MATH 251 Algebra I
Problems

(1) Lang Chapter 1 exercises 3,4,5,15,16
(2) Read about monoids if needed. Show that for any ring R (i.e. $\mathbb{Z}, \mathbb{R}, \mathbb{C} \ldots$) we have $(R, \times, 1)$, where $\times$ is multiplication in R, is a monoid.
(3) Let $M = \mathbb{Z} \times \mathbb{Z}$ with the operation

$$((x_1, y_1), (x_2, y_2)) \mapsto (x_1x_2 + 2y_1y_2, x_1y_2 + x_2y_1).$$

Show that with the appropriate unit this is a monoid. Can you identify it as a submonoid of $(\mathbb{R}, \times, 1)$?
(4) A semigroup (S, m) is a set with an associative binary operation. Let (S, m) be a semigroup, and choose $e \notin S$. Let $S' = S \cup \{e\}$ and $m' : S' \times S' \to S'$ defined by $(a, b) \mapsto ab$ if $a, b \in S$ and $ae = ea = a$. Show that
  (a) S' is a monoid,
  (b) the inclusion $S \to S'$ is a semigroup homomorphism, and
  (c) if M a monoid and $f : S \to M$ a semigroup homomorphism, there is a unique way to extend it to a monoid homomorphism $f' : S \to M$.
(5) Suppose M is a commutative monoid. Consider the relation on $M \times M$ given by $(a, b) \sim (c, d)$ if there is f such that $a + d + f = b + c + f$ (so they have "the same difference"). Show that
  (a) this is an equivalence relation;
  (b) the set M^{gr} of eqivalence classes is a commutative group;
  (c) the map $u : M \to M^{gr}$ given by $a \mapsto (a, e)/\sim$ is a monoid homomorphism; and
  (d) if $f : M \to G$ i any monoid homomorphism to an abelian group, there is a unique group homomorphism $f^{gr} : M^{gr} \to G$ such that $f = f^{gr} \circ u$.
  (e) What similar group construction can you give for a commutative semigroup instead of a monoid?
(6) Recall or read somewhere on the cycle decomposition for elements of S_n. For an m cycle $\sigma \in S_n$, what powers σ^i are also m-cycles? What is the cycle decomposition of σ^i in general?
(7) If $\sigma \in S_n$ has cycle decomposition with cycles of lengths $k_1, \ldots, k_r$, what is the order of σ?
(8) Show that the group of rigid motions of the cube acts faithfully on the set of pairs of opposite vertices. Deduce that this group is isomorphic to S_4. What about the action on pairs of opposite faces?
(9) Show that if G finite, $[G : H] = n$, then H contains a normal subgroup K of G of index $\leq n!$ (consider the action of G on G/H).