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Generating functions

Given Ω, $X : \Omega \to \mathbb{R}$.

Definition:

Moment generating function: $M_X(t) = E(e^{tX})$.

Characteristic function: $\varphi_X(t) = E(e^{itX})$.

Here $i = \sqrt{-1} \in \mathbb{C}$.

Evidently $\varphi_X(t) = M_X(it)$

Advantage of $M_X(t)$: real valued.

Advantage of $\varphi_X(t)$: always exists for real variables.

Interpretation:

Definition: $\mu_k = E(X^k)$ is the k-th moment of X.

For instance $\mu_1 = E(X), \mu_2 = V(X) + \mu^2$

Then $M_X(t) = E(e^{tX}) = E(\sum \frac{X^k t^k}{k!})$

$M_X(t) = \sum \frac{\mu_k t^k}{k!}$.

Similarly $\varphi_X(t) = \sum \frac{\mu_k (it)^k}{k!}$.

Note: $\mu_k = \frac{\partial^k}{(\partial x)^k}(\varphi_X(t))$

Ideas:

0. $M_X(t)$ is computable

1. $M_X(t)$ holds enough information to recover the distribution of X.

2. $M_X(t)$ behaves well under natural operations: rescaling, sums.

Laundry list:

Bernoulli: $M_{Bern(p)}(t) = q + p \cdot e^t$

Discrete Uniform $U : \{1, \ldots n\}$**:**

$M_U(t) = (1/n) \sum_{k=1}^{n} e^{kt} = \frac{1}{n} \frac{e^{(n+1)t} - e^t}{e^t - 1}$

Binomial:

$M_{Binom(n,p)}(t) = \sum_{k=0}^{n} e^{kt} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (p \cdot e^t)^k q^{n-k} = (q + p \cdot e^t)^n$

Geometric:

$M_{G(p)}(t) = \sum_{k=1}^{\infty} (e^t)^j q^{j-1} p = \frac{pe^t}{1 - qe^t}$

Poisson:

$M_{P(\lambda)}(t) = e^{-\lambda} \sum_{k=0}^{\infty} (e^t)^k \lambda^k / k! = e^{\lambda(e^t - 1)}$.

Uniform $[a, b]$:

$$M_{u(a,b)}(t) = \frac{\int_a^b e^{tx} dx}{b-a} = \frac{e^{tb}-e^{ta}}{t(b-a)}$$

Exponential

$$M_{Exp(\lambda)}(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(t-\lambda)x} dx = \frac{\lambda}{t-\lambda}$$

Standard Normal:

$$M_{N(0,1)}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2+tx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2+t^2/2} dx = e^{t^2/2}.$$

The book derives the moments on the way - read!

Moment problem: Given some moments $\mu_k(X)$, or given $M_X(t)$, can you recover X?

Finite discrete on $\{x_1, \ldots x_n\}$:
$M_X(t) = \sum e^{t \cdot x_k} p_k$

Claim: $M_X(m), m = 0, \ldots, n-1$ suffices to determine!

Write $A_{m,k} = e^{x_k \cdot m}$; P for the column vector of p_k; M for the column vector of $M_X(m)$. Then $M = AP$. But A is a Vandermonde matrix with determinant $\prod_{k<l}(e^l - e^k)$. ♣

Continuous case: $\varphi_X(t)$ determines $f_X(t)$.

$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx$

then by Fourier analysis

$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi_X(x)dx$

Try your hand at our examples!

Properties:

1. $M_{X+b}(t) = e^{tb}M_X(t)$. (just pull out the term)

2. $M_{aX}(t) = M_X(at)$. (replace t by at)

3. $M_{X^*}(t) = e^{(-\mu/\sigma)t}M_X(t/\sigma)$

4. X, Y independent then

$$M_{X+Y}(t) = M_X(t)M_Y(t).$$

Proof: e^{tX} and e^{tY} are indenpendent!

$$M_{Binom(n,p)}(t) = M_{Bern(p)}(t)^n$$

$$M_{NegBin(k,p)}(t) = M_{Geom(p)}(t)^k = \left(\frac{pe^t}{1-qe^t}\right)^k$$

Now you have X_i independent with finite μ and finite $\sigma > 0$. We can replace X_i by $(X_i - \mu)/\sigma$ so that the new $\mu = 0$ and $\sigma = 1$. If we prove this case of course the general case follows, since S_n^* remains the same.

