MATH 251 PROBLEMS

Solve 15, at least 4 from 1-8 and at least 7 from the rest, and return by Monday Nov 3

(1) (Goursat's Lemma, from Lang ex. 5) Let G_1, G_2 be groups, $H < G_1 \times G_2$ such that the two projections $p_i : H \to G_i$ are surjective. Write $N_1 = Ker(p_2)$ and $N_2 = Ker(p_1)$.
   (a) Show that p_i maps N_i isomorphically to a normal subgroup of G_i.
   (b) Show that the image of H in $G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism
$$G_1/N_1 \to G_2/N_2.$$

(2) Let M and N be normal subgroups of G and assume $G = MN$. prove that $G/(M \cap N) \simeq G/M \times G/N$.

(3) Suppose G is a group of odd order. Show that if $x \in G$ is not the identity, then x and x^{-1} are not conjugate.

(4) Prove that the number of p-Sylow subgroups of $GL_2(\mathbb{F}_p)$ is $p+1$.

(5) Find a composition series for A_4.

(6) Describe the conjugacy classes in the dihedral group D_{2n} and write the class formula explicitly.

(7) (a) Find a composition series for $GL_2(\mathbb{Z}/5\mathbb{Z})$
   (b) Find a composition series for $GL_2(\mathbb{Z}/25\mathbb{Z})$
   (to what extent is 5 important here?)

(8) Let H be a normal subgroup of G with H having order p. Show that H is contained in every p-Sylow subgroup of G.

(9) Let $\mathcal{A}$ be a category, $X, Y \in Ob(\mathcal{A})$ and $\phi \in Hom(X, Y)$. For any $S \in Ob(\mathcal{A})$ consider the map of sets
$$\begin{array}{ccc} Hom(S,X) & \xrightarrow{\phi_*^S} & Hom(S,Y) \\ f & \mapsto & \phi \circ f \end{array}$$
   (a) Show that if ϕ is an isomorphism then ϕ_*^S is bijective.
   (b) Suppose that ϕ_*^Y is bijective. Show that there is a map $g \in Hom(Y, X)$ such that $\phi \circ g = id_Y$. Conclude that $\phi = \phi \circ g \circ \phi$.
   (c) Suppose now both ϕ_*^X and ϕ_*^Y are bijective, and let g be as above. Show that $g \circ \phi = id_X$. Conclude that ϕ is an isomorphism if and only if ϕ_*^S for all S.

(10) prove that for an object $A \in Ob(C)$ the identity $id_A \in Hom(A, A)$ is unique.

(11) prove that the inverse of an isomorphism is unique

(12) Show that for objects S, T in a category, if $Isom(S, T)$ is nonempty then it is a principal homogeneous space for the group $Aut(T)$ (i.e. the group acts simply transitively).

(13) Complete the proof that for a fixed object Y, we have that $X \mapsto Hom(X, Y)$ is a contravariant functor and $X \mapsto Hom(Y, X)$ a covariant functor.

(14) Recall that a functor $F : A \to B$ is an equivalence of categories if it has a quasi inverse $G : B \to B$ such that the compositions are isomorphic to the identity functors. Prove that any two quasi inverses G, G' of a functor F are isomorphic.

(15) Consider a partially ordered set X and let $Cat(X)$ be the associated categories (a unique arrow $x \to y$ for each pair $x \leq y$). Show that the product of x and y in $Cat(X)$, if exists, is the greatest lower bound of x, y. Identify similarly the coproduct.

(16) Use the previous exercise to cook up a category where products and coproducts don't always exist.

(17) Let Y be a set and $P(Y)$ be the set of all subsets of Y, partially ordered by inclusion. Identify explicitly products and coproducts in $Cat(P(Y))$.

(18) Let $A \to B$ be an abelian group homomorphism. What is the fibered product $A \times_B 0$ in elementary terms? What is the cofibered coproduct $0 \sqcap^A B$?

(19) Prove that a group object in *Groups* is an *abelian* group.

(20) We say that a category $\mathcal{A}$ is *pre-additive* if for any $X, Y \in Ob(\mathcal{A})$ we are given an abelian group structure on $Hom(X, Y)$, such that composition

$$\begin{array}{ccc} Hom(W,X) & \longrightarrow & Hom(W,Y) \\ f & \mapsto & \phi \circ f \end{array}$$

is a group homomorphsim for any $\phi : X \to Y$, and similarly

$$\begin{array}{ccc} Hom(X,Y) & \longrightarrow & Hom(W,Y) \\ f & \mapsto & f \circ \phi \end{array}$$

is a group homomorphism for any $\phi : W \to X$.

(a) Show that if $\mathcal{A}$ is a pre-additive category then $\mathcal{A}^{op}$ is also a pre-additive category with the same group structures.

(b) Show that the rule $(f + g)(x) := f(x) + g(x)$ gives the category $\mathcal{A}b$ of abelian groups the structure of a pre-additive category.

(21) Given a pre-additive category $\mathcal{A}$, objects $X, Y \in Ob(\mathcal{A})$ and given $f : X \to Y)$, we say $\phi : K \to X$ is a kernel for f if it satisfies the following universal property:

- $f \circ \phi = 0 \in Hom(K, Y)$.
- for any $\psi : S \to X$ such that $f \circ \psi = 0 \in Hom(S, Y)$, there is a unique $h : S \to K$ such that $\psi = \phi \circ h$.

(a) Consider $\mathcal{A} = \mathcal{A}b$. Show that if $f : X \to Y$ is an abelian group homomorphism, then the embedding $\mathrm{K}er f \hookrightarrow X$ is a kernel for f.

(b) What is a kernel for f in $\mathcal{A}b^{op}$?