

MATH 251 ALGEBRA I PROBLEMS

- (1) Lang Chapter 1 exercises 3,4,5,15,16
- (2) Read about monoids if needed. Show that for any ring R (i.e. $\mathbb{Z}, \mathbb{R}, \mathbb{C} \dots$) we have $(R, \times, 1)$, where \times is multiplication in R , is a monoid.
- (3) Let $M = \mathbb{Z} \times \mathbb{Z}$ with the operation

$$((x_1, y_1), (x_2, y_2)) \mapsto (x_1x_2 + 2y_1y_2, x_1y_2 + x_2y_1).$$

Show that with the appropriate unit this is a monoid. Can you identify it as a submonoid of $(\mathbb{R}, \times, 1)$?

- (4) A semigroup (S, m) is a set with an associative binary operation. Let (S, m) be a semigroup, and choose $e \notin S$. Let $S' = S \cup \{e\}$ and $m' : S' \times S' \rightarrow S'$ defined by $(a, b) \mapsto ab$ if $a, b \in S$ and $ae = ea = a$. Show that
 - (a) S' is a monoid,
 - (b) the inclusion $S \rightarrow S'$ is a semigroup homomorphism, and
 - (c) if M a monoid and $f : S \rightarrow M$ a semigroup homomorphism, there is a unique way to extend it to a monoid homomorphism $f' : S' \rightarrow M$.
- (5) Suppose M is a commutative monoid. Consider the relation on $M \times M$ given by $(a, b) \sim (c, d)$ if $a + d = b + c$ (so they have “the same difference”). Show that
 - (a) this is an equivalence relation;
 - (b) the set M^{gr} of equivalence classes is a commutative group;
 - (c) the map $u : M \rightarrow M^{gr}$ given by $a \mapsto (a, e)/\sim$ is a monoid homomorphism; and
 - (d) if $f : M \rightarrow G$ is any monoid homomorphism to an abelian group, there is a unique group homomorphism $f^{gr} : M^{gr} \rightarrow G$ such that $f = f^{gr} \circ u$.
 - (e) What similar group construction can you give for a commutative semigroup instead of a monoid?
- (6) Recall or read somewhere on the cycle decomposition for elements of S_n . For an m cycle $\sigma \in S_n$, what powers σ^i are also m -cycles? What is the cycle decomposition of σ^i in general?
- (7) If $\sigma \in S_n$ has cycle decomposition with cycles of lengths k_1, \dots, k_r , what is the order of σ ?
- (8) Show that the group of rigid motions of the cube acts faithfully on the set of pairs of opposite vertices. Deduce that this group is isomorphic to S_4 . What about the action on pairs of opposite faces?
- (9) Show that if G finite, $[G : H] = n$, then H contains a normal subgroup K of G of index $\leq n!$ (consider the action of G on G/H).