QUESTIONS ABOUT PUNCTURED MAPS

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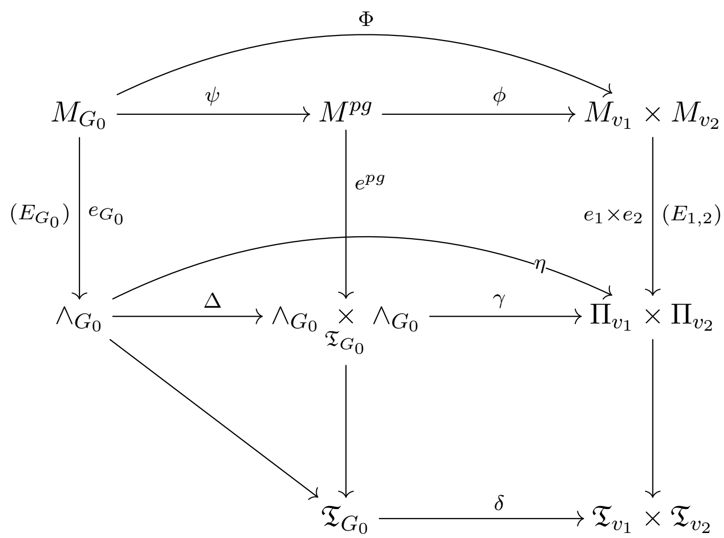
Diagram:



Notation:

M_{G_0} is the moduli of log maps with graph

$$G_0 = v_1 \bullet \!\!\xrightarrow[]{\quad e \quad}\!\! \bullet v_2.$$

The image in the stack of log structures lies in $\mathfrak{T}_{G_0}$, the stack obtained by gluing the various toric stacks corresponding to the monoids appearing in M_{G_0}.

M_{v_i} = moduli of punctured maps for v_i. The image in the stack of log structures lies in $\mathfrak{T}_{v_i}$, obtained similarly.

$\wedge_{G_0}$ = moduli of log nodes in X with log structure enhanced by a map of type G_0. It is obtained as the fibered product $\wedge^n(X) \times_{\mathfrak{T}_\wedge} \mathfrak{T}_{G_0}$, where $\wedge^n(X)$ is the "usual" moduli of log nodes in X and $\mathfrak{T}_\wedge$ is its log base. The map $\mathfrak{T}_{G_0} \to \mathfrak{T}_\wedge$ used in the fibered product is the map of log stacks underlying the evaluation map $M_{G_0} \to \wedge^n(X)$.

Π_{v_i} = moduli of punctured points in X enhanced by log structure of punctured maps of type v_i. There is (or at least should be) a moduli space $\Pi(X)$ of punctured points in X, which is similar to the space $\wedge(X)$ of "usual" log points in X but with components for both positive and negative contact

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orders. It has a base log stack $\mathfrak{T}_\Pi$. Underlying the evaluation map $M_{v_i} \to \Pi(X)$ there is a map $\mathfrak{T}_{v_i} \to \mathfrak{T}_\Pi$, and $\Pi_{v_i} = \Pi(X) \times_{\mathfrak{T}_\Pi} \mathfrak{T}_{v_i}$.

$M^{pg} = (M_{v_1} \times M_{v_2}) \underset{\mathfrak{T}_{v_1}\times\mathfrak{T}_{v_2}}{\times} \mathfrak{T}_{G_0}$, so the big rectangle on the right is cartesian by definition.

Discussion: $[M_{v_i}]^{\mathrm{vir}}$ is defined using the log obstruction theory, which is an obstruction theory relative to $\mathfrak{T}_{v_i}$. One should be able to similarly define the fundamental class $[M_{v_i}]^{\mathrm{vir}}$ using an obstruction theory relative to Π_{v_i}. This is well known in the non-log case (defined using sections of the tangent bundle vanishing at the marked point), and should be similar here. The indicated $E_{1,2}$ is the product obstruction theory. In Manolache's terminology,

$$[M_{v_1} \times M_{v_2}]^{\mathrm{vir}} \quad = \quad (e_1 \times e_2)^!_{E_{1,2}} \, [\Pi_{v_1} \times \Pi_{v_1}].$$

The class $[M^{pg}]^{\mathrm{vir}}$ is defined using the pullback of the obstruction theory of $M_{v_1} \times M_{v_2}$ over $\mathfrak{T}_{v_1} \times \mathfrak{T}_{v_2}$, which is perfect relative to $\mathfrak{T}_{G_0}$. If the previous statement holds, it should also be definable using tha pullback of the obstruction theory of $M_{v_1} \times M_{v_2}$ over $\Pi_{v_1} \times \Pi_{v_2}$. In Manolache's terminology,

$$[M^{pg}]^{\mathrm{vir}} \quad = \quad (e^{pg})^!_{\gamma^* E_{1.2}} \, [\wedge_{G_0} \underset{\mathfrak{T}_{G_0}}{\times} \wedge_{G_0}].$$

The class $[M_{G_0}]^{\mathrm{vir}}$ is defined by the usual log obstruction theory, giving a perfect obstruction theory relative to $\mathfrak{T}_{G_0}$. Again there should be a compatible perfect obstruction theory E_{G_0} relative to $\wedge_{G_0}$ giving the same class. So $[M_{G_0}]^{\mathrm{vir}} = (e_{G_0})^!_{E_{G_0}}[\wedge_{G_0}]$.

Qile and Mark have discussed the left square. It is cartesian, and Behrend–Fantechi's results say that $\Delta^![M^{pg}]^{\mathrm{vir}} = [M_{G_0}]^{\mathrm{vir}}$. It should also hold that

$$\psi_*[M_{G_0}]^{\mathrm{vir}} \quad = \quad (e^{pg})^!_{\gamma^* E_{1.2}} \, \Delta_*[\wedge_{G_0}].$$

Our trouble is the right part. The map δ is neither l.c.i. nor proper. The map γ is similarly problematic. There is however hope in pushing forward classes with proper support: the point is that $\Delta_*[\wedge_{G_0}]$ has proper support relative to $\Pi_{v_1} \times \Pi_{v_1}$. It is not clear current technology gives such a push-forward in general, but we may be in good shape in our case.

Questions:

(1) Can you define the punctured evaluation spaces Π_{v_i} so that the two small right squares are cartesian?
(2) Is it true that $(e_1 \times e_2)^!_{E_{1,2}} \, \gamma_*\Delta_*[\wedge_{G_0}] \;=\; \phi_*\left((e^{pg})^!_{\gamma^* E_{1.2}} \, \Delta_*[\wedge_{G_0}]\right)$?
(3) Similarly, does it follow that $(e_1 \times e_2)^!_{E_{1,2}} \, \eta_*[\wedge_{G_0}] \;=\; \Phi_*[M_{G_0}]^{\mathrm{vir}}$?

If the answer to that question is positive, then the gluing formula boils down to understanding the class $\eta_*[\wedge_{G_0}]$ in terms of some sort of Künneth formula.