proof, global sections of the log structure on $Y \times \mathbb{P}(P)$ are given by $\mathcal{M}_Y(Y) \oplus P$. By the adjointness properties of Spec and the associated log structure functors,

$$\begin{aligned} \operatorname{Hom}_{\mathbf{LogSch}}(Y \times \mathbb{P}(P), X) &= \operatorname{Hom}_{\mathbf{Mon}}(Q, \mathcal{M}_Y(Y) \oplus P) \\ &= \operatorname{Hom}_{\mathbf{Mon}}(Q, \mathcal{M}_Y(Y)) \times \operatorname{Hom}_{\mathbf{Mon}}(Q, P) \\ &= \operatorname{Hom}_{\mathbf{LogSch}}(Y, Q) \times P^Q. \end{aligned}$$

The choice of $h \in P^Q$ specifies the component of $\wedge_P X$ to which Y will map. Taking the possibility of disconnected $\underline{Y}$ into account yields the result. $\square$

More concretely, when $X = \operatorname{Spec}(Q \to \mathbb{Z}[Q])$, $\operatorname{Hom}_{\mathbf{LogSch}}(Y \times \mathbb{P}(P), X)$ is just the set of commutative diagrams

$$\begin{tikzcd} \mathcal{M}_X(X) \oplus P \arrow[r, "{(\alpha,0)}"] & \Gamma(X, \mathcal{O}_X) \\ Q \arrow[u, "\begin{pmatrix} f \\ h \end{pmatrix}"] \arrow[r, "{q \mapsto [q]}"] & \mathbb{Z}[Q] \arrow[u] \end{tikzcd}$$

and this certainly splits as a disjoint union over the possible $h \in P^Q$. Note that commutativity forces the right vertical arrow to be uniquely determined from the left arrow and given by:

$$[q] \mapsto \begin{cases} \alpha(f(q)), & h(q) = 0 \\ 0, & h(q) \neq 0 \end{cases}$$

The above commutative diagram fits into a larger commutative diagram

$$\begin{tikzcd}
Q \oplus P \arrow[r] \arrow[d, "\begin{pmatrix} f & 0 \\ 0 & \mathrm{Id}_P \end{pmatrix}"] & \mathbb{Z}[Q] \arrow[d, "q \mapsto \alpha(f(q))"] \\
\mathcal{M}_X(X) \oplus P \arrow[r, "(\alpha, 0)"] & \Gamma(X, \mathcal{O}_X) \\
Q \arrow[r] \arrow[u, "\begin{pmatrix} f \\ h \end{pmatrix}"] \arrow[uu, bend left=60, "\begin{pmatrix} \mathrm{Id}_Q \\ h \end{pmatrix}"] & \mathbb{Z}[Q] \arrow[u] \arrow[uu, bend right=60, "u_h"']
\end{tikzcd}$$

where the top horizontal arrow is given by

$$\begin{array}{rcl} Q \oplus P & \to & \mathbb{Z}[Q] \\ (q, p) & \mapsto & \begin{cases} [q], & p = 0 \\ 0, & p \neq 0, \end{cases} \end{array}$$

the top vertical arrows are uniquely determined by the rest of the diagram and where

$$u_h([q]) := \begin{cases} [q], & h(q) = 0 \\ 0, & h(q) \neq 0 \end{cases}$$

The universal map

$$\begin{tikzcd}
(\wedge_P X)_P \arrow[r, "u"] \arrow[d] & X \\
\wedge_P X &
\end{tikzcd}$$

in this case is the disjoint union of the maps

$$\begin{tikzcd}
(\wedge_P^h X)_P \arrow[r, "u_h"] \arrow[d] & X \\
\wedge_P^h X &
\end{tikzcd}$$

given by (Spec($\_ \to \_$) of) the diagrams:

$$\begin{tikzcd}
& Q \arrow[r] \arrow[dl, "(\mathrm{Id}_Q, h)"'] & \mathbb{Z}[Q] \arrow[dl, "u_h"] \\
Q \oplus P \arrow[r] & \mathbb{Z}[Q] \arrow[d, equal] & \\
Q \arrow[r] \arrow[u, "(\mathrm{Id}_Q, 0)"] & \mathbb{Z}[Q] &
\end{tikzcd}$$

Note that the component $\wedge_P^h X$ of $\wedge_P X$ indexed by $h \in P^Q$ is isomorphic to X, but the map u_h from this component to X is an isomorphism only on the component where $h = 0$. However, if Q is finitely generated, then u_h is finitely presented, so u

is locally of finite presentation. Moreover, if Q is integral, then $\wedge_P X$ is a fine log scheme, so the theorem is proved in this case.

2.7. **Bootstrapping.** Here we use the explicit constructions of the previous section and the results of Section 2.5 to prove the representability of $\wedge_h^{\mathrm{rig}} X$ by a log algebraic space for various X, h.

Lemma 7. *Let $h : Q \hookrightarrow P$ be as in Lemma 5. Then for any integral monoid R, the presheaf $\wedge_h^{\mathrm{rig}} \operatorname{Spec}(R \to \mathbb{Z}[R])$ is representable by log scheme. If R and Q are fine, then the scheme underlying $\wedge_h^{\mathrm{rig}} \operatorname{Spec}(R \to \mathbb{Z}[R])$ is of locally finite type over $\mathbb{Z}$ and its log structure is fine.*

Proof. For $R = \mathbb{N}$, this is clear from Lemma 5. For any set n, if

$$X = \operatorname{Spec}(\oplus_n \mathbb{N} \to \mathbb{Z}[\oplus_n \mathbb{N}]),$$

then we see that

$$\wedge_h^{\mathrm{rig}} X \quad = \quad \coprod_{S^n} (\mathbb{P}(Q) \times X),$$

either by arguing directly as in the proof of Lemma 5, or by using Lemma 5, Lemma 4(1) and the fact that

$$X \quad = \quad \prod_n \operatorname{Spec}(\mathbb{N} \to \mathbb{Z}[\mathbb{N}]).$$

We find that the lemma holds when R is free.

For arbitrary R, we can find a coequalizer diagram

$$\oplus_m \mathbb{N} \rightrightarrows \oplus_n \mathbb{N} \to R$$

in **Mon**, and we can take m, n finite if R is fine. Since

$$\operatorname{Spec}(\,\_\, \to \,\_\,) : \mathbf{Mon}^{\mathrm{op}} \quad \to \quad \mathbf{LogSch}$$

preserves inverse limits, we have an equalizer diagram

$$X \to \operatorname{Spec}(\oplus_n \mathbb{N} \to \mathbb{Z}[\oplus_n \mathbb{N}]) \rightrightarrows \operatorname{Spec}(\oplus_m \mathbb{N} \to \mathbb{Z}[\oplus_m \mathbb{N}])$$

in **LogSch**, with $X = \operatorname{Spec}(R \to \mathbb{Z}[R])$. By Lemma 4(1) we have an equalizer diagram

$$\wedge_h^{\mathrm{rig}} X \to \wedge_h^{\mathrm{rig}} \operatorname{Spec}(\oplus_n \mathbb{N} \to \mathbb{Z}[\oplus_n \mathbb{N}]) \rightrightarrows \wedge_h^{\mathrm{rig}} \operatorname{Spec}(\oplus_m \mathbb{N} \to \mathbb{Z}[\oplus_m \mathbb{N}])$$

in **PShLogSch**/$\mathbb{P}(Q)$. But **LogSch** *has* equalizers, and we know the terms on the right are representable, so we get the representability result. Furthermore, an equalizer of locally finite type schemes over $\mathbb{Z}$ is again locally finite type, and similarly for fineness, so we get the finiteness results. $\square$

Lemma 8. *Let $h : Q \hookrightarrow P$ be as in Lemma 5. Suppose X is a log scheme (resp. algebraic space) with a global chart $X \to \operatorname{Spec}(R \to \mathbb{Z}[R])$. Then $\wedge_h^{\mathrm{rig}} X$ is representable by a log scheme (resp. algebraic space). If Q and R are fine, then $\wedge_h^{\mathrm{rig}} X$ is of locally finite presentation over X and fine.*

Proof. To say that $X \to \operatorname{Spec}(R \to \mathbb{Z}[R])$ is a global chart is to say that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec}(R \to \mathbb{Z}[R]) \arrow[d] \\ \underline{X} \arrow[r] & \operatorname{Spec} \mathbb{Z}[R] \end{tikzcd}$$

is cartesian in **LogSch**. By Lemma 4(1)(3), we obtain a cartesian diagram

$$\begin{tikzcd} \wedge_h^{\mathrm{rig}} X \arrow[r] \arrow[d] & \wedge_h^{\mathrm{rig}} \operatorname{Spec}(R \to \mathbb{Z}[R]) \arrow[d] \\ \mathbb{P}(Q) \times \underline{X} \arrow[r] & \mathbb{P}(Q) \times \operatorname{Spec} \mathbb{Z}[R] \end{tikzcd}$$

in **PShLogSch**. All the claims follow from the previous lemma using standard stability under base change properties. □

Theorem 9. *Let $h : Q \hookrightarrow P$ be as in Lemma 5. Suppose $X \in$ **PShLogSch** is a log algebraic space. Then so is $\wedge_h^{\mathrm{rig}} X$. If X is fine and Q is fine, then $\wedge_h^{\mathrm{rig}} X$ is fine and of locally finite presentation over X.*

Proof. First, $\wedge_h^{\mathrm{rig}} X$ is a sheaf in the strict étale topology by Lemma 4(5). Next, choose a strict étale cover $Y \to X$ with Y fine if X is fine. After refining the cover if necessary, we can assume Y has a global chart $Y \to \operatorname{Spec}(R \to \mathbb{Z}[R])$ with R fine when X is fine. By definition of a strict étale cover, we have a cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \underline{Y} \arrow[d] \\ X \arrow[r] & \underline{X} \end{tikzcd}$$

where the right vertical arrow is an étale cover of the algebraic space $\underline{X}$ by the scheme $\underline{Y}$. Applying $\wedge_h^{\mathrm{rig}}$ and using Lemma 4(1)(3), and getting rid of a product with $\mathbb{P}(Q)$, we obtain a cartesian diagram

$$\begin{tikzcd} \wedge_h^{\mathrm{rig}} Y \arrow[r] \arrow[d] & \underline{Y} \arrow[d] \\ \wedge_h^{\mathrm{rig}} X \arrow[r] & \underline{X} \end{tikzcd}$$

in **PShLogSch**. Since the right vertical arrow is a strict étale cover, so is the left vertical arrow. Since $\wedge_h^{\mathrm{rig}} Y$ is a log scheme by the previous lemma, this proves that $\wedge_h^{\mathrm{rig}} X$ is a log algebraic space.

