Brown University
MA251: Algebra

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Simplicity of A_n

Proposition 1. The group A_5 is simple.

Proof. There are 5 conjugacy classes of A_5. The table below contains a representative and the order of each one:

Representative from Conjugacy Class	(1)	(12345)	(21345)	(12)(34)	(123)
Order of Conjugacy Class	1	12	12	15	20

Any normal subgroup $N \triangleleft A_5$ must be a union of these conjugacy classes, including (1). Further, the order of N would divide the order A_5. However the only divisors of $|A_5| = 60$ that are possible by adding up 1 and any combination of $\{12, 12, 15, 20\}$ are 60 and 1. Thus $N = \{1\}$ or $N = A_n$, and A_n is simple. □

Proposition 2. The group A_6 is simple.

Proof. The proof is much the same as in Proposition 1. We consider the order of each conjugacy class of A_6:

Representative	(1)	(123)	(123)(456)	(12)(34)	(12345)	(23456)	(1234)(56)
Order	1	40	40	45	72	72	90

No combination of the orders which includes 1 gives a divisor of $|A_6| = 360$ except 1 and the sum of the orders of *all* of the conjugacy classes. So $N = A_6$ or $N = \{1\}$. A_6 is simple. □

Lemma 3. For $n \geq 5$, any two 3-cycles in A_n are conjugate in A_n.

Proof. It is sufficient to show that any 3-cycle in A_n is conjugate to (123). Let σ be a 3-cycle in A_n. There is some $\pi \in S_n$ such that

$$(123) = \pi\sigma\pi^{-1}$$

If $\pi \in A_n$, then we're done. If $\pi \notin A_n$, then let $\pi' = (45)\pi$, so $\pi' \in A_n$. We finish by conjugating σ with π':

$$\pi'\sigma\pi'^{-1} = (45)\pi\sigma\pi^{-1}(45) = (45)(123)(45) = (123)$$

□

Lemma 4. A_n, $n \geq 3$, is generated by 3-cycles.

Proof. Let the set of all 3-cycles of A_n be

$$T = \{(abc) \mid 1 \leq a, b, c \leq n\}$$

and note that $\langle T \rangle \subset A_n$. Since any $\tau \in A_n$ is the *even* product of 2-cycles (transpositions), we consider pairs of transpositions σ in the decomposition of τ into transpositions. For any pair, there are two cases:

(i) If $\sigma = (ab)(cd)$, $a \neq b \neq c \neq d$, then $\sigma = (acb)(acd)$.

(ii) If $\sigma = (ab)(ac)$, then $\sigma = (acb)$.

Thus $A_n \subset \langle T \rangle$, and we can conclude that $A_n = \langle T \rangle$. □

We will show that any nontrivial normal subgroup $N \triangleleft A_n$ contains a 3-cycle. Since if N contains a 3-cycle, then N contains all 3-cycles (Lemma 3). If N contains every 3-cycle, then $N = A_n$ (Lemma 4).

Theorem 1. For $n \geq 7$, A_n is a simple group.

Proof. Our goal will be to prove that a normal subgroup of $A_{n \geq 7}$ contains a 3-cycle.
Let $N \triangleleft A_n$, $n \geq 7$ be a non-trivial normal subgroup of A_n, and suppose $e \neq \sigma \in N$ such that (by relabeling if needed) $\sigma(1) \neq 1$. Let $\tau = (ijk)$ where $i, j, k \neq 1$ and $\sigma(1) \in \{i, j, k\}$. Note that

$$\tau\sigma\tau^{-1}(1) = \tau(\sigma(1)) \neq \sigma(1) \implies \tau\sigma\tau^{-1} \neq \sigma.$$

Let $\phi = \tau\sigma\tau^{-1}\sigma$ and note that $\phi(1) \neq 1$. First, $\phi \in N$ since

$$\phi = (\tau\sigma\tau^{-1})\sigma^{-1}$$

Note that τ is a 3-cycle. $\sigma\tau^{-1}\sigma^{-1}$ is also a 3-cycle since τ^{-1} is a 3-cycle. So $\phi = \tau(\sigma\tau^{-1}\sigma^{-1})$ is the product of two 3-cycles; thus ϕ permutes at most 6 elements of $\{1, 2, ..., n\}$. Let S be a set of precisely 6 elements in $\{1, 2, ..., n\}$, including those permuted by ϕ. Let $H \subset A_n$ be the group corresponding to the even permutations of S, and note that H is an isomorphic copy of A_6 inside A_n.
Since $N \triangleleft A_n$ by hypothesis and $H \subset A_n$ is a subgroup, we have $N \cap H \triangleleft H$ and is nonempty since it contains ϕ. Since $H \cong A_6$, which we have shown to be simple, $N \cap H = H$ and $H \subset N$, so N contains a 3-cycle. □

Corollary 5. Together with Proposition 1 and Proposition 2, the theorem shows that A_n is simple when $n \geq 5$. As needed.

Based on notes by Keith Conrad at UConn, who in turn references J. Rotman's *Advanced Modern Algebra*.