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MA 2530 Final Talk

FUNCTIONAL EQUATION FOR GENERALIZED ZETA FUNCTION

1. Intro

Recall, last week, David introduced the Riemann Zeta Function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 + p^{-s})^{-1}$$

which converges for $Re(s) > 1$, and he gave us an analytic continuation to the right half plane $Re(s) > 0$.

But we want our zeta function to be meromorphic in all of $\mathbb{C}$, so we need some sort of functional equation to extend it in the other direction. We start by defining the gamma function

$$\Gamma(s) = \int_0^{\infty} e^{-y} y^s \frac{dy}{y}$$

which converges for $Re(s) > 0$. You may recall, either from undergrad number theory, or from last lecture, that ζ satisfies the functional equation in $K = \mathbb{Q}$

$$\zeta(s) = \pi^{\frac{s-1}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) \Gamma\left(\frac{s}{2}\right)^{-1}$$

which we can also write as:

$$\pi^{\frac{-1}{2}s} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{-1}{2}(1-s)} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$$

so that you can see the symmetry. Thus, we can extend ζ meromorphically across all of $\mathbb{C}$, where the only pole is at $s = 1$ of residue 1. Note: this is the Zeta function for $\mathbb{Q}$, so we want to define an analogue for higher number fields.

2. Quick Proof of $\mathbb{Q}$ Case

We're going to do a super-quick proof of the functional equation in $\mathbb{Q}$, just to give you a sense of how it works, because the generalization works similarly. We define

$$F(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

Then, we note that we can scale our variable y, like

$$\int_0^{\infty} e^{-y} y^s \frac{dy}{y} = \int_0^{\infty} e^{-ay} (ay)^s \frac{dy}{y}$$

for $a > 0$, because $e^{-y} y^{s-1}$ is absolutely integrable from 0 to ∞. You might wonder, why do we care? So, now we have

$$\begin{aligned}
\frac{\Gamma\left(\frac{s}{2}\right)}{a^s} &= \int_0^{\infty} e^{-y} \left(\frac{y}{a^2}\right)^{s/2} \frac{dy}{y} \\
&= \int_0^{\infty} e^{-a^2 y} \left(\frac{a^2 y}{a^2}\right)^{s/2} \frac{dy a^2}{a^2 y} \\
&= \int_0^{\infty} e^{-a^2 y} y^{s/2} \frac{dy}{y}
\end{aligned}$$

Well, since

$$\begin{aligned} F(s) &= \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) \\ &= \sum_{n=1}^{\infty} \pi^{-s/2} n^{-s} \Gamma\left(\frac{s}{2}\right) \\ &= \int_0^\infty \sum_{n=1}^{\infty} e^{-\pi n^2 y} y^{s/2} \frac{dy}{y} \end{aligned}$$

And we define $g(y) = \sum_{n=1}^{\infty} e^{-\pi n^2 y}$ for notational brevity.

$$\begin{aligned} &= \int_0^1 g(y) y^{s/2} \frac{dy}{y} + \int_1^\infty g(y) y^{s/2} \frac{dy}{y} \\ &= I + II \end{aligned}$$

We do a change of variables on I, to get

$$I = \int_0^1 g(y) y^{s/2} \frac{dy}{y} = \int_1^\infty g(1/y) y^{-s/2} \frac{dy}{y}$$

Here, we introduce the theta function

$$\theta(y) = \sum_{-\infty}^{\infty} e^{-n^2 \pi y}$$

First, we note that $2g(y) = \theta(y) - 1$, so $g(y) = \frac{1}{2}(\theta(y) - 1)$, and immediately from the Poisson summation formula, we have $\theta(y^{-1}) = y^{1/2}\theta(y)$. Then,

