MATH 251 PROBLEMS

(1) Prove that the number of p-Sylow subgroups of $GL_2(\mathbb{F}_p)$ is $p+1$.
(2) Find a composition series for A_4.
(3) Describe the conjugacy classes in the dihedral group D_{2n} and write the class formula explicitly.
(4) (a) Find a composition series for $GL_2(\mathbb{Z}/5\mathbb{Z})$
    (b) Find a composition series for $GL_2(\mathbb{Z}/25\mathbb{Z})$
    (to what extent is 5 important here?)
(5) Let $\mathcal{A}$ be a category, $X, Y \in Ob(\mathcal{A})$ and $\phi \in Hom(X, Y)$. For any $S \in Ob(\mathcal{A})$ consider the map of sets

$$\begin{array}{ccc} Hom(S,X) & \xrightarrow{\phi_*^S} & Hom(S,Y) \\ f & \mapsto & \phi \circ f \end{array}$$

    (a) Show that if ϕ is an isomorphism then ϕ_*^S is bijective.
    (b) Suppose that ϕ_*^Y is bijective. Show that there is a map $g \in Hom(Y, X)$ such that $\phi \circ g = id_Y$. Conclude that $\phi = \phi \circ g \circ \phi$.
    (c) Suppose now both ϕ_*^X and ϕ_*^Y are bijective, and let g be as above. Show that $g \circ \phi = id_X$. Conclude that ϕ is an isomorphism if and only if ϕ_*^S for all S.
(6) prove that for an object $A \in Ob(C)$ the identity $id_A \in Hom(A, A)$ is unique.
(7) Show that for objects S, T in a category, if $Isom(S, T)$ is nonempty then it is a principal homogeneous space for the group $Aut(T)$ (i.e. the group acts simply transitively).
(8) Complete the proof that for a fixed object Y, we have that $X \mapsto Hom(X, Y)$ is a contravariant functor and $X \mapsto Hom(Y, X)$ a covariant functor.
(9) Recall that a functor $F : A \to B$ is an equivalence of categories if it has a quasi inverse $G : B \to A$ such that the compositions are isomorphic to the identity functors. Prove that any two quasi inverses G, G' of a functor F are isomorphic.