Then $M_{S_n^*}(t) = (M_X(t/\sqrt{n}))^n$.

We claim that $\lim_{n\to\infty} M_{S_n^*}(t) = M_{N(0,1)}(t) = e^{t^2/2}$.

Now $M_X(t) = 1 + t^2/2 + R_3(t)$. So $M_{S_n^*}(t) = (1 + \left(t^2/2 + R(t/\sqrt{n})\right)/n)^n$ where $\lim_{n\to\infty} = 0$. So indeed $\lim_{n\to\infty} M_{S_n^*}(t) = e^{t^2/2}$.

Branching processes.

You want to know how long your lineage will last. Historically, kings and aristocrats cared about their male lineage. Today you might be interested in knowing how long your DNA (or Y chomosomes for males, or mitochondrial DNA for females) will last.

A very rough approximation is this: you assume that generations appear in sync. In each generation you have a certain distribution on the number X of offspring, a positive integer, given by p_k where $\sum_{k=0}^{\infty} p_k = 1$. You assume that this is identically distributed and independent among individuals - certainly not the case in reality! Also $p_0 > 0$.

Say d_m is the probability of dying out by generation m. Then $d_1 = p_0$. What is d_2? If we had k offspring in generation 1, then each has d_1 probability of not having offspring, and they are independent.

What about dying out in m generations? same argument works:

$d_m = p_0 + p_1 d_{m-1} + p_2(d_{m-1})^2 + \ldots = \sum_{k=0}^{\infty} p_k (d_{m-1})^k$

We use a tool which is a variant of M_X simply called the generating function $h(z) := E(z^X)$. Clearly $h(e^t) = M_X(t)$.

Then $h(z) = \sum_{k=0}^{\infty} p_k z^k$.

This means precisely that we have the recursive relation

$$d_m = h(d_{m-1}). \tag{1}$$

Note: $0 = d_0 \leq d_1 \leq d_2 \ldots \leq 1$. So this sequence converges to some d, the probability of dying out at some point.

Since h is a poltynomial it is continuous. The equation (1) has a limit

$$d = h(d).$$

The solution $d = 1$ is always there, and $h(0) = p_0 > 0$. Note that $h' > 0$ and $h'' > 0$, so there are at most two solutions, and the other one could be $0 < d < 1$ or $1 < d$.

This is precisely determined by whether or not $h'(1) < 1$ or $h'(1) > 1$.

What is it?

$h'(1) = p_1 + 2p_2 + 3p_3 + \ldots$
$= m := E(X)$.

Now if $d_i < d$ then also $d_{i+1} < d$ (looking at the graph).

Conclusion:

Theorem. if $E(X) < 1$ your lineage will die with probability $d < 1$, otherwise will die with probability 1.

How about the distribution of offspring?

Z_n = number of offspring in n-th generation. Not really calculable, but limiting behavior is.

Write $h_n(z) = E(z^{Z_n})$.

then $h_{n+1}(z) = \sum_k p_k \cdot E\left(z^{Z_{n+1}} \mid X = k\right)$

But $E\left(z^{Z_{n+1}} \mid X = k\right) = E\left(z^{\sum_{r=1}^k Z_n}\right)$ because Z_{n+1} is the sum of number of offspring in n generations of each one of the k first generation offspring.

So $E\left(z^{Z_{n+1}} \mid X = k\right) = h_n(z)^k$.

So $h_{n+1}(z) = \sum_k p_k h_n(z)^k = h(h_n(z))$.

So $h_n(z) = h(h(\cdots h(z) \cdots))$.

Take the derivative to get $h'_{n+1}(z) = h'(h_n(z)) \cdot h'_n(z)$.

Plug in $z = 1$ to get the expected value of Z_m. Get

$E(Z_{n+1}) = h'(1) \cdot E(Z_n)$, so $m_{n+1} = m \cdot m_n$.

So $m_n = m^n$!

Makes sense?