Will Kochanski

The Jordan-Hölder Theorem

Lemma. *Let G be a group with $A \neq B$ normal in G such that $G/A, G/B$ are simple then:*

$$G/A \simeq B/(A \cap B) \qquad G/B \simeq A/(A \cap B)$$

Proof. Suppose that $A \subset B$ then B/A is normal in the simple group G/A. Since A is not equal to B the quotient is not trivial, and by the assumption that G/B is simple neither is it the whole group. This is a contradiction, so we can assume $A \not\subset B$ and by symmetry $B \not\subset A$.

Consider AB a normal subgroup of G, its image under the quotient map, AB/A will be a normal subgroup of G/A. However from $B \not\subset A$ we have that $AB/A \neq \{e\}$ and so since G/A is simple we must have $AB/A = G/A$. Finally from the second isomorphism theorem we conclude:

$$B/(A \cap B) \simeq AB/A = G/A$$

By symmetry that $A/(A \cap B) \simeq G/B$. $\square$

Theorem. *Let G be a group and assume G has a decomposition series. Let*

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_r = \{e\}$$

$$G = H_0 \triangleright H_1 \triangleright \ldots \triangleright H_s = \{e\}$$

Be any two decomposition series for G then $r = s$ and there exists $\sigma \in S_r$ such that $\forall$ k:

$$G_k/G_{k+1} \simeq H_{\sigma(k)}/H_{\sigma(k)+1}$$

Proof. We use induction over the length of shortest decomposition series for G. It is sufficient to show that any decomposition series is equivalent to a minimal series, and therefore that any two series are equivalent. If G is simple then it has a unique decomposition series $G \triangleright \{e\}$. For the inductive case assume that:

$$G = G_0 \triangleright G_1 \triangleright \ldots \triangleright G_r = \{e\}$$

is a minimal composition series for G. Suppose that $G_1 = H_1$ then by induction the series starting from G_1 will be equivalent to the series starting from H_1, and therefore the whole series will be as well. Otherwise let $K = H_1 \cap G_1$ which is normal in G. By the lemma we have that $G_1/K \simeq G/H_1$ and $H_1/K \simeq G/G_1$ are simple.

Let $K_i := K \cap G_i$ then $G_i \triangleright K_i$ and $K_i \triangleright K_{i+1}$. Consider the homomorphism $K_i \to G_i/G_{i+1}$ given by the quotient map. The image is normal and the kernel is K_{i+1}, therefore by the isomorphism theorems we have that K_i/K_{i+1} is a normal subgroup of G_i/G_{i+}. Furthermore since G_i/G_{i+1} is simple for each K_i, K_{i+1} either $K_i = K_{i+1}$ or the quotient K_i/K_{i+1} is simple. By removing duplicates we get two decomposition series for G_1:

$$G_1 \triangleright G_2 \triangleright \ldots \triangleright G_r = \{e\}$$

$$G_1 \triangleright K_1 \triangleright \ldots \triangleright K_r = \{e\}$$

By induction on G_1 these series are equivalent, and in particular must have the same length, $r - 1$, so exactly one of the groups K_i/K_{i+1} is trivial.

We have already shown that $H_1 \triangleright K_1$ with a simple quotient and therefore we also have two composition series for H_1:

$$H_1 \triangleright H_2 \triangleright ... \triangleright H_s = \{e\}$$

$$H_1 \triangleright K_1 \triangleright ... \triangleright K_r = \{e\}$$

Since exactly one of the groups K_i/K_{i+1} is trivial we conclude that H_1 also has a decomposition series of length $r-1$ which is less than that of G. Therefore by induction these series are equivalent with $s-1 = r-1$.

It is therefore sufficient to show that the series:

$$G \triangleright G_1 \triangleright K_1 \triangleright ... \triangleright K_r = \{e\}$$

$$G \triangleright H_1 \triangleright K_1 \triangleright ... \triangleright K_r = \{e\}$$

Are equivalent. By the lemma $G/G_1 \simeq H_1/K_1$ and $G/H_1 \simeq G_1/K_1$ and clearly $K_i/K_{i+1} \simeq K_i/K_{i+1}$ therefore this is the case.

$\square$

References

1. "The Jordan-Holder Theorem - Notes for Wednesday January 23". David E Speyer. University of Michigan. http://www.math.lsa.umich.edu/ speyer/594/JordanHolder.pdf

2. "The Jordan-Hölder Theorem". Dan Abramovich. Brown University. Sept 2013.