$$\begin{aligned} F(s) &= \int_1^\infty g(1/y) y^{-s/2} \frac{dy}{y} + \int_1^\infty g(y) y^{s/2} \frac{dy}{y} \\ &= \frac{1}{2}\int_1^\infty (\theta(1/y) - 1) y^{-s/2} \frac{dy}{y} + \frac{1}{2}\int_1^\infty (\theta(y) - 1) y^{s/2} \frac{dy}{y} \\ &= \frac{1}{2}\int_1^\infty (y^{1/2}\theta(y) - 1) y^{-s/2} \frac{dy}{y} + \frac{1}{2}\int_1^\infty (\theta(y) - 1) y^{s/2} \frac{dy}{y} \\ &= \frac{1}{2}\int_1^\infty (\theta(y) - 1)(y^{(1-s)/2} + y^{s/2}) \frac{dy}{y} + \frac{1}{2}\int_1^\infty (y^{(1-s)/2} - y^{-s/2}) \frac{dy}{y} \\ &= \int_1^\infty (y^{(1-s)/2} + y^{s/2}) g(y) \frac{dy}{y} - \frac{1}{1-s} - \frac{1}{s} \end{aligned}$$

which is clearly symmetric under $s \mapsto 1-s$, so

$$\begin{aligned} F(s) &= F(1-s) \\ \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s) \end{aligned}$$

3. Complicating Things A Bit: Number Fields

So, for a number field K, with $[K : \mathbb{Q}] = n$, we define the Zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p} \text{ prime}} (1 - (N\mathfrak{p})^{-s})^{-1}$$

where we range $\mathfrak{a}$ over the non-zero ideals of K and $\mathfrak{p}$ over the prime ideals in K. David showed us last week that this converges when $\zeta_{\mathbb{Q}}$ does.

So, a natural question arises: is there a similar functional equation for ζ_K? Yes, but it's not quite as nice. To get there, we're going to restrict the zeta function a little bit: (don't worry, we'll bring it back to normal at the end)

4. Ideal Class Group

Recalling that I is the multiplicative group of non-zero fractional ideals and P the subset of principal ideals, we define I/P as the ideal class group. Letting $\mathfrak{K}$ be an ideal class of I/P, we define

$$\zeta(s, \mathfrak{K}) = \sum_{\mathfrak{a} \in \mathfrak{K}} \frac{1}{N\mathfrak{a}^s}$$

Note, that then we have

$$\zeta_K(s) = \sum_{\mathfrak{K}} \zeta(s, \mathfrak{K})$$

Then, let $\mathfrak{a} \in \mathfrak{K}^{-1}$. Then, we have the map $\phi : \mathfrak{b} \to \mathfrak{b}\mathfrak{a}^{-1} = (\xi)$, where $\mathfrak{b} \in \mathfrak{K}, \xi \in \mathfrak{a}$, since two ideals are in the same class if they satisfy $\mathfrak{a} = (\alpha)\mathfrak{b}$. We can easily see, then, that ϕ is a bijection between ideals $\mathfrak{b} \in \mathfrak{K}$ and the equivalence classes of nonzero elements of $\mathfrak{a}$. Let $R(\mathfrak{a})$ be a set of these representatives: then,

$$\begin{aligned}
N\mathfrak{a}^{-s}\zeta(s, \mathfrak{K}) &= N\mathfrak{a}^{-s} \sum_{\mathfrak{b} \in \mathfrak{K}} \frac{1}{N\mathfrak{b}^s} \\
&= \sum_{\mathfrak{b} \in \mathfrak{K}} \frac{1}{(N\mathfrak{b}N\mathfrak{a})^s} \\
&= \sum_{\mathfrak{b} \in \mathfrak{K}} \frac{1}{N(\mathfrak{a}\mathfrak{b})^s} \text{ since norm is multiplicative} \\
&= \sum_{\xi \in R(\mathfrak{a})} \frac{1}{N\xi^s} \text{ since } \phi \text{ is a bijection}
\end{aligned}$$

So,

$$\zeta(s, \mathfrak{K}) = N\mathfrak{a}^s \sum_{\xi \in R(\mathfrak{a})} \frac{1}{N\xi^s}$$

Note, from here, we can write $\xi = x_1\alpha_1 + ... + x_n\alpha_n$ for $\{\alpha_1, ..., \alpha_n\}$ a basis of $\mathfrak{a}$ over $\mathbb{Z}$. We'll be writing $\xi_v = \sigma_v \xi$, where σ_v are the embeddings of K in $\mathbb{R}$ or $\mathbb{C}$.