When X and Q are fine, then we can take R fine, so we know from the previous lemma that $\wedge_h^{\mathrm{rig}} Y$ is fine and of l.f.p. over $\underline{Y}$. But the left vertical arrow is a strict étale cover, so $\wedge_h^{\mathrm{rig}} X$ is also fine, and the bottom horizontal arrow is of l.f.p. because this is true of the top horizontal arrow and being of l.f.p. is étale local on the codomain. □

2.8. **Transition functions.** Before defining log nodes, it is helpful to recall some generalities about fiber bundles and transition functions. Let $\mathbf{C}$ be a site with fiber products, X an object of $\mathbf{C}$. A *fiber bundle* with fiber X is a $\mathbf{C}$ morphism $Y' \to Y$ such that there exists a cartesian diagram

$$\begin{tikzcd} U \times X \arrow[r, "b"] \arrow[d, "\pi_1"'] & Y' \arrow[d] \\ U \arrow[r, "a"] & Y \end{tikzcd} \tag{11}$$

(called a *trivialization* of $Y' \to Y$) where $a : U \to Y$ is a cover of Y. Given a trivialization (11), one sees easily that both diagrams

$$\begin{tikzcd} (U \times_Y U) \times X \arrow[r, "b(\pi_1 \times X)"] \arrow[d, "\pi_{12}"'] & Y' \arrow[d] \\ U \times_Y U \arrow[r, "a\pi_1 = a\pi_2"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (U \times_Y U) \times X \arrow[r, "b(\pi_2 \times X)"] \arrow[d, "\pi_{12}"'] & Y' \arrow[d] \\ U \times_Y U \arrow[r, "a\pi_1 = a\pi_2"] & Y \end{tikzcd}$$

are cartesian, so by the universal property of fibered products, there is a unique $\mathbf{C}$ morphism

$$g : (U \times_Y U) \times X \to (U \times_Y U) \times X$$

making the diagram

$$\begin{tikzcd} (U \times_Y U) \times X \arrow[dr, "g"', "\cong"] \arrow[drr, bend left, "b(\pi_2 \times X)"] \arrow[ddr, bend right, "\pi_{12}"'] & & \\ & (U \times_Y U) \times X \arrow[r, "b(\pi_1 \times X)"] \arrow[d, "\pi_{12}"'] & Y' \arrow[d] \\ & U \times_Y U \arrow[r, "a\pi_1 = a\pi_2"] & Y \end{tikzcd}$$

commute. Reversing the roles of the π_i shows that g is invertible, so

$$g \in \operatorname{Aut}((U \times_Y U) \times X / U \times_Y U) = \operatorname{Aut}(X)(U \times_Y U),$$

where $\operatorname{Aut}(X)$ is the presheaf

$$\begin{array}{rcl} \operatorname{Aut}(X) : \mathbf{C}^{\mathrm{op}} & \to & \mathbf{Sets} \\ U & \mapsto & \operatorname{Aut}(U \times X/U). \end{array}$$

This element g is called the *transition function* of the trivialization (11). It enjoys various properties, such as a cocycle condition on the three pullbacks to

$$\operatorname{Aut}(X)(U \times_Y U \times_Y U),$$

naturality with respect to the trivialization, and naturality with respect to base change of Y and base change of trivialization.

Given a subpresheaf of groups $G \subseteq \operatorname{Aut}(X)$, a *fiber bundle* with fiber X and *structure group* G is a $\mathbf{C}$ morphism $Y' \to Y$ admitting a trivialization (11) whose transition function g lies in the subset

$$G(U \times_Y U) \subseteq \operatorname{Aut}(X)(U \times_Y U).$$

Fiber bundles with fiber X and structure group G are stable under base change.

When we study log nodes, we will be interested in a more general type of "fiber bundle". Fix a *model* $\mathbf{C}$ morphism $h : M' \to M$. Then a *fiber bundle* with *fiber* h (or *modelled on* h) is a $\mathbf{C}$ morphism $Y' \to Y$ such that there exists a cartesian diagram

$$\begin{tikzcd} M' \arrow[d, "h"'] & U' \arrow[l, "d"'] \arrow[r, "b"] \arrow[d] & Y' \arrow[d] \\ M & U \arrow[l, "c"'] \arrow[r, "a"] & Y \end{tikzcd} \tag{12}$$

(again called a trivialization) where $a : U \to Y$ is a cover of Y. The isomorphism

$$\begin{array}{rcl} (U \times_Y U) \times_M^{c\pi_1} M' & \to & (U \times_Y U) \times Y' \\ (u_1, u_2, m') & \mapsto & (u_1, u_2, b(u_1, m')) \end{array}$$

with inverse $(u_1, u_2, y') \mapsto (u_1, u_2, d(u_1, y'))$ (and a similar variant when $i = 2$) shows that both diagrams

$$\begin{tikzcd} (U \times_Y U) \times_M^{c\pi_1} M' \arrow[r, "b(\pi_1 \times M')"] \arrow[d, "\pi_{12}"'] & Y' \arrow[d] \\ (U \times_Y U) \arrow[r, "a\pi_1 = a\pi_2"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (U \times_Y U) \times_M^{c\pi_2} M' \arrow[r, "b(\pi_2 \times M')"] \arrow[d, "\pi_{12}"'] & Y' \arrow[d] \\ (U \times_Y U) \arrow[r, "a\pi_1 = a\pi_2"] & Y \end{tikzcd}$$

are cartesian, so the universal property yields an isomorphism

$$g : (U \times_Y U) \times_M^{c\pi_2} M' \to (U \times_Y U) \times_M^{c\pi_1} M'$$

of objects over[2] $U \times_Y U$ making the diagrams

$$\begin{tikzcd} (U \times_Y U) \times_M^{c\pi_2} M' \arrow[dr, "g", "\cong"'] \arrow[drr, bend left, "b(\pi_2 \times M')"] \arrow[ddr, bend right, "\pi_{12}"'] & & \\ & (U \times_Y U) \times_M^{c\pi_1} M' \arrow[r, "b(\pi_1 \times M')"] \arrow[d, "\pi_{12}"'] & Y' \arrow[d] \\ & U \times_Y U \arrow[r, "a\pi_1 = a\pi_2"] & Y \end{tikzcd}$$

[2]Note that the twist map $(u_1, u_2, m') \mapsto (u_2, u_1, m')$ would also furnish such an isomorphism, but not *over* $U \times_Y U$.

and

$$
\begin{tikzcd}
(U \times_Y U) \times_M^{c\pi_2} M' \arrow[r, "g", "\cong"'] \arrow[d, "\pi_{12}"'] \arrow[rr, bend left, "\pi_3"] & (U \times_Y U) \times_M^{c\pi_1} M' \arrow[r, "\pi_3"] \arrow[d, "\pi_{12}"'] & M' \arrow[d, "h"] \\
U \times_Y U \arrow[rr, bend right, "c\pi_2"'] & U \times_Y U \arrow[r, "c\pi_1"] & M
\end{tikzcd}
\tag{13}
$$

commute.

The previous notion of fiber bundle is recovered as the case where $M' = X$ and M is the terminal object; our general notion of transition function reduces to the usual one in that case. In the general setup, it is difficult to say where the transition function lives, but we can formulate a notion of reduction of structure group which is adequate for our purposes.

Definition 4. Suppose $z : G' \to G$ is a morphism of group objects of $\mathbf{C}$. Then an *action* a of $z : G' \to G$ on $h : M' \to M$ consists of an action of G on M, denoted $(g, f) \mapsto g \cdot f$, and the following data: For each object Y of $\mathbf{C}$ and each $f \in M(Y)$, $g \in G'(Y)$, there is an isomorphism $a(g) : Y \times_M^{z(g)\cdot f} M' \to Y \times_M^f M'$ of objects of $\mathbf{C}$ over Y making the diagram

$$
\begin{tikzcd}
Y \times_M^{z(g)\cdot f} M' \arrow[r, "a(g)", "\cong"'] \arrow[d, "\pi_1"'] \arrow[rr, bend left, "\pi_2"] & Y \times_M^f M' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & M' \arrow[d, "h"] \\
Y \arrow[rr, bend right, "z(g)\cdot f"'] & Y \arrow[r, "f"] & M
\end{tikzcd}
\tag{14}
$$

commute. This data should be natural in Y, f, and the assignment $g \mapsto a(g)$ should be a group homomorphism in the sense that $a(\mathrm{Id}) : Y \times_M^{z(\mathrm{Id})\cdot f = f} M' \to Y \times_M^f M'$ is the identity and $a(g_1 g_2) = a(g_1)a(g_2)$ under the identification

$$
Y \times_M^{z(g_1 g_2)\cdot f} M' = Y \times_M^{z(g_1)\cdot(z(g_2)\cdot f)} M'.
$$

Given an action a of $z : G' \to G$ on $h : M' \to M$ in the above sense, we say that a $\mathbf{C}$ morphism $Y' \to Y$ is an *h fiber bundle with structure group a* (or z if a is clear from context) if there is a trivialization (11) whose transition function diagram (13) is isomorphic to the action diagram (14) for some $g \in G'(U \times_Y U)$ (called the *transition function* of the trivialization) after making the substitutions $Y \mapsto U \times_Y U$, $f \mapsto c\pi_1$ in the latter.

When M is the terminal object, the data of a z action on h is nothing more or less than the data of a G' action on M' and the corresponding notion of fiber bundle with structure group z is just the usual notion of fiber bundle with fiber M' and structure group G'.

2.9. **Log nodes.** Let $h : Q \to P$ be an **SMon** morphism. In the terminology of the previous section, an h *log node* is a map of log schemes $Y' \to Y$ which is a $\mathbb{P}(h) : \mathbb{P}(P) \to \mathbb{P}(Q)$ fiber bundle (in the strict étale topology on **LogSch**) with reduction of structure group to $\mathbb{G}(h) : \mathbb{G}(P) \to \mathbb{G}(Q)$.

Definition 5. For an **SMon** morphism $h : Q \to P$ and a presheaf $X \in$ **PShLogSch**, let $\wedge_h X$ be the category whose objects are pairs $(Y' \to Y, F)$, where $Y' \to Y$ is an h log node and $F \in X(Y')$. A morphism

$$(Y' \to Y, F) \to (Z' \to Z, G)$$

in $\wedge_h X$ is a cartesian diagram

$$\begin{tikzcd} Y' \arrow[r, "b"] \arrow[d] & Z' \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

in **LogSch** such that $F = X(b)(G)$.

The category $\wedge_h X$ is called the category of *h log nodes in X*. The forget functor

$$(Y' \to Y, F) \mapsto Y$$

fibers $\wedge_h X$ in groupoids over **LogSch**. It is easily seen to be a stack in the strict étale topology when X is a sheaf in this topology.

2.10. **Log points.** When $h = 0 : 0 \to P$, we call an h log node a *P log point*. A P log point is a map $Y' \to Y$ of log schemes which is a $\mathbb{P}(P)$ fiber bundle with structure group $\mathbb{G}(P)$. Let us spell out the following special case of Definition 5.

Definition 6. For a sharp monoid P, and a presheaf $X \in$ **PShLogSch**, the category $\wedge_P X$ of *P log points* in X is the category whose objects are pairs $(Y' \to Y, F)$ consisting of a P log point $Y' \to Y$ and an element $F \in X(Y')$. Morphisms in $\wedge_P X$ are as in Definition 5.

2.11. **Quotient presentation.** Fix an **SMon** morphism $h : Q \to P$. The point of this section is to present the category of h log nodes in X as a global quotient of the category of *rigid* h log nodes in X under appropriate hypotheses on X.

