Math 161 0 - Probability, Fall Semester 2012-2013 Dan Abramovich

Selected distributions Geometric distribution

T = number of independent bernoulli trials of prob p till first success:

$$P(T=n) = q^{n-1}p.$$

Have seen: adds up to 1.

Memoryless:

$$\begin{split} P(T = n + k | T > n) &= q^{n + k - 1} p / q^n \\ &= q^{k - 1} p = P(T = k) \end{split}$$

Variant: "Negative binomial distribution"

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Independent bernoulli-*p*-trials till we get k successes. X = time. $u(x, k, p) := P(X = x) = {\binom{x-1}{k-1}}p^kq^{x-k}$

Poisson distribution

This is a good way to approximate binomial distributions, since those are usually a lot of work to sum up.

If you do n = 100 trials, one per hour, with probability p, you expect roughly the same number of successes as doing n = 200 trials, two per hour, but with probability p/2. These converge rather rapidly to the following limit:

Write
$$\lambda = np$$
. Think of λ as the
average number of successes per hour.
So $p = \lambda/n$. As $n \to \infty$ we have
 $p \to 0$, so
 $b(n, p, k) = {n \choose k} p^k q^{n-k}$
 $= \frac{(n)_k p^k ((1-p)^{(1/p)})^{\lambda}}{(1-\lambda/n)^k} \xrightarrow[n \to \infty]{} \frac{\lambda^k}{k!} e^{-\lambda}$
We set
Poisson $(\lambda, k) = \frac{\lambda^k}{k!} e^{-\lambda}$

hypergeometric distribution.

Urn has 100 balls, 40 blue balls and the rest red balls. What's the probability of getting 5 blue balls and the rest red balls, drawing 20 balls at random?

This is just counting:

$$h(100, 40, 20, 5) = \frac{\binom{40}{5}\binom{100-40}{20-5}}{\binom{100}{20}}$$

In general

$$h(N,k;n,x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}.$$

Read about benford $f(k) = \log_{10}(k+1) - \log_{10}(k)$. Comes from taking $\log x \mod 1$.

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Densities Uniform

on an interval [a, b] set f(x) = 1/(b - a).

Exponential

on $x \ge 0$ set $f(x) = \lambda e^{-\lambda x}$. We have calculated $F_X(x) = 1 - e^{-\lambda x}$. The more natural function is $G(x) = P(X > x) = e^{-\lambda x}$.

We have seen it is memoryless.

Note that if we set $k = \lfloor x \rfloor$, then k is geometric with $p = 1 - e^{-\lambda}$. This explains the memoryless property of geometric distributions.

Functions of random variables

Theorem: X continuous random variable, $\phi(x)$ a *strictly increasing* function. Write $Y = \phi(X)$. Then

$$F_Y(y) = F_X(\phi^{-1}(y)).$$

$$F_Y(\phi(x)) = F_X(x).$$

If ψ strictly decreasing and $Z = \psi(x)$ then

$$F_Z(z) = 1 - F_X(\psi^{-1}(z)).$$

Proof: write out events:

$$F_Y(y) = P(Y \le y)$$

= $P(\phi(X) \le y) = P(X \le \phi^{-1}(y))$
= $F_X(\phi^{-1}(y))$

Similar for ψ !

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Corollary

 $f_Y(y) = f_X(\phi^{-1}(y)) \cdot \frac{d}{dy}\phi^{-1}(y)$

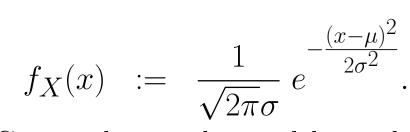
 $f_Z(z) = -f_X(\psi^{-1}(z)) \cdot \frac{d}{dz} \psi^{-1}(z)$

What if W = |X|?

$$F_W(w) = F_X(x) - F_X(-x)$$

$$f_W(w) = (f_X(x) + f_X(-x)).$$

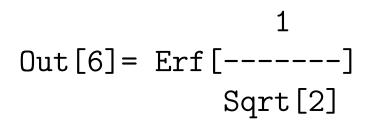
Normal density $X \sim Normal(\mu, \sigma)$



Centered around μ , width regulated by σ .

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^x e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt.$$

In[6]:= 1/Sqrt[2*Pi]
*Integrate[Exp[-x^2/2],{x,-1,1}]



Note: if $Z \sim Normal(0, 1)$ then $X = \sigma Z + \mu$.

$$F_X(x) = F_Z\left(\frac{x-\mu}{\sigma}\right)$$

indeed

$$f_X(x) = f_Z\left(\frac{x-\mu}{\sigma}\right)/\sigma$$

$$= \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

READ REST OF LAUNDRY LIST