5. The Theorem, for one ideal class

As above, let $\mathfrak{K}$ be an ideal class, where we'll keep $[K : \mathbb{Q}] = n = r_1 + 2r_2$, where r_1 is the number of real absolute values of K and r_2 is the number of conjugate pairs of complex embeddings. We'll write D_K for the absolute value of the discriminant of K. Define

$$A = 2^{-r_2} D_K^{-1/2} \cdot \pi^{-n/2}$$

Then, we define

$$F(s, \mathfrak{K}) = A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta(s, \mathfrak{K})$$

Let $\mathfrak{d}$ be the different of $K/\mathbb{Q}$, and let $\mathfrak{K}'$ be the ideal class denoted by $\mathfrak{d}^{-1}\mathfrak{K}^{-1}$. Then, F is analytic except for simple poles at $s = 0$ and $s = 1$, and F satisfies the functional equation

$$F(s, \mathfrak{K}) = F(1 - s, \mathfrak{K}').$$

6. The Proof, for one ideal class

6.1. The Gamma Function. We start by remembering the gamma function

$$\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}$$

which converges for $Re(s) > 0$. Also, recall that we can scale our variable y, like

$$\int_0^\infty e^{-y} y^s \frac{dy}{y} = \int_0^\infty e^{-ay}(ay)^s \frac{dy}{y}$$

for $a > 0$. So, now we have

$$\frac{\Gamma\left(\frac{s}{2}\right)}{a^s} = \int_0^\infty e^{-a^2 y} y^{s/2} \frac{dy}{y}$$

We then recall that $D_{\mathfrak{a}} = N\mathfrak{a}^2 D_K$, (this is the absolute value of the discriminant of $\mathfrak{a}$. Now, we're going to separate by our absolute values, putting real and complex as separate cases.

6.2. Case 1: Real. Let v be a real valuation. Set $a^2 = \pi D_{\mathfrak{a}}^{-1/n} |\xi_v|^2$. Then, we substitute this a^2 in our equation above. So, we have

$$\left(\frac{D_K^{1/2n} N\mathfrak{a}^{1/n}}{\sqrt{\pi}|\xi_v|}\right)^s \Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-\pi D_{\mathfrak{a}}^{-1/n}|\xi_v|^2 y} y^{s/2} \frac{dy}{y}$$

6.3. Case 2: Complex. Let v be a complex valuation. Set $a = 2\pi D_{\mathfrak{a}}^{1/n} |\xi_v|^2$. Then, we substitute again, (also, we're going to substitute $2s$ for s here, so that Γ will be over s, not $s/2$, and we find,

$$\left(\frac{D_K^{1/n} N\mathfrak{a}^{2/n}}{2\pi|\xi_v|^2}\right)^s \Gamma(s) = \int_0^\infty e^{-2\pi D_{\mathfrak{a}}^{-1/n}|\xi_v|^2 y} y^s \frac{dy}{y}$$

Now, (noticing that we have almost the same expressions on the right,) we'll use this to split up F. First, we let $y = \prod_v y_v$ be our variable over the $(r_1 + r_2)$-space we're creating, and this allows $\frac{dy}{y} = \prod_v \frac{dy_v}{y_v}$ to be our measure. Then, we have

$$\begin{aligned} A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \frac{N\mathfrak{a}^2}{N\xi^s} &= (2^{-r_2} D_K^{1/2} \pi^{-n/2})^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \\ &= \left(\Gamma\left(\frac{s}{2}\right) \frac{D_K^{s/2n} N\mathfrak{a}^{s/n}}{\pi^{s/2} N\xi^{s/n}}\right)^{r_1} \cdot \left(\Gamma(s) \frac{D_K^{s/n} N\mathfrak{a}^{2s/n}}{2^s \pi^s N\xi^{2s/n}}\right)^{r_2} \end{aligned}$$

at this point, I'm going to switch from writing e^y to $exp(y)$, for clarity's sake

$$\begin{aligned} &= \int_0^\infty \cdots \int_0^\infty exp(\textstyle\sum \text{that stuff above}) ||y||^{s/2} \frac{dy}{y} \\ &= \int_0^\infty \cdots \int_0^\infty exp(-\pi D_{\mathfrak{a}}^{-1/n} \sum_v N_v |\xi_v|^2 y_v) ||y||^{s/2} \frac{dy}{y} \end{aligned}$$

where here we have that $N_v =$ the number of valuations conjugate to v, so $N_v = 1$ if v is real and 2 if v is complex.