Proposition 10. *Suppose X has the sheaf property for h log nodes. Then there is an equivalence*

$$\wedge_h X \quad = \quad [\wedge_h^{\mathrm{rig}} X / \mathbb{G}(P)]$$

of groupoid fibrations over **LogSch**.

Theorem 11. *Suppose X is a log algebraic space and h is as in Lemma 3. Then $\wedge_h X$ is a log algebraic stack (admits a strict smooth cover by a log scheme). If X and Q are fine, then $\wedge_h X$ is a fine log algebraic stack admitting a strict smooth cover by a log scheme of locally finite presentation over X.*

For concreteness, we spell out a particular case of the previous theorem:

Theorem 12. *Suppose X is a fine log algebraic space. Then for any integral monoid P, $\wedge_P X$ is a fine log algebraic stack admitting a strict smooth cover by a log scheme of locally finite presentation over X.*

3. The DF(1) case

In this section we will give a largely self-contained treatment of log points and log nodes in a DF(1) log scheme X. In this situation we have relatively simple notions of *contact order* and a relatively simple global construction for log points and log nodes in X. In particular, we can easily describe the cohomology of these spaces. The discussion of this section generalizes easily to the case of a DF(n) log scheme, and also fairly easily to the case of a DF(P) log scheme for an arbitrary monoid P.

3.1. Scholium on DF(1) log structures.

The material in this section is merely a slight elaboration of some things M. Olsson mentioned in his Pisa lectures on log geometry.

Consider a scheme $\underline{X}$ (we drop the underline from various notation if there is no chance of confusion) equipped with the data of a line bundle (invertible sheaf of $\mathcal{O}_X$ modules) L and a section $s : \mathcal{O}_X \to L$. To this data we will associate a log structure $\alpha : \mathcal{M}(L,s) \to \mathcal{O}_X$ on $\underline{X}$. This log structure will be natural under pullback in the following sense: Given a map of schemes $f : \underline{X}' \to \underline{X}$, there will be a natural identification

$$f^*\mathcal{M}(L,s) \quad = \quad \mathcal{M}(f^*L, f^*s).$$

The log structure $\mathcal{M}(L,s)$ is also natural in (L,s) in the following sense: Given another line bundle L' with section $s' : \mathcal{O}_X \to L'$ and an isomorphism of line bundles $\phi : L \to L'$ satisfying $\phi s = s'$, we will obtain an isomorphism

$$\phi : \mathcal{M}(L', s') \quad \to \quad \mathcal{M}(L,s)$$

of log structures on $\underline{X}$ (note the abusive notation and the contravariant nature of the isomorphism).

The log structure $\alpha : \mathcal{M}(L,s) \to \mathcal{O}_X$ is equipped with various additional structure: It comes with a map

$$\beta : \underline{\mathbb{N}} \quad \to \quad \overline{\mathcal{M}}(L,s)$$

of sheaves of monoids (this is equivalent to the data of the global section $\beta(1) \in \overline{\mathcal{M}}(L,s)(X)$ of the characteristic monoid) which locally lifts to a chart for $\mathcal{M}(L,s)$. It also comes with an $\mathcal{O}_X^*$ equivariant map

$$\eta : L^\vee \setminus \mathbf{0}_X \quad \to \quad \mathcal{M}(L,s).$$

Here $L^\vee \setminus \mathbf{0}_X$ is the $\mathcal{O}_X^*$ torsor associated to the *dual* of L (the complement of the zero section in $L^\vee$) and $\mathcal{O}_X^*$ acts on $\mathcal{M}(L,s)$ through the inclusion of monoids $\mathcal{O}_X^* \subseteq \mathcal{M}(L,s)$. This action is faithful because every log structure considered in this manuscript (including $\mathcal{M}(L,s)$) will be integral. These additional structures are related by a commutative diagram

$$\begin{tikzcd} \underline{\mathbb{N}} \arrow[d, "\beta"'] & L^\vee \setminus \mathbf{0}_X \arrow[l, "1"'] \arrow[d, "\eta"] \arrow[dr, "s^\vee"] & \\ \overline{\mathcal{M}}(L,s) & \mathcal{M}(L,s) \arrow[l] \arrow[r, "\alpha"] & \mathcal{O}_X \end{tikzcd}$$

of $\mathcal{O}_X^*$ equivariant morphisms (with $\mathcal{O}_X^*$ acting trivially on $\underline{\mathbb{N}}$ and $\overline{\mathcal{M}}(L,s)$). The *universal property* of $\mathcal{M}(L,s)$ is the following: If $\alpha' : \mathcal{M}' \to \mathcal{O}_X$ is a log structure on $\underline{X}$, then giving a map of log structures

$$g : \mathcal{M}(L,s) \quad \to \quad \mathcal{M}'$$

is the same thing as giving an $\mathcal{O}_X^*$ equivariant map

$$h : L^\vee \setminus \mathbf{0}_X \quad \to \quad \mathcal{M}'$$

making

$$\begin{tikzcd} L^\vee \arrow[d, "h"] \arrow[dr, "s^\vee"] & \\ \mathcal{M}' \arrow[r, "\alpha'"] & \mathcal{O}_X \end{tikzcd}$$

commute. More precisely,

$$\eta^* : \mathrm{Hom}_{\mathbf{LogStr}(\underline{X})}(\mathcal{M}(L,s), \mathcal{M}') \quad \to \quad \mathrm{Hom}_{\mathcal{O}_X^*/\mathcal{O}_X}(L^\vee \setminus \mathbf{0}_X, \mathcal{M}')$$

is bijective. Here, $\mathrm{Hom}_{\mathcal{O}_X^*/\mathcal{O}_X}$ means morphisms in the category of sheaves with $\mathcal{O}_X^*$ action over $\mathcal{O}_X$.

Now, having made all these assertions about $\mathcal{M}(L,s)$, we still need to construct it! The point is that, in light of the universal property, we can construct it locally. The locally constructed log structures will then glue to a global log structure because they both satisfy the same universal property on overlaps. To perform the local construction of $\mathcal{M}(L,s)$, we can assume there is a section σ of the $\mathcal{O}_X^*$ torsor $L^\vee \setminus \mathbf{0}_X$. We then declare $\mathcal{M}(L,s)$ to be the log structure associated to the prelog structure

$$\begin{aligned} a : \underline{\mathbb{N}} \quad &\to \quad \mathcal{O}_X \\ 1 \quad &\mapsto \quad s^\vee(\sigma). \end{aligned}$$

That is:

$$\mathcal{M}(L,s) \quad := \quad \underline{\mathbb{N}} \oplus_{a^{-1}\mathcal{O}_X^*} \mathcal{O}_X^*,$$

with structure map

$$\begin{aligned} \alpha : \underline{\mathbb{N}} \oplus_{a^{-1}\mathcal{O}_X^*} \mathcal{O}_X^* \quad &\to \quad \mathcal{O}_X \\ [n,u] \quad &\mapsto \quad us^\vee(\sigma)^n. \end{aligned}$$

Since we have the section σ, any other section of $L^\vee \setminus \mathbf{0}_X$ can be written uniquely as $u\sigma$ for some $u \in \mathcal{O}_X^*$. We can therefore define the map η locally by the formula:

$$\begin{array}{rcl} \eta : L^\vee \setminus \mathbf{0}_X & \to & \underline{\mathbb{N}} \oplus_{a^{-1}\mathcal{O}_X^*} \mathcal{O}_X^* \\ u\sigma & \mapsto & [1, u]. \end{array}$$

This map is clearly $\mathcal{O}_X^*$ equivariant and the diagram

$$\begin{tikzcd} L^\vee \setminus \mathbf{0}_X \arrow[d, "\eta"] \arrow[dr, "s^\vee"] & \\ \underline{\mathbb{N}} \oplus_{a^{-1}\mathcal{O}_X^*} \mathcal{O}_X^* \arrow[r, "\alpha"] & \mathcal{O}_X \end{tikzcd}$$

clearly commutes. One can check that these locally defined η glue to a global η by examining the effect of a different choice of section σ'. The local version of $\beta : \underline{\mathbb{N}} \to \overline{\mathcal{M}}(L, s)$ is just the natural map from the prelog structure a to the characteristic monoid of the associated log structure. These local β clearly glue to a global β because changing σ by a unit does not effect the latter map.

To check that $\mathcal{M}(L, s)$ has the desired universal property, consider an $\mathcal{O}_X^*$ equivariant map

$$h : L^\vee \setminus \mathbf{0}_X \;\; \to \;\; \mathcal{M}'$$

making

$$\begin{tikzcd} L^\vee \arrow[d, "h"] \arrow[dr, "s^\vee"] & \\ \mathcal{M}' \arrow[r, "\alpha'"] & \mathcal{O}_X \end{tikzcd}$$

commute. From this, we define a map of log structures $g : \mathcal{M}(L, s) \to \mathcal{M}'$ on $\underline{X}$ by the formula:

$$\begin{array}{rcl} \underline{\mathbb{N}} \oplus_{a^{-1}\mathcal{O}_X^*} \mathcal{O}_X^* & \to & \mathcal{M}' \\ [n, u] & \mapsto & uh(\sigma)^n. \end{array}$$

The composition

$$\eta^* g = g\eta : L^\vee \setminus \mathbf{0}_X \;\; \to \;\; \mathcal{M}'$$

is clearly just the original map h.

This concludes the construction of $\mathcal{M}(L, s)$.

We can reformulate our discussion in the language of stacks as follows: consider the data of $\underline{X}$, L, s as above. This may be viewed as a map of algebraic stacks

$$(L, s) : \underline{X} \;\; \to \;\; [\underline{\operatorname{rig} A}^1/\mathbb{G}_m].$$

Indeed, the scheme

$$L^\vee \setminus \mathbf{0}_X \;\; = \;\; \operatorname{Spec}_X \left(\oplus_{n \in \mathbb{Z}} L^n\right)$$

comes with a natural $\mathbb{G}_m$ action (relative to $\underline{X}$) corresponding to the indicated $\mathbb{Z}$ grading of its structure sheaf making it a $\mathbb{G}_m$ torsor over $\underline{X}$. Furthermore, the section $s(1) \in L$ is a section of the degree 1 part of the structure sheaf of $L^\vee \setminus \mathbf{0}_X$, hence it corresponds to a $\mathbb{G}_m$ equivariant map

$$\text{“}s\text{”} : L^\vee \setminus \mathbf{0}_X \quad \to \quad \underline{\operatorname{rig} A^1}$$

for the usual action of $\mathbb{G}_m$ on $\underline{\operatorname{rig} A^1}$.

So far we have only considered $\underline{\operatorname{rig} A^1}$ with the trivial log structure. If we consider $\operatorname{rig} A^1$ with its natural log structure, then it represents the presheaf

$$\begin{array}{rcl} \mathbf{LogSch}^{\mathrm{op}} & \to & \mathbf{Sets} \\ Y & \mapsto & \mathcal{M}_Y(Y). \end{array}$$

The group scheme $\mathbb{G}_m$ (with the trivial log structure) represents the presheaf

$$\begin{array}{rcl} \mathbf{LogSch}^{\mathrm{op}} & \to & \mathbf{Sets} \\ Y & \mapsto & \mathcal{O}_Y^*(Y). \end{array}$$

and the natural action of $\mathbb{G}_m$ on $\operatorname{rig} A^1$ corresponds to the natural (effective!) action of $\mathcal{O}_Y^*(Y)$ on $\mathcal{M}_Y(Y)$. The quotient $[\operatorname{rig} A^1/\mathbb{G}_m]$ (taken either in the category of stacks in the strict étale topology or sheaves in the strict étale topology—the result is the same since the action is effective) therefore represents the presheaf

$$\begin{array}{rcl} \mathbf{LogSch}^{\mathrm{op}} & \to & \mathbf{Sets} \\ Y & \mapsto & \overline{\mathcal{M}}_Y(Y). \end{array}$$

The natural map

$$[\operatorname{rig} A^1/\mathbb{G}_m] \quad \to \quad [\underline{\operatorname{rig} A^1}/\mathbb{G}_m]$$

then has the following interpretation: Given a global section $c \in \overline{\mathcal{M}}_Y(Y)$, one obtains an $\mathcal{O}_Y^*$ torsor

$$T_c \quad := \quad \{m \in \mathcal{M}_Y : \overline{m} = c \text{ in } \overline{\mathcal{M}}_Y\}$$

and an $\mathcal{O}_Y^*$ equivariant map

$$\alpha_Y | T_c : T_c \quad \to \quad \mathcal{O}_Y$$

from T_c to $\mathcal{O}_Y$, which is the data of a Y point of $[\underline{\operatorname{rig} A^1}/\mathbb{G}_m]$. (We saw earlier how to convert from this to principal $\mathbb{G}_m$ bundles and $\mathbb{G}_m$ equivariant maps to $\underline{\operatorname{rig} A^1}$.)

Given the data $\underline{X}$, L, s as above, if we let $X := (\underline{X}, \mathcal{M}(L,s))$, then I claim:

Proposition 13. *The diagram*

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d] & {[\operatorname{rig} A^1/\mathbb{G}_m]} \arrow[d] \\ \underline{X} \arrow[r, "{(L,s)}"] & {[\underline{\operatorname{rig} A^1}/\mathbb{G}_m]} \end{tikzcd}$$

is a cartesian diagram of log algebraic stacks.

The top horizontal arrow β is (pullback of) the natural global section

$$\beta(1) \in \overline{\mathcal{M}}(L,s)(X)$$

discussed above. Indeed, for a log scheme Y, the data of a map from Y to the cartesian product stack

$$\underline{X} \times_{[\underline{\mathrm{rig}\, A}^1/\mathbb{G}_m]} [\mathrm{rig}\, A^1/\mathbb{G}_m]$$

consists of:

A1. A map of schemes $f : \underline{Y} \to \underline{X}$.
A2. A global section $e \in \overline{\mathcal{M}}_Y(Y)$.
A3. An isomorphism $h : f^*L^\vee \setminus \mathbf{0}_Y \to T_e$ of $\mathcal{O}_Y^*$ torsors making the diagram

$$\begin{tikzcd} f^*L^\vee \setminus \mathbf{0}_Y \arrow[rr, "h"] \arrow[dr, "f^*s^\vee"'] & & T_e \arrow[dl, "\alpha_Y"] \\ & \mathcal{O}_Y & \end{tikzcd}$$

commute.

By the universal property of

$$f^*\mathcal{M}(L,s) \quad = \quad \mathcal{M}(f^*L, f^*s)$$

discussed above, the composition of h with the inclusion $T_e \subseteq \mathcal{M}_Y$ determines a map $f^\dagger : f^*\mathcal{M}(L,s) \to \mathcal{M}_Y$ of log structures on $\underline{Y}$, and hence the pair $(f, f^\dagger)$ determines a map of log schemes $f : Y \to X$. This shows how to go from a Y point of the cartesian product to a Y point of X. To go the other way, suppose we have a map of log schemes $(f, f^\dagger) : Y \to X$. Then of course we have the underlying map of schemes $f : \underline{Y} \to \underline{X}$. The map of log structures $f^\dagger : f^*\mathcal{M}(L,s) \to \mathcal{M}_Y$ sits inside a commutative diagram

$$\begin{tikzcd} & f^*L^\vee \setminus \mathbf{0}_Y \arrow[r, "f^*s^\vee"] \arrow[d, "f^*\eta"] \arrow[dl, "1"'] & \mathcal{O}_Y \\ & f^*\mathcal{M}(L,s) \arrow[r, "f^\dagger"] \arrow[ur, "f^*\alpha"] \arrow[d] & \mathcal{M}_Y \arrow[u, "\alpha_Y"'] \arrow[d] \\ \underline{\mathbb{N}} \arrow[r, "\beta"] & f^{-1}\overline{\mathcal{M}}(L,s) \arrow[r, "\overline{f}^\dagger"] & \overline{\mathcal{M}}_Y \end{tikzcd} \tag{15}$$

from which we can read off the other pieces of data constituting a map to the cartesian product: The global section $e \in \overline{\mathcal{M}}_Y(Y)$ is the composition of the global section $\beta(1)$ (really $f^{-1}\beta(1)$) of $f^{-1}\overline{\mathcal{M}}(L,s)$ and the map $\overline{f}^\dagger$, and the isomorphism h of $\mathcal{O}_Y^*$ torsors is the composition $f^\dagger f^*\eta$ (which lands in T_e by commutativity of the above diagram).

Example. Consider the case $\underline{X} = \underline{\operatorname{rig} A^1}$ with the line bundle $L = \mathcal{O}_{\operatorname{rig} A^1}(0)$ whose sections are the "meromorphic functions on $\operatorname{rig} A^1$ with at most a simple pole at the origin". This L comes with the tautological section $s = 1$. The dual map $s^\vee : L^\vee \to \mathcal{O}_{\operatorname{rig} A^1}$ is of course nothing but the inclusion of the ideal of functions vanishing at the origin; this map takes the nowhere vanishing section

$$z \in (L^\vee \setminus \mathbf{0}_{\operatorname{rig} A^1})(\operatorname{rig} A^1)$$

to $z \in \mathcal{O}_{\operatorname{rig} A^1}(\operatorname{rig} A^1) = \mathbb{Z}[z]$. According to our above local recipe (which applies globally here), the log structure $\mathcal{M}(L, s)$ is the one associated to the chart

$$\begin{array}{rcl} \mathbb{N} & \to & \mathcal{O}_{\operatorname{rig} A^1}(\operatorname{rig} A^1) = \mathbb{Z}[z] \\ 1 & \mapsto & z. \end{array}$$

This is the "usual" log structure on $\operatorname{rig} A^1$. This example readily generalizes to the case of a scheme $\underline{X}$ with a Cartier divisor $\underline{D} \subseteq \underline{X}$ by taking $L = \mathcal{O}_X(D)$ with the tautological section $s_D \in \mathcal{O}_X(D)(X)$ (pardon the notation!) vanishing along D. The dual map $s^\vee : \mathcal{O}_X(-D) \to \mathcal{O}_X$ is the inclusion of the ideal sheaf of $\underline{D}$, and a (local) section of the $\mathcal{O}_X^*$ torsor $\mathcal{O}_X(-D) \setminus \mathbf{0}_X$ is a (local) equation f for $\underline{D}$ and the corresponding log structure $\mathcal{M}(\mathcal{O}_X(D), s_D)$ is (locally) the one associated to the prelog structure

$$\begin{array}{rcl} \mathbb{N} & \mapsto & \mathcal{O}_X \\ 1 & \mapsto & f. \end{array}$$

(This is *not* generally the same as the "divisorial" log structure

$$\mathcal{M}_X \;\; := \;\; \{ f \in \mathcal{O}_X : f|_{X \setminus D} \in \mathcal{O}^*_{X \setminus D} \}$$

associated to $\underline{D} \subseteq \underline{X}$, though they agree when $\underline{X}$ and $\underline{D}$ are smooth varieties.)

3.2. P **log points in a DF(1) log scheme.** Let P be a fine, sharp monoid. Let $\mathbb{P}(P) := \operatorname{Spec}(0 : P \to \mathbb{Z})$. Let $X = (\underline{X}, \mathcal{M}(L, s))$ be a log scheme with a DF(1) log structure (a log structure of the form discussed in the previous section). Consider the presheaf

$$\begin{array}{rcl} \wedge_P^{\operatorname{rig}} X : \mathbf{LogSch}^{\mathrm{op}} & \to & \mathbf{Sets} \\ Y & \mapsto & \operatorname{Hom}_{\mathbf{LogSch}}(Y \times \mathbb{P}(P), X). \end{array}$$

Note that

$$\underline{Y \times \mathbb{P}(P)} \;\; = \;\; \underline{Y}$$

and the log structure of $Y \times \mathbb{P}(P)$ is given by

$$\begin{array}{rcl} \alpha_{Y \times \mathbb{P}(P)} : \mathcal{M}_Y \oplus \underline{P} & \to & \mathcal{O}_Y \\ (m, p) & \mapsto & \begin{cases} \alpha_Y(m), & p = 0 \\ 0, & \text{otherwise.} \end{cases} \end{array} \tag{16}$$

3.2.1. *Contact order.* Since X is DF(1) it comes with the global section $\beta(1) \in \overline{\mathcal{M}}_X(X)$. Given a map

$$(f, f^\dagger) : Y \times \mathbb{P}(P) \to X$$

(i.e. an element of $(\wedge_P^{\mathrm{rig}} X)(Y)$), we can compose $\beta(1)$ (really $f^{-1}\beta(1)$) with

$$\overline{f}^\dagger : f^{-1}\overline{\mathcal{M}}_X \quad \to \quad \overline{\mathcal{M}}_Y \oplus \underline{P}$$

and the second projection π_2 onto $\underline{P}$ to obtain a global section

$$c \quad := \quad \pi_2 \overline{f}^\dagger \beta(1) \in \underline{P}(Y),$$

that is, a locally constant function $c : \underline{Y} \to P$. This locally constant function c is called the *contact order function.* The contact order function c is clearly natural in Y, so it yields a coproduct decomposition

$$\wedge_P^{\mathrm{rig}} X \quad = \quad \coprod_{p \in P} \wedge_{P,p}^{\mathrm{rig}} X,$$

where $\wedge_{P,p}^{\mathrm{rig}} X$ is the subpresheaf of $\wedge_P^{\mathrm{rig}} X$ consisting of P log points in X with contact order function given by the constant function p ("contact order p" for short). That is, $(\wedge_{P,p}^{\mathrm{rig}} X)(Y)$ consists of those

$$(f, f^\dagger) \in (\wedge_P^{\mathrm{rig}} X)(Y) = \mathrm{Hom}_{\mathbf{LogSch}}(Y \times \mathbb{P}(P), X)$$

where $\pi_2 \overline{f}^\dagger \beta \in \underline{P}(Y)$ is the constant function p.