We're now going to sum over $\xi \in R(\mathfrak{a})$, so no two ξ are equivalent. Notice that this sum will be absolutely and uniformly convergent for $Re(s) > 1$. Then, since $\zeta(s, \mathfrak{K}) = N\mathfrak{a}^s \sum_{\xi \in R(\mathfrak{a})} \frac{1}{N\xi^s}$, we have that

$$\begin{aligned} F(s, \mathfrak{K}) &= A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta(s, \mathfrak{K}) \\ &= \int_0^\infty \cdots \int_0^\infty \sum_{\xi \in R(\mathfrak{a})} exp(-\pi D_{\mathfrak{a}}^{-1/n} \sum_v N_v |\xi_v|^2 y_v) \cdot ||y||^{s/2} \frac{dy}{y} \end{aligned}$$

6.4. **Some Algebraic Manipulation.** We then want to simplify, so instead of integrating n times over $\mathbb{R}^+$, we're going to integrate once over $G = \mathbb{R}^+ \times \cdots \times \mathbb{R}^+ = \prod \mathbb{R}_v^+$. (via chalkboard magic...) We're also going to change $\frac{dy}{y}$ to the appropriate measure d^*y.

$$F(s, \mathfrak{K}) = \int_G \sum_{\xi \in R(\mathfrak{a})} exp(-\pi D_{\mathfrak{a}}^{-1/n} \sum_v N_v |\xi_v|^2 y_v) \cdot ||y||^{s/2} d^*y$$

We then investigate G: as usual, let $U \subset K$ be the group of units, and let $V \subset G$ be the image of U under the map $\psi : K \to G$ defined by $\psi(x) = (|x_v|)$. (If we think about the kernel of this map, it's pretty easy to see that it'll be the roots of unity.) If we let $G_0 = \{y \in G : ||y|| = 1\}$ be the set of elements in G of norm 1, we have that V is a subgroup of G_0, and G_0/V is compact.

However, given any element $y = (y_1, ..., y_n) \in G$, we can also write $y = ||y|| \cdot c$, where $||c|| = 1$, so we know that any $y \in G$ can be rewritten as $y = t^{1/n}c = (t^{1/n}c_v)$, where $t \in R^+, c = (c_v) \in G_0$. So, G can also be written as the product $G = R^+ \times G_0$, and $||y|| = t$. Then, we have that

$$F(s, \mathfrak{K}) = \int_0^\infty \int_{G_0} \sum_{\xi \in R(\mathfrak{a})} exp(-\pi D_{\mathfrak{a}}^{-1/n} \sum_v N_v |\xi_v|^2 t^{1/n} c_v) \cdot t^{s/2} d^*c \frac{dt}{t}$$

noting that d^*c is the appropriate measure on G_0. We then need one more manipulation (aside: one of the best descriptions of number theory proofs: A number theorist sits down with an expression. Rearranges it once. Rearranges it twice. Rearranges it a third time and exclaims, I've proved something!) We can define E as the fundamental domain of V^2 in G_0. (a fundamental domain is a collection of representatives, one each from the orbits created by multiplying by elements in V^2) So, by definition,

$$G_0 = \bigcup_{\eta \in V} \eta^2 E$$

So, we can change the integral over G_0 to be over E: let w be the number of roots of unity in K, we now switch our c to be in E

$$F(s, \mathfrak{K}) = \int_0^\infty \int_E \frac{1}{w} \sum_{\eta \in V} \sum_{\xi \in R(\mathfrak{a})} exp(-\pi D_{\mathfrak{a}}^{-1/n} \sum_v N_v |\xi_v|^2 t^{1/n} (\eta^2 c_v)) \cdot t^{s/2} d^*c \frac{dt}{t}$$