In addition to the contact order function, we can also compose $\beta(1)$ (really $f^{-1}\beta(1)$) with $\overline{f}^\dagger : f^{-1}\overline{\mathcal{M}}_X \to \overline{\mathcal{M}}_Y \oplus \underline{P}$ and the *first* projection π_1 onto $\overline{\mathcal{M}}_Y$ to obtain a global section

$$e \quad := \quad \pi_1 \overline{f}^\dagger \beta(1) \in \overline{\mathcal{M}}_Y(Y).$$

The function e is also clearly natural in Y so it determines a morphism

$$e : \wedge_P^{\mathrm{rig}} X \quad \to \quad [\mathrm{rig}\, A^1 / \mathbb{G}_m]$$

and hence, by restriction to components, morphisms

$$e_p : \wedge_{P,p}^{\mathrm{rig}} X \quad \to \quad [\mathrm{rig}\, A^1 / \mathbb{G}_m]$$

for each $p \in P$.

3.2.2. *Definition of v_p.* Consider an element $p \in P$. If $p = 0$, let

$$v_p : [\mathrm{rig}\, A^1 / \mathbb{G}_m] \quad \to \quad [\underline{\mathrm{rig}\, A^1} / \mathbb{G}_m]$$

be the natural map (the one induced by the unique map $\mathrm{rig}\, A^1 \to \underline{\mathrm{rig}\, A^1}$ which is the identity on underlying schemes—this is $\mathbb{G}_m$ equivariant for the natural actions). If $p \neq 0$, let $v_p : [\mathrm{rig}\, A^1 / \mathbb{G}_m] \to [\underline{\mathrm{rig}\, A^1} / \mathbb{G}_m]$ be the map induced by the unique map $\mathrm{rig}\, A^1 \to \underline{\mathrm{rig}\, A^1}$ which is the contraction to the origin on underlying schemes—this is also $\mathbb{G}_m$ equivariant. When $p = 0$, the map $v_p : [\mathrm{rig}\, A^1 / \mathbb{G}_m] \to [\underline{\mathrm{rig}\, A^1} / \mathbb{G}_m]$ factors naturally through the inclusion of the "origin"

$$\underline{B\mathbb{G}_m} \quad \hookrightarrow \quad [\underline{\mathrm{rig}\, A^1} / \mathbb{G}_m]$$

(this map is a strict closed embedding). In either case ($p = 0$ or $p \neq 0$), we have

Proposition 14. *The diagram*

$$\begin{tikzcd} \wedge^{\mathrm{rig}}_{P,p} X \arrow[r, "e_p"] \arrow[d] & {[\mathrm{rig}\, A^1/\mathbb{G}_m]} \arrow[d, "v_p"] \\ \underline{X} \arrow[r, "(L,s)"] & {[\underline{\mathrm{rig}\, A}^1/\mathbb{G}_m]} \end{tikzcd}$$

is cartesian.

Before proving Proposition 14, let us spell out the "modular interpretation" of v_p in the $p \neq 0$ case (the $p = 0$ case was discussed already). Given a log scheme Y and a global section $c \in \overline{\mathcal{M}}_Y(Y)$ (that is, an element of $[\mathrm{rig}\, A^1/\mathbb{G}_m](Y)$), the map v_p associates the "usual" $\mathcal{O}_Y^*$ torsor

$$T_c \;:=\; \{m \in \mathcal{M}_Y : \overline{m} = c \text{ in } \overline{\mathcal{M}}_Y\}$$

and the $\mathcal{O}_Y^*$ equivariant map $0 : T_c \to \mathcal{O}_Y$ (as opposed to α_Y). Note

$$(T_c, 0) \in \underline{B\mathbb{G}_m}(Y) \subseteq [\underline{\mathrm{rig}\, A}^1/\mathbb{G}_m](Y).$$

We first prove Proposition 14 in the $p = 0$ case. Recall from above that an element of

$$(\underline{X} \times_{[\underline{\mathrm{rig}\, A}^1/\mathbb{G}_m]} [\mathrm{rig}\, A^1/\mathbb{G}_m])(Y)$$

consists of data f, e, h as in (A1), (A2), (A3) above. Given this data, if we compose h with the inclusion $T_e \subseteq \mathcal{M}_Y$ and use the universal property of $f^*\mathcal{M}(L,s)$ we obtain a morphism $f^\dagger : f^*\mathcal{M}(L,s) \to \mathcal{M}_Y$ of log structures on $\underline{Y}$. Looking at the formula (16) for $\alpha_{Y \times \mathbb{P}(P)}$, we then see that the formula

$$\begin{array}{rcl} (f^\dagger, 0) : f^*\mathcal{M}(L,s) & \to & \mathcal{M}_Y \oplus P \\ m & \mapsto & (f^\dagger(m), 0) \end{array}$$

defines a map of log structures on $\underline{Y}$ from $f^*\mathcal{M}(L,s)$ to the log structure of $Y \times \mathbb{P}(P)$. That is,

$$(f, (f^\dagger, 0)) \in \mathrm{Hom}_{\mathbf{LogSch}}(Y \times \mathbb{P}(P), X) = (\wedge^{\mathrm{rig}}_P X)(Y),$$

and, in fact, we clearly have

$$(f, (f^\dagger, 0)) \in (\wedge^{\mathrm{rig}}_{P,0} X)(Y) \subseteq (\wedge^{\mathrm{rig}}_P X)(Y).$$

This discussion is natural in Y so we obtain a map

$$\underline{X} \times_{[\underline{\mathrm{rig}\, A}^1/\mathbb{G}_m]} [\mathrm{rig}\, A^1/\mathbb{G}_m] \to \wedge^{\mathrm{rig}}_{P,0} X.$$

The inverse map

$$\wedge^{\mathrm{rig}}_{P,0} X \to \underline{X} \times_{[\underline{\mathrm{rig}\, A}^1/\mathbb{G}_m]} [\mathrm{rig}\, A^1/\mathbb{G}_m]$$

is defined as follows: An element of $(\wedge^{\mathrm{rig}}_{P,0} X)(Y)$ consists of a map of schemes $f : \underline{Y} \to \underline{X}$ and a map

$$f^\dagger : f^*\mathcal{M}(L,s) \to \mathcal{M}_Y \oplus \underline{P}$$

of log structures on $\underline{Y}$ with the property that $\pi_2 f^\dagger$ is the zero map. The key point here is that, by examining the formula (16) for the structure map

$$\alpha_{Y\times\mathbb{P}(P)} : \mathcal{M}_Y \oplus \underline{P} \ \ \to \ \ \mathcal{O}_Y$$

and using $\pi_2 f^\dagger = 0$, we in fact see that the map

$$\pi_1 f^\dagger : f^*\mathcal{M}(L,s) \to \mathcal{M}_Y$$

is also a map of log structures on Y! Hence, by the universal property of $f^*\mathcal{M}(L,s)$, we have a commutative diagram

$$\begin{tikzcd}
 & f^*L^\vee \setminus \mathbf{0}_Y \arrow[r, "f^*s^\vee"] \arrow[d, "f^*\eta"] \arrow[dl, "1"'] & \mathcal{O}_Y \\
 & f^*\mathcal{M}(L,s) \arrow[r, "\pi_1 f^\dagger"] \arrow[d] \arrow[ur, "f^*\alpha"] & \mathcal{M}_Y \arrow[u, "\alpha_Y"'] \arrow[d] \\
\underline{\mathbb{N}} \arrow[r, "\beta"] & f^{-1}\overline{\mathcal{M}}(L,s) \arrow[r, "\pi_1 \overline{f}^\dagger"] & \overline{\mathcal{M}}_Y
\end{tikzcd} \tag{17}$$

reminiscent of the diagram (15) we obtained in the proof of Proposition 13. From this we obtain a global section $e := \pi_1 \overline{f}^\dagger \beta \in \overline{\mathcal{M}}_Y(Y)$ (i.e. $e \in [\operatorname{rig} A^1/\mathbb{G}_m](Y)$); we also obtain an isomorphism $h : f^*L^\vee \setminus \mathbf{0}_Y \to T_e$ of $\mathcal{O}_Y^*$ torsors making

$$\begin{tikzcd}
f^*L^\vee \setminus \mathbf{0}_Y \arrow[rr, "h"] \arrow[dr, "f^*s^\vee"'] & & T_e \arrow[dl, "\alpha_Y"] \\
 & \mathcal{O}_Y &
\end{tikzcd}$$

commute by restricting $\pi_1 f^\dagger f^*\eta$ and arguing that this lands in $T_e \subseteq \mathcal{M}_Y$ by commutativity of (17) just as we did in the proof of Proposition 13. The data of f, e, h furnish a map

$$(\wedge^{\mathrm{rig}}_{P,0} X)(Y) \to (\underline{X} \times_{[\underline{\operatorname{rig} A^1}/\mathbb{G}_m]} [\operatorname{rig} A^1/\mathbb{G}_m])(Y)$$

which is clearly inverse to the map constructed in the previous paragraph, thus we complete the proof of Proposition 14 in the case $p = 0$.

Combining Proposition 13 and the case of Proposition 14 we just proved (and noting that $v_0 : [\operatorname{rig} A^1/\mathbb{G}_m] \to [\underline{\operatorname{rig} A^1}/\mathbb{G}_m]$ is the same as the arrow in Proposition 13) we obtain:

Proposition 15. *For a DF(1) log scheme X and a sharp, fine monoid P, the presheaf $\wedge^{\mathrm{rig}}_{P,0} X$ of rigid P log points in X with contact order zero is represented by X itself.*

We still need to prove Proposition 14 in the case $p \neq 0$. The proof is very similar to the $p = 0$ case, so let me just highlight the differences: An element of

$$(\underline{X} \times_{[\underline{\text{rig}\,A^1}/\mathbb{G}_m]} [\text{rig}\,A^1/\mathbb{G}_m])(Y)$$

in *this* case consists of the following data: $f : \underline{Y} \to \underline{X}$, $e \in \overline{\mathcal{M}}_Y(Y)$ just as in (A1), (A2), but instead of h as in (A3), we have

B3. An $\mathcal{O}_Y^*$ equivariant isomorphism $h : f^*L^\vee \setminus \mathbf{0}_Y \to T_e$ making the diagram

$$\begin{tikzcd} f^*L^\vee \setminus \mathbf{0}_Y \arrow[rr, "h"] \arrow[dr, "f^*s^\vee"'] & & T_e \arrow[dl, "0"] \\ & \mathcal{O}_Y & \end{tikzcd}$$

commute. The only difference between this and the h of (A3) is that α_Y has been replaced by 0—c.f. the modular interpretation of $v_{p\neq 0}$ above.

Let's notice right off the bat that the very existence of the h in (B3) requires $f^*s^\vee = 0$, which requires f to factor through the zero locus of s. By examining formula (16), we see that the map $0 : T_e \to \mathcal{O}_Y$ is part of a commutative $\mathcal{O}_Y^*$ equivariant triangle

$$\begin{tikzcd} T_e \arrow[rr, "(\subseteq{,}p)"] \arrow[dr, "0"'] & & \mathcal{M}_Y \oplus \underline{P} \arrow[dl, "\alpha_{Y\times\mathbb{P}(P)}"] \\ & \mathcal{O}_Y & \end{tikzcd}$$

because $p \neq 0$. Concatenating this triangle and the one in (B3) yields an $\mathcal{O}_Y^*$ equivariant commutative diagram

$$\begin{tikzcd} f^*L^\vee \setminus \mathbf{0}_Y \arrow[rr, "g=h(\subseteq{,}p)"] \arrow[dr, "f^*s^\vee"'] & & \mathcal{M}_Y \oplus \underline{P} \arrow[dl, "\alpha_{Y\times\mathbb{P}(P)}"] \\ & \mathcal{O}_Y & \end{tikzcd}$$

and hence, by the universal property of $\mathcal{M}(L,s)$, a map $f^\dagger : \mathcal{M}(L,s) \to \mathcal{M}_Y \oplus \underline{P}$ of log structures on $\underline{Y}$. Thus we obtain an element $(f, f^\dagger) \in (\wedge^{\text{rig}}_{P,p} X)(Y)$.

The inverse map is constructed exactly as in the $p = 0$ case, except this time an element of $(\wedge^{\text{rig}}_{P,0} X)(Y)$ consists of a map of schemes $f : \underline{Y} \to \underline{X}$ and a map

$$f^\dagger : f^*\mathcal{M}(L,s) \to \mathcal{M}_Y \oplus \underline{P}$$

of log structures on $\underline{Y}$ with the property that $\pi_2 f^\dagger = p \neq 0$. Examining formula (16) we find that

$$\begin{tikzcd} f^*L^\vee \setminus \mathbf{0}_Y \arrow[r, "f^*s^\vee"] \arrow[d, "f^*\eta"'] & \mathcal{O}_Y \\ f^*\mathcal{M}(L,s) \arrow[r, "f^\dagger"] \arrow[ur, "0"] & \mathcal{M}_Y \oplus \underline{P} \arrow[u, "\alpha_{Y\times\mathbb{P}(P)}"'] \end{tikzcd}$$

commutes, so the usual formula $h := \pi_1 f^\dagger f^* \eta$ identifies the $\mathcal{O}_Y^*$ torsor $f^*L^\vee \setminus \mathbf{0}_Y$ with the $\mathcal{O}_Y^*$ torsor

$$T_e \;\; := \;\; \{m \in \mathcal{M}_Y : \overline{m} = \pi_1 \overline{f}^\dagger \beta(1)\}$$

in a manner identifying $f^*s^\vee : f^*L^\vee \setminus \mathbf{0}_Y \to \mathcal{O}_Y$ with the *zero* map to $\mathcal{O}_Y$ (as opposed to identifying it with $\alpha_Y : T_e \to \mathcal{O}_Y$ as in the $p = 0$ case).

This completes the proof of Proposition 15.

The next step is to notice that, in the $p \neq 0$ case we can also understand the fiber product $\underline{X} \times_{[\underline{\operatorname{rig} A^1}/\mathbb{G}_m]} [\operatorname{rig} A^1/\mathbb{G}_m]$ "directly" just as we did in the $p = 0$ case via Proposition 13. We mentioned already that

$$v_{p \neq 0} : [\operatorname{rig} A^1/\mathbb{G}_m] \to [\underline{\operatorname{rig} A^1}/\mathbb{G}_m]$$

factors as the composition of the map $\pi : [\operatorname{rig} A^1/\mathbb{G}_m] \to \underline{B\mathbb{G}_m}$ and the inclusion of the origin $\underline{B\mathbb{G}_m} \hookrightarrow [\underline{\operatorname{rig} A^1}/\mathbb{G}_m]$. On the level of underlying stacks,

$$\underline{\pi} : [\underline{\operatorname{rig} A^1}/\mathbb{G}_m] \to \underline{B\mathbb{G}_m}$$

is just the universal line bundle. Indeed, the universal principal $\mathbb{G}_m$ bundle is $\operatorname{Spec} \mathbb{Z} \to \underline{B\mathbb{G}_m}$ and the universal line bundle is obtained from the universal $\mathbb{G}_m$ bundle by the usual twisted quotient construction

$$\operatorname{Spec} \mathbb{Z} \times_{\mathbb{G}_m} \underline{\operatorname{rig} A^1} \;\; = \;\; [\underline{\operatorname{rig} A^1}/\mathbb{G}_m].$$

(The $\times$ here is not the fibered product.) The map π itself is interpreted as the universal line bundle, equipped with the DF(1) log structure from the zero section. Let $\underline{D} \subseteq \underline{X}$ be the zero locus of s. Then we have a cartesian diagram:

$$\begin{tikzcd}
L|_D \arrow[r, "\beta"] \arrow[d] & {[\operatorname{rig} A^1/\mathbb{G}_m]} \arrow[d] \\
\underline{L|_D} \arrow[r] \arrow[d] & {[\underline{\operatorname{rig} A^1}/\mathbb{G}_m]} \arrow[d] \\
\underline{D} \arrow[r, "{(L,0)}"] \arrow[d] & \underline{B\mathbb{G}_m} \arrow[d] \\
\underline{X} \arrow[r, "{(L,s)}"] & {[\underline{\operatorname{rig} A^1}/\mathbb{G}_m]}
\end{tikzcd} \tag{18}$$

Combining this with Proposition 15, we find:

Proposition 16. *For a DF(1) log scheme $X = (\underline{X}, \mathcal{M}(L,s))$ a fine, sharp monoid P, and a nonzero element $p \in P$, the presheaf $\wedge^{\mathrm{rig}}_{P,p} X$ of rigid P log points in X with contact order p is represented by $L|_D$ (the total space of the restriction of the line bundle L to D, with the DF(1) log structure from the zero section).*

For the sake of completeness, let me explain directly how to construct the isomorphsim

$$\wedge^{\mathrm{rig}}_{P,p}X \quad \to L|_D.$$

An element of $(\wedge^{\mathrm{rig}}_{P,p}X)(Y)$ consists of a map of schemes $f : \underline{Y} \to \underline{X}$ and a map

$$f^\dagger : f^*\mathcal{M}(L,s) \quad \to \quad \mathcal{M}_Y \oplus \underline{P}$$

of log structures on $\underline{Y}$ with $\pi_2 \overline{f}^\dagger \beta(1) = p \in \underline{P}(Y)$. The composition

$$\alpha_Y \pi_1 f^\dagger f^* \eta : f^*L \setminus \mathbf{0}_Y \quad \to \quad \mathcal{O}_Y \tag{19}$$

is an $\mathcal{O}_Y^*$ equivariant map—one checks easily that this is the same thing as a a global section $t \in (f^*L)(Y)$, which is the same thing as a map $t : \underline{Y} \to \underline{L}$ factoring f through $\underline{L} \to \underline{X}$. (Note that, although the composition

$$\alpha_{Y \times \mathbb{P}(P)} f^\dagger f^* \eta : f^*L \setminus \mathbf{0}_Y \to \mathcal{O}_Y$$

is just the zero map, the map (19) is not generally zero.) We already saw that f has to factor through $\underline{D}$, so we may as well view t as a map $t : \underline{Y} \to \underline{L}|_{\underline{D}}$. The lifting of t to a map of log schemes $Y \to L|_D$ corresponds exactly to the map $\pi_1 f^\dagger f^* \eta$ factoring $\alpha_Y \pi_1 f^\dagger f^* \eta$ through α_Y.

Recall that we can pass from rigid P log points to log points by taking quotients by $\mathbb{G}(P)$. The action of $\mathbb{G}(P)$ on $\wedge^{\mathrm{rig}}_{P,0}X = X$ is the trivial one, while the action of $\mathbb{G}(P)$ on $\wedge^{\mathrm{rig}}_{P,p}X = L|_D$ is the one induced by the natural scaling action of $\mathbb{G}_m$ on $L|_D$ and the map of group schemes

$$\mathbb{G}(p) : \mathbb{G}(P) \quad \to \quad \mathbb{G}(\mathbb{N}) = \mathbb{G}_m.$$

For example, in the case $P = \mathbb{N}$, we find

$$\begin{aligned} \wedge_{\mathbb{N},0}X &= X \times B\mathbb{G}_m \\ \wedge_{\mathbb{N},p\neq 0}X &= [L|_D/_p\mathbb{G}_m], \end{aligned}$$

where the subscript p indicates that the action in question is the p^{th} power of the natural scaling action.

3.3. **Cohomology.** Of course this description allows us to compute the cohomology of the various $\wedge_{\mathbb{N},p}X$, but it seems more meaningful to compute the "log cohomology", which might mean one of the following:

(1) the cohomology of the "interior" $(\wedge_{\mathbb{N},p}X)^\circ$ (the open locus where the log structure is trivial)
(2) the cohomology of the Kato-Nakayama space
(3) the log deRham cohomology

Let's assume that X is a smooth variety with the DF(1) log structure from a smooth divisor $D \subseteq X$. Then X is log smooth, and each of the stacks $\wedge_{\mathbb{N},p}X$ is log smooth (over a point with trivial log structure), so these three cohomologies will be the same and we can unambiguously denote this cohomology $\mathrm{H}^*_{\log}$. Explicitly, we have:

$$\begin{aligned}
\mathrm{H}^*_{\log}(\wedge_{\mathbb{N},0}X) &= \mathrm{H}^*_{\log}(X \times B\mathbb{C}^*) \\
&= \mathrm{H}^*(X^\circ) \otimes \mathrm{H}^*(B\mathbb{C}^*) \\
&= \mathrm{H}^*(X \setminus D)[t] \\
&= \mathrm{H}^*(X^{\mathrm{KN}})[t] \\
&= \mathrm{H}^*(\mathrm{OBl}_D X)[t] \\
\mathrm{H}^*_{\log}(\wedge_{\mathbb{N},p\neq 0}X) &= \mathrm{H}^*(N^\circ_{D/X}/_p\mathbb{C}^*) \\
&= \mathrm{H}^*((N_{D/X} \setminus \mathbf{0}_D)/_p\mathbb{C}^*) \\
&= \mathrm{H}^*(\mathrm{OBl}_{\mathbf{0}_D} N_{D/X}/_p\mathbb{C}^*) \\
&= \mathrm{H}^*(D)[t]/(t^p - c_1(N_{D/X})).
\end{aligned}$$

3.4. **Log nodes in a DF(1) log scheme.** The case of log nodes in a DF(1) log scheme is very similar to the case of log points, so we can be quite brief. We fix an integral monomorphism $h : Q \hookrightarrow P$ of sharp, fine monoids with trivial kernel: $h^{-1}(0) = \{0\}$. Recall that a *splitting set* for h is a sub*set* $S \subseteq P$ such that $(s, q) \mapsto s + h(q)$ defines a bijection of sets $S \times Q \cong P$. Recall that S remains a splitting set for any pushout of h along a map $Q \to Q'$ (as long as Q' is also integral). Notice that a splitting set *must* contain zero because P is sharp. Every h as above has a splitting set consisting of the Q-*primitive* elements of P.

For example,

$$S := \{(a, b) \in \mathbb{N}^2 : a = 0 \text{ or } b = 0\}$$

is a splitting set for $\Delta : \mathbb{N} \to \mathbb{N}^2$.