and since $\eta_v = |u_v|$, we can pull this term inside our absolute value

$$= \int_0^\infty \int_E \frac{1}{w} \sum_{u \in U} \sum_{\xi \in R(\mathfrak{a})} exp(-\pi D_{\mathfrak{a}}^{-1/n} \sum_v N_v |u_v \xi_v|^2 t^{1/n} c_v) \cdot t^{s/2} d^*c \frac{dt}{t}$$

6.5. **The Theta Function: Hecke.** Maybe we're going to rewrite this expression more than just three times...to rewrite it this time, we're going to need to introduce a new function, thanks to Hecke: Let $c_1, ..., c_n$ be positive real numbers, such that $c_{r_1+v} = c_{r_1+v+r_2}$, so we have r_1 individual values and r_2 pairs of values. We keep the same definition of $D_{\mathfrak{a}}$. Recall that $\xi = x_1\alpha_1 + ... + x_n\alpha_n$ for α_i a basis of $\mathfrak{a}$ over $\mathbb{Z}$. Furthermore, we're going to let $j \in (1, r_1 + 2r_2)$ be the indices for the conjugates of K in $\mathbb{C}$, so we can write $\xi^{(j)} = x_1\alpha_1^{(j)} + ... + x_n\alpha_n^{(j)}$. Then, define

$$\Theta(c, \mathfrak{a}) = \sum_{x \in \mathbb{Z}^n} exp\left(-\pi D_a^{-1/n} \sum_{j=1}^n c_j |\xi^{(j)}|^2\right)$$

Thus, Θ satisfies the functional equation

$$\Theta(c, \mathfrak{a}) = \frac{1}{\sqrt{c_1 \cdots c_n}} \Theta(c^{-1}, \mathfrak{a}')$$

where we use c^{-1} to denote $c_1^{-1}, ..., c_n^{-1}$. Do you want to see the proof? If not (please), it's just a quick application of the Poisson summation formula.

6.6. **Back to Our Formula.** So, since we're summing over u and ξ both, we're pretty much summing over almost everything, with the exception of $\xi = 0$. Thus, we can rewrite our formula as,

$$F(s,\mathfrak{K}) = \int_0^\infty \int_E \frac{1}{w}[\Theta(t^{1/n}c,\mathfrak{a}) - 1] \cdot t^{s/2} d^*c \frac{dt}{t}$$

(make sure you keep theta function on the board). We're then going to split up our integral into two parts, since we want to use this new functional equation we got for the Theta function:

$$\begin{aligned} F(s,\mathfrak{K}) &= \int_0^1 \int_E \frac{1}{w}[\Theta(t^{1/n}c,\mathfrak{a}) - 1] \cdot t^{s/2} d^*c \frac{dt}{t} + \int_1^\infty \int_E \frac{1}{w}[\Theta(t^{1/n}c,\mathfrak{a}) - 1] \cdot t^{s/2} d^*c \frac{dt}{t} \\ &= \int_0^1 \int_E \frac{1}{w}\Theta(t^{1/n}c,\mathfrak{a}) \cdot t^{s/2} d^*c \frac{dt}{t} + \int_0^1 \frac{1}{w} t^{s/2} d^*c \frac{dt}{t} + \int_1^\infty \int_E \frac{1}{w}[\Theta(t^{1/n}c,\mathfrak{a}) - 1] \cdot t^{s/2} d^*c \frac{dt}{t} \\ &= \int_0^1 \int_E \frac{1}{w}[\Theta(t^{1/n}c,\mathfrak{a})] \cdot t^{s/2} d^*c \frac{dt}{t} + \frac{\mu^*(E)2}{ws} + \int_1^\infty \int_E \frac{1}{w}[\Theta(t^{1/n}c,\mathfrak{a}) - 1] \cdot t^{s/2} d^*c \frac{dt}{t} \\ &= I + II + III \end{aligned}$$