Recall that $\mathbb{P}(P) := \mathrm{Spec}(0 : P \to \mathbb{Z})$ and that an h as above induces a map $\mathbb{P}(h) : \mathbb{P}(P) \to \mathbb{P}(Q)$. To give a map from a log scheme Y to a $\mathbb{P}(Q)$ is to give a map of monoids $z : Q \to \mathcal{M}_Y(Y)$ satisfying $\alpha_Y(z(q)) = 0 \in \mathcal{O}_Y(Y)$ for all $q \in Q \setminus \{0\}$. The fibered product $Y \times^z_{\mathbb{P}(Q)} \mathbb{P}(P)$ is just $\underline{Y}$ with the log structure $\mathcal{M}_{Y,z} := \mathcal{M}_Y \oplus_{\underline{Q}} \underline{P}$.

Fix a DF(1) log scheme $X = (\underline{X}, \mathcal{M}(L, s))$. The presheaf of *rigid h log nodes* in X is the presheaf

$$\wedge^{\mathrm{rig}}_h X : \mathbf{LogSch}^{\mathrm{op}} \to \mathbf{Sets}$$

taking a log scheme Y to the set of pairs (z, f) where $z : Y \to \mathbb{P}(Q)$ is a map of log schemes and f is an element of $\mathrm{Hom}_{\mathbf{LogSch}}(Y \times_{\mathbb{P}(Q)} \mathbb{P}(P), Z)$. Such an f consists of a map of schemes $f : \underline{Y} \to \underline{X}$ together with a map

$$f^\dagger : f^*\mathcal{M}(L, s) \to \mathcal{M}_{Y,z}$$

of log structures on $\underline{Y}$. The chosen splitting set S gives natural isomorphisms

$$\begin{aligned}
\mathcal{M}_{Y,z} &= \mathcal{M}_Y \times \underline{S} \\
\overline{\mathcal{M}}_{Y,z} &= \overline{\mathcal{M}}_Y \times S
\end{aligned}$$

of objects of the étale topos of $\underline{Y}$. Composing $\beta(1) \in \overline{\mathcal{M}}(L,s)$ with $\overline{f}^{\dagger}$, the second isomorphism above, and the projection $\pi_2 : \mathcal{M}_Y \times \underline{S} \to \underline{S}$ yields an element of $\underline{S}(Y)$, i.e. a locally constant function from $\underline{Y}$ to S, which we call the *contact order function.* It is natural in Y, hence it determines a coproduct decomposition

$$\wedge_h^{\mathrm{rig}} X \quad = \quad \coprod_{s \in S} \wedge_{h,s}^{\mathrm{rig}} X$$

of rigid log nodes into components with fixed contact order $s \in S$.

On the component with contact order zero, an element of $(\wedge_{h,0}^{\mathrm{rig}} X)(Y)$ consists of a map $z : Y \to \mathbb{P}(Q)$, a map of schemes $f : \underline{Y} \to \underline{X}$, and a map

$$f^{\dagger} : f^*\mathcal{M}(L,s) \to \mathcal{M}_{Y,z} = \mathcal{M}_Y \oplus_{\underline{Q}} \underline{P}$$

of log structures on $\underline{Y}$ fitting in a commutative diagram

$$\begin{tikzcd} f^*\mathcal{M}(L,s) \arrow[r, "f^{\dagger}"] \arrow[dr, "g \times 0"'] & \mathcal{M}_Y \oplus_{\underline{Q}} \underline{P} \arrow[d, equal] \\ & \mathcal{M}_Y \times S \end{tikzcd}$$

It is easy to see that the map g here is a map of log structures, so that an element of $(\wedge_{h,0}^{\mathrm{rig}} X)(z : Y \to \mathbb{P}(Q))$ is nothing more or less than a map of log schemes $Y \to X$, hence $\wedge_{h,0}^{\mathrm{rig}} X$ is represented by $X \times \mathbb{P}(Q)$. On a component with nonzero contact order, we have the same story as in the log points situation: the underlying map of schemes f must factor through $D := Z(s)$, and the log data is the same thing as a map from Y to the total space of $L|_D$ (with the log structure from the zero section), so $\wedge_{h,s \neq 0}^{\mathrm{rig}} X = L|_D \times \mathbb{P}(Q)$.

To pass from rigid to nonrigid log nodes, we take a quotient by $\mathbb{G}(P)$. The action of

$$\mathbb{G}(P)(Y) \quad = \quad \mathrm{Hom}_{\mathbf{Ab}}(P^{\mathrm{gp}}, \mathcal{O}_Y^*(Y))$$

on $(\wedge_{h,s}^{\mathrm{rig}} X)(Y)$ is described in §2.4. Following this through our isomorphisms, we see that $\mathbb{G}(P)$ acts on $\wedge_{h,0}^{\mathrm{rig}} X = X \times \mathbb{P}(Q)$ through the natural action of $\mathbb{G}(Q)$ on $\mathbb{P}(Q)$ and the group homomorphism $\mathbb{G}(h) : \mathbb{G}(P) \to \mathbb{G}(X)$ (the action here is only on the $\mathbb{P}(Q)$ factor; the action on X is trivial). On a component $\wedge_{h,s \neq 0}^{\mathrm{rig}} X = L|_D \times \mathbb{P}(Q)$ parameterizing log nodes with nontrivial contact order, $\mathbb{G}(P)$ acts on the $\mathbb{P}(Q)$ factor as before, but it also acts on the $L|_D$ factor through the natural $\mathbb{G}_m = \mathbb{G}(\mathbb{N})$ action scaling the fibers of $L|_D$ and the group homomorphism $\mathbb{G}(s) : \mathbb{G}(P) \to \mathbb{G}(\mathbb{N})$ (recall that s is an element of the splitting set $S \subseteq P$, so we can view it as a monoid homomorphism $s : \mathbb{N} \to P$).

For example, if $h = \Delta : \mathbb{N} \to \mathbb{N}^2$, then $\mathbb{G}(\Delta) : \mathbb{G}_m^2 \to \mathbb{G}_m$ is the multiplication map m for $\mathbb{G}_m$. For a nonzero element of S, say $(a,0)$, the action of $\mathbb{G}_m^2$ on $\wedge_{\Delta,(a,0)}^{\mathrm{rig}} X = L|_D \times \mathbb{P}(\mathbb{N})$ is given by

$$(\lambda_1, \lambda_2) \cdot (l, p) \quad = \quad (\lambda_1^a \cdot l, \lambda_1 \lambda_2 \cdot p).$$

4. Topological View

In this section, we discuss our evaluation spaces for log stable maps in terms of Kato-Nakayama spaces. We work over $\mathbb{C}$; all monoids are assumed finitely generated. Recall that the Kato-Nakayama log structure

$$\begin{array}{rcl} \alpha : S^1 \times \mathbb{R}_{\geq 0} & \to & \mathbb{C} \\ (u, r) & \mapsto & ur \end{array}$$

is just the usual multiplication map, where we think of S^1 and $\mathbb{R}_{\geq 0}$ as subsets (submonoids in fact) of $\mathbb{C}$ in the usual way.

Recall that the Kato-Nakayama space [KN] is an inverse limit preserving (left exact) functor

$$\begin{array}{rcl} \mathbf{LogAnEsp} & \to & \mathbf{Top} \\ X & \mapsto & X^{\mathrm{KN}} \end{array}$$

from log analytic spaces to topological spaces.[3] The space X^{KN} comes with a natural structure map $X^{\mathrm{KN}} \to X$. We often view $X \mapsto X^{\mathrm{KN}}$ as a functor from log schemes of finite type over $\mathbb{C}$ to topological spaces by composing with the analytification functor. Notation for analytification is suppressed.

Recall that

$$\begin{array}{rcl} \operatorname{Spec}(Q \to \mathbb{C}[Q])^{\mathrm{KN}} & = & \operatorname{Hom}_{\mathbf{Mon}}(Q, S^1 \times \mathbb{R}_{\geq 0}) \\ & =: & (S^1 \times \mathbb{R}_{\geq 0})(Q), \end{array}$$

and that, under the natural bijection

$$\begin{array}{rcl} (\operatorname{Spec} \mathbb{C}[Q])(\mathbb{C}) & = & \operatorname{Hom}_{\mathbf{Mon}}(Q, \mathbb{C}) \\ & =: & \mathbb{C}(Q), \end{array}$$

the structure map

$$\operatorname{Spec}(Q \to \mathbb{C}[Q])^{\mathrm{KN}} \to (\operatorname{Spec} \mathbb{C}[Q])(\mathbb{C})$$

is identified with

$$\alpha_* : (S^1 \times \mathbb{R}_{\geq 0})(Q) \to \mathbb{C}(Q).$$

Note also that

$$\begin{array}{rcl} \mathbb{P}(P)^{\mathrm{KN}} & = & S^1(P) \\ & = & S^1(P^{\mathrm{gp}}) \end{array}$$

is the Pontryagin dual of the finitely generated abelian group P^{gp}.

For $X \in \mathbf{LogAnEsp}$, we have

$$(X \times \mathbb{P}(P))^{\mathrm{KN}} = X^{\mathrm{KN}} \times S^1(P)$$

[3] In fact it may be viewed as a functor to locally ringed spaces, but we will not need the structure sheaf of X^{KN} in this discussion.

Note

$$\begin{aligned}\mathbb{G}(P)(\mathbb{C}) &= (\operatorname{Spec}\mathbb{C}[P^{\mathrm{gp}}])(\mathbb{C}) \\ &= \mathbb{C}^*(P^{\mathrm{gp}}).\end{aligned}$$

The argument map

$$\begin{aligned}\mathbb{C}^* &\to S^1 \\ z &\mapsto z/||z||\end{aligned}$$

induces a map $\mathbb{G}(P)(\mathbb{C}) \to S^1(P^{\mathrm{gp}})$. A map $g \in \mathbb{G}(P)(X)$, yields a map

$$g(\mathbb{C}) : X \to \mathbb{C}^*(P^{\mathrm{gp}}),$$

and composing with the argument map yields a map

$$X \to S^1(P^{\mathrm{gp}}).$$

Upon applying the Kato-Nakayama space functor, a P log point $Y' \to Y$ becomes an $S^1(P^{\mathrm{gp}})$ torsor over Y^{KN}. In particular, an $\mathbb{N}$ log point $Y' \to Y$ becomes a circle bundle over Y^{KN}. If $Y' \to Y$ is trivialized on a (strict) étale cover $U \to Y$, with gluing cocyle data $g \in G(U \times_Y U)$, then the $S^1(P^{\mathrm{gp}})$ torsor $(Y')^{\mathrm{KN}} \to Y^{\mathrm{KN}}$ is trivialized on the (classical) cover $U^{\mathrm{KN}} \to Y^{\mathrm{KN}}$ and has gluing cocycle data given by the composition of the structure map

$$(U \times_Y U)^{\mathrm{KN}} \to U \times_Y U,$$

and the map

$$g(\mathbb{C}) : (U \times_Y U)(\mathbb{C}) \to S^1(P^{\mathrm{gp}})$$

mentioned above.

Recall from Lemma 6 that when $X = \operatorname{Spec}(Q \to \mathbb{C}[Q])$, we have

$$\wedge_P^{\mathrm{rig}} X = \coprod_{h \in P(Q)} X$$

and that, over the component indexed by $h : Q \to P$, the universal diagram

$$\begin{tikzcd} X \times \mathbb{P}(P) \arrow[r] \arrow[d] & X \\ X & \end{tikzcd}$$

is Spec(_ $\to$ _) of

$$\begin{tikzcd} & Q \arrow[r] \arrow[dl, "(\mathrm{Id}_Q, h)"'] & \mathbb{C}[Q] \arrow[dl, "u_h"] \\ Q \oplus P \arrow[r] & \mathbb{C}[Q] \arrow[d, equal] & \\ Q \arrow[u, "(\mathrm{Id}_Q, 0)"] \arrow[r] & \mathbb{C}[Q] & \end{tikzcd}$$

where

$$u_h([q]) := \begin{cases} [q], & h(q) = 0 \\ 0, & h(q) \neq 0 \end{cases}$$

Taking Kato-Nakayama spaces here yields a diagram

$$\begin{tikzcd}
(S^1 \times \mathbb{R}_{\geq 0})(Q) \times S^1(P) \arrow[r, "u_h^{\mathrm{KN}}"] \arrow[d, "\pi_1"'] \arrow[drr, "\alpha_* \pi_1"] & (S^1 \times \mathbb{R}_{\geq 0})(Q) \arrow[drr, "\alpha_*"] & & \\
(S^1 \times \mathbb{R}_{\geq 0})(Q) \arrow[drr, "\alpha_*"] & & \mathbb{C}(Q) \arrow[r, "u_h"] \arrow[d, equal] & \mathbb{C}(Q) \\
& & \mathbb{C}(Q) &
\end{tikzcd}$$

where

$$u_h(f)(q) \quad = \quad \begin{cases} f(q), & h(q) = 0 \\ 0, & h(q) \neq 0 \end{cases}$$

and

$$\begin{aligned} u_h^{\mathrm{KN}} : S^1(Q) \times (\mathbb{R}_{\geq 0})(Q) \times S^1(P) \quad &\to \quad (S^1 \times \mathbb{R}_{\geq 0})(Q) \\ v_h(f_1, f_2, g)(q) \quad &:= \quad \begin{cases} (f_1(q), f_2(q)), & h(q) = 0 \\ (g(h(q)) f_1(q), 0), & h(q) \neq 0 \end{cases} \end{aligned}$$

For example, when $P = Q = \mathbb{N}$ and $h : \mathbb{N} \to \mathbb{N}$ sends 1 to $h(1) =: h$ in $\mathbb{N}_{>0}$, the above diagram looks like:

$$\begin{tikzcd}
S^1 \times \mathbb{R}_{\geq 0} \times S^1 \arrow[r, "(u,r,v) \mapsto (v^h u, 0)"] \arrow[d, "\pi_{12}"] \arrow[drr] & S^1 \times \mathbb{R}_{\geq 0} \arrow[drr] & & \\
S^1 \times \mathbb{R}_{\geq 0} \arrow[drr] & & \mathbb{C} \arrow[r, "0"] \arrow[d, equal] & \mathbb{C} \\
& & \mathbb{C} &
\end{tikzcd}$$

In particular, over the origin this diagram looks like:

$$\begin{tikzcd}
S^1 \times S^1 \arrow[r, "(u,v) \mapsto v^h u"] \arrow[d, "\pi_1"] \arrow[drr] & S^1 \arrow[drr] & & \\
S^1 \arrow[drr] & & \{0\} \arrow[r, equal] \arrow[d, equal] & \{0\} \\
& & \{0\} &
\end{tikzcd}$$

Now let's pass from this local study to the global situation. Suppose X is a smooth variety with log structure from a smooth divisor $\underline{D} \subseteq \underline{X}$. Then étale locally,

X looks like

$$\operatorname{Spec}(\mathbb{N} \to \mathbb{C}[\mathbb{N}]) \times \underline{D},$$

so the stack $\wedge_{\mathbb{N}} X$ of $\mathbb{N}$ log points in X is étale locally

$$\wedge_{\mathbb{N}}(\operatorname{Spec}(\mathbb{N} \to \mathbb{C}[\mathbb{N}])) \times \underline{D}.$$

Suppose $(T' \to T, f : T' \to D)$ is a standard log point in D (with the log structure inherited from X) with "contact order" $h \in \mathbb{N}_{>0}$.[4] Let us now proceed naively, forgetting our study of the stack of log points, and attempt to find a topological stack which naturally receives an "evaluation morphism" from T^{KN} in this situation. After taking Kato-Nakayama spaces, $(T')^{\mathrm{KN}} \to T^{\mathrm{KN}}$ is a circle bundle, and

$$f^{\mathrm{KN}} : (T')^{\mathrm{KN}} \to D^{\mathrm{KN}} = S^1 N_{D/X}$$

becomes a "map of circle bundles," but not in the usual sense: instead it is S^1 equivariant for the usual action on $(T')^{\mathrm{KN}}$ and the h^{th} power of the usual action on $S^1 N_{D/X}$.[5] In other words, f^{KN} is naturally viewed as a map of topological stacks

$$T^{\mathrm{KN}} \to [S^1 N_{D/X} /_h S^1] = \underline{D} \times B\mu_h.$$

On the other hand, our log point in X over T may also be viewed as a map of stacks $T \to \wedge D \subseteq \wedge X$, and we may take Kato-Nakayama spaces everywhere to find an alternative ("less naive") topological stack receiving an evaluation morphism from T^{KN}. The component of $\wedge X$ where this is mapping looks locally like

$$\underline{D} \times [(\operatorname{Spec} \mathbb{N} \to \mathbb{C}[\mathbb{N}]) /_h \mathbb{G}_m]$$

(where the h subscripts always indicate that the action is through the h^{th} power of the usual action) and hence its Kato-Nakayama space looks locally like

$$\underline{D} \times [S^1 \times \mathbb{R}_{\geq 0} /_h \mathbb{C}^*],$$

and the (contact order h component of the) Kato-Nakayama space of $\wedge D$ sits inside this as

$$\underline{D} \times [S^1 \times \{0\} /_h \mathbb{C}^*] = \underline{D} \times B\mu_n \times B\mathbb{R}_{>0}.$$

Notice that this "evaluation space" is basically the same as the one we arrived at from the naive study, up to the $B\mathbb{R}_{>0}$ factor. This factor might be considered trivial since $\mathbb{R}_{>0}$ is contractible, and every $\mathbb{R}_{>0}$ torsor on a paracompact space is trivial.[6] The presence of this extra factor is ultimately accounted for as follows: Recall that we had an action

$$b : \mathbb{G}_m \times \operatorname{Spec}(0 : \mathbb{N} \to \mathbb{C}) \to \operatorname{Spec}(0 : \mathbb{N} \to \mathbb{C}).$$

Taking Kato-Nakayama spaces, we get an action

$$b^{\mathrm{KN}} : \mathbb{C}^* \times S^1 \to S^1,$$

which is given by $b^{\mathrm{KN}}(z, u) = (z/||z||)u$, so the subgroup $\mathbb{R}_{>0} \subseteq \mathbb{C}^*$ is acting trivially.

[4]Meaning: for every $t \in \underline{T}$, $\overline{f}^{\dagger} : \overline{\mathcal{M}}_{D,f(t)} \to \overline{\mathcal{M}}_{T'/T}$ is multiplication by h.

[5]This is a local question, so to see this, we can assume $T' = T_{\mathbb{N}}$ is the trivial $\mathbb{N}$ log point, then we know this is true from our study of the universal standard $\mathbb{N}$ log point. On the other hand, one could also see this directly from the definitions of "contact order" and Kato-Nakayama spaces.

[6]The long line, however, has lots of nontrivial $\mathbb{R}_{>0}$ torsors over it.

References

[AC10] D. Abramovich and Q. Chen, *Logarithmic stable maps to Deligne-Faltings paris II*, in preparation (2010).

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-witten theory of deligne-mumford stacks*, American Journal of Mathematics **130** (2008), no. 5, 1337–1398.

[AV02] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75 (electronic). MR MR1862797 (2002i:14030)

[Che10a] Q. Chen, *Logarithmic stable maps to Deligne-Faltings paris I*, arXiv:1008.3090v2 [math.AG] (2010).

[Che10b] Qile Chen, *The degeneration formula for logarithmic expanded degenerations*, arXiv:1009.4378v1 [math.AG] (2010).

[Edi10] Dan Edidin, *Equivariant geometry and the cohomology of the moduli space of curves*, arXiv:1006.2364v1 [math.AG] (2010).

[IP03] E.-N. Ionel and T. H. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. (2) **157** (2003), no. 1, 45–96.

[IP04] ———, *The symplectic sum formula for Gromov-Witten invariants*, Ann. of Math. (2) **159** (2004), no. 3, 935–1025.

[FK] F. Kato, *Log smooth deformation and moduli of log smooth curves.* Int. J. Math. 11 (2000) 215-232.

[G] W. D. Gillam, *Logarithmic stacks and minimality.*

[Kat89] Kazuya Kato, *Logarithmic structures of Fontaine-Illusie*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191–224. MR 1463703 (99b:14020)

[KN] Kazuya Kato and Chikara Nakayama, *Log Betti cohomology, log étale cohomology, and log de Rham cohomology of log schemes over* $\mathbb{C}$. Kodai Math. J. **22** (1999), 161-186.

[Kim09] B. Kim, *Logarithmic stable maps*, arXiv:0807.3611V2 [math.AG], to appear in the proceedings volume of the conference "New developments in Algebraic Geometry, Integrable Systems and Mirror symmetry", RIMS, Kyoto. (2009).

[Kre99] Andrew Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), no. 3, 495–536. MR 1719823 (2001a:14003)

[Li01] Jun Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom. **57** (2001), no. 3, 509–578. MR MR1882667 (2003d:14066)

[Li02] ———, *A degeneration formula of GW-invariants*, J. Differential Geom. **60** (2002), no. 2, 199–293. MR MR1938113 (2004k:14096)

[LR01] An-Min Li and Yongbin Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. **145** (2001), no. 1, 151–218. MR 1839289 (2002g:53158)

[Man08] C. Manolache, *Virtual pull-backs*, arXiv:0805.2065v1 [math.AG] (2008).

[Ogu06] A. Ogus, *Lectures on logarithmic algebraic geometry*, texed notes, 2006.

[Ols03] Martin C. Olsson, *Logarithmic geometry and algebraic stacks*, Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 5, 747–791. MR 2032986 (2004k:14018)

[Ols08] ———, *Compactifying moduli spaces for abelian varieties*, Lecture Notes in Mathematics, vol. 1958, Springer-Verlag, Berlin, 2008. MR 2446415 (2009h:14072)

[Sie01] B. Siebert, *Gromov-Witten invariants in relative and singular cases*, Lecture given at the workshop on algebraic aspects of mirror symmetry, Universita t Kaiserslautern, Germany, Jun. 26 (2001).

Department of Mathematics, Brown University, Box 1917, Providence, RI 02912, U.S.A.

E-mail address: abrmovic, q.chen, wgillam, ssmarcus@math.brown.edu