where we're setting $\mu^*(E)$ to be the measure of E under d^*c. Then, we use the functional equation for Theta, noting that

$$\Theta(t^{1/n}c,\mathfrak{a}) = \frac{1}{\sqrt{t}}\Theta(t^{-1/n}c^{-1},\mathfrak{a}')$$

since $c \in G_0$, so $||c|| = 1$. So, looking just at I, we have

$$\begin{aligned} \int_0^1 \int_E \frac{1}{w}\Theta(t^{1/n}c,\mathfrak{a}) \cdot t^{s/2} d^*c \frac{dt}{t} &= \int_0^1 \int_E \frac{1}{w}\left[\frac{1}{\sqrt{t}}\Theta(t^{-1/n}c^{-1},\mathfrak{a}')\right] \cdot t^{s/2} d^*c \frac{dt}{t} \\ &= \int_0^1 \int_E \frac{1}{w}\left[\frac{1}{\sqrt{t}}\Theta(t^{-1/n}c^{-1},\mathfrak{a}') - 1\right] \cdot t^{s/2} d^*c \frac{dt}{t} + \frac{\mu^*(E)2}{w(1-s)} \end{aligned}$$

We then do a change of variables, letting $\tau = 1/t$, so we have $d\tau = -dt/t^2$. Note that we can change c^{-1} to c, since d^*c is invariant under that change. (doesn't care)

$$\begin{aligned} &= -\int_\infty^1 \int_E \frac{1}{w}\sqrt{\tau}\Theta(\tau^{1/n}c,\mathfrak{a}') \cdot \tau^{-s/2} d^*c \tau \frac{-d\tau}{\tau^2} && + \frac{\mu^*(E)2}{w(1-s)} \\ &= \int_\infty^1 \int_E \frac{1}{w}\Theta(t^{1/n}c,\mathfrak{a}') \cdot t^{(1-s)/2} d^*c \frac{dt}{t} && + \frac{\mu^*(E)2}{w(1-s)} \\ &= \int_\infty^1 \int_E \frac{1}{w}[\Theta(t^{1/n}c,\mathfrak{a}') - 1] \cdot t^{(1-s)/2} d^*c \frac{dt}{t} && + \frac{\mu^*(E)2}{w(1-s)} \end{aligned}$$

Then, we have that

$$\begin{aligned} F(s,\mathfrak{K}) = &\int_1^\infty \int_E \frac{1}{w}[\Theta(t^{1/n}c,\mathfrak{a}) - 1] \cdot t^{s/2} d^*c \frac{dt}{t} + \frac{\mu^*(E)2}{ws} \\ &+ \int_\infty^1 \int_E \frac{1}{w}[\Theta(t^{1/n}c,\mathfrak{a}') - 1] \cdot t^{(1-s)/2} d^*c \frac{dt}{t} + \frac{\mu^*(E)2}{w(1-s)} \end{aligned}$$

since this is the same for $s \mapsto 1-s$ and $\mathfrak{a} \mapsto \mathfrak{a}'$, we have the functional equation $F(s,\mathfrak{K}) = F(1-s,\mathfrak{K}')$! Note here that both integrals converge absolutely for all complex s, with the exception of the two simple poles at $s = 0$ and $s = 1$.

7. Going Back to General

Recall, a while back, we had

$$\zeta_K(s) = \sum_{\mathfrak{K}} \zeta(s,\mathfrak{K})$$

Similarly, we'll define

$$F_K(s) = \sum_{\mathfrak{K}} F(s, \mathfrak{K}) = A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s)$$

From our proof above, we can easily see that

$$F_K(s) = \sum_{\mathfrak{K}} F(s, \mathfrak{K}) = \sum_{\mathfrak{K}} F(1-s, \mathfrak{K}') = \sum_{\mathfrak{K}} F(1-s, \mathfrak{K}) = F_K(1-s)$$

by reordering the ideal classes. So,

$$A^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_K(s) = A^{1-s} \Gamma\left(\frac{1-s}{2}\right)^{r_1} \Gamma(1-s)^{r_2} \zeta_K(1-s)$$