Math 52 0 - Linear algebra, Spring Semester 2012-2013
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Linear equations

The first case of a linear equation you learn is in one variable, for instance:

$$2x = 5.$$

We learned in school that this is equivalent to what you get when you divide both sides by 2: ←why?

$$x = 5/2.$$

This gives us the solution.

It also drives home the fact that fractions - rational numbers - are a necessary part of life.

Some of these linear equations are special. The equation

$$0 \cdot x = 5$$

has no solution, while for

$$0 \cdot x = 0$$

"all x are solutions".

At least we need to decide which solutions we want - rational, real, complex...

For the time being we work with *real* equations and solutions.

A real linear equation in real variables $x_1, \ldots, x_n$ is of the form

$$a_1x_1 + a_2x_2 + \cdots a_nx_n = b$$

where b and the coefficients a_i are real.

We usually are concerned with systems of linear equations.

Here is a system of two linear equations with two unknowns:

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ x_1 - x_2 &= -1 \end{aligned}$$

We'll review soon how to solve it. It has the solution

$$x_1 = 1, \quad x_2 = 2$$

check it!!→ which is in fact unique.

You can graph the solution of each equation as a line, and the two lines have different slopes, so the intersect in a single point.

The system

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

has no solution - two parallel lines, whereas in

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 4x_2 &= 10 \end{aligned}$$

there is a whole line of solutions. ←is there another case?

Linear algebra: you can learn a whole lot by thinking deeply about systems of linear equations

Key questions:

(consistency) does a solution exist?

(uniqueness) if it exists, is it unique?

of which we will have a variant:

(dimension) “how many solutions?”

Finally, the practical question:

(Algorithm) find the solutions!

It is costomary to encode a system of linear equations in a **matrix:**

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & = & 5 \\ x_1 & - & x_2 & = & -1 \end{array} \Rightarrow \begin{bmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

since the coefficients and the right hand side hold all the data.

Some people put a bar to separate the right hand side, since it has a different meaning:

$$\left[\begin{array}{rr|r} 1 & 2 & 5 \\ 1 & -1 & -1 \end{array}\right]$$

which is OK.

Let us solve something a bit more challenging:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 8 \\ x_1 & - & x_2 & + & x_3 & = & 0 \\ 2x_1 & & & - & x_3 & = & 1 \end{array}$$

which becomes

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 2 & 3 & 8 \\ 2 & 0 & -1 & 1 \end{bmatrix}$$

write it!!→ we can subtract the first equation from the second:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 8 \\ 2 & 0 & -1 & 1 \end{bmatrix}$$

write it!!→ and subtract *twice* the first from the third:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 8 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

We have *eliminated* the first variable from the second and third equations.

Divide 2nd equation by 3:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2/3 & 8/3 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

subtract twice second equation from third

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2/3 & 8/3 \\ 0 & 0 & -13/3 & -13/3 \end{bmatrix}$$

eliminating x_2 from last equation. rescale last equation

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 2/3 & 8/3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

which means $x_3 = 1$.

To get the rest, subtract 2/3 equation 3 from equation 2:

$$\begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

so $x_2 = 2$; subtract third from first

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and add second to first

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

so

$$x_1 = 1, \quad x_2 = 2, \quad x_3 = 1.$$

The steps, called *elementary row operations*, are

- add a multiple of one row to another
- rescale a row by a nonzero number
- (not used here) switch the order of rows.

Definition. Two matrices are *row equivalent* if one is obtained from the other by a sequence of elementary row operations.

Key fact. Row equivalent matrices have the same set of solutions.

The process we used is called
row reduction
or *Gaussian elimination*
or *Gauss–Jordan elimination*
or *bringing a matrix to echelon form.*

ask class→

Observations:

- all steps are *reversible*. The resulting system after each step has the same solution set.
- we went through fractions even though solutions are integers
- We can test our solution at the end (and had better do it!)

this is unfortunate→

- can't really check in the middle. . .

Echelon form = step form

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In the black squares we have nonzero numbers.

They are called *pivots*, they are positioned in *pivot columns*.

The stars hold any numbers.

Reduced echelon form:

the black squares are now 1 and above them we have zeros:

$$\begin{bmatrix} 0 & 1 & * & 0 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem: each matrix is equivalent to a *unique* matrix in reduced echelon form.

Below we show the existence of a reduced echelon form for any martix. We do this by formalizing an algorithm.

The uniqueness is a bit deeper. We'll see it later.

Algorithm for existence of an echelon form of a matrix A:

←do a running example 2

1. Find the minimum i such that the i-th column is nonzero. This is the first pivot column.

$$\begin{bmatrix} 0 & * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * & * \\ 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * & * \end{bmatrix}$$

2. There is some position j in the ith column such that a_{ij} is not zero. Switch the 1st and j-th rows of the matrix:

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * & * \end{bmatrix}$$

3. Subtract a_{ji}/a_{1i} times the first row from the j-th row to get zeros, for $j > 1$:

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \end{bmatrix}$$

4. If there is just one row, you got your echelon form.

Otherise, do 1-4 on the submatrix with the first row deleted

$$\begin{bmatrix} 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \\ 0 & 0 & * & * & * & * & * & * & * \end{bmatrix}$$

which brings it to an echelon form

$$\begin{bmatrix} 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and together with the first row we got gives an echelon form for A:

$$\begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Algorithm for existence of a reduced echelon form of a matrix A in echelon form:

1. Rescale pivot rows to make the pivots 1:

$$\begin{bmatrix} 0 & \boxed{1} & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \boxed{1} & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Starting (for instance) from the row of the rightmost pivot and going to the left, add a multiple of the row to each previous row to make the entry above the pivot 0: first

$$\begin{bmatrix} 0 & 1 & * & * & * & * & \boxed{0} & * & * \\ 0 & 0 & 0 & 1 & * & * & \boxed{0} & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

then

$$\begin{bmatrix} 0 & 1 & * & \boxed{0} & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(etc.)

The variables where the pivots appear are “fixed” or “basic” variables. The other are *free* variables.

One can find the *general* solution by allowing free variables to take any value and solving for the basic variables.

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ \boxed{1} & 4 & 5 & -9 & -7 \end{bmatrix} \Rightarrow \begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & \boxed{2} & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-5} & 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & \boxed{-5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow$$

Note: could have made the pivots 1 earlier!

Reduced form:

$$\begin{bmatrix} \boxed{1} & 4 & 5 & -9 & -7 \\ 0 & \boxed{1} & 2 & -3 & -3 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 4 & 5 & \boxed{0} & -7 \\ 0 & 1 & 2 & \boxed{0} & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \boxed{0} & -3 & 0 & 5 \\ 0 & 1 & 2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Can we answer the questions?

Consistency: Suppose A is an extended martix of a system of equations, and U is an echelon form. Then the system is consistent if an only if U and no row of the form

$$\begin{bmatrix}0 & \cdots & 0 & b\end{bmatrix}$$

with $b \neq 0$.

If there is such a row we have an equivalent equation of the form $0 = b$, which has no solutions.

For the other direction, and the other key questions, we need to write the solutions. Let us assume U is in reduced echelon form.

(Algorithm) You may be familiar with the method of "back substitution".

Suppose there are r nonzero rows in U. Each stands for an equation of the form:

$$x_i + a_{k\,i+1}x_{i+1} + \dots a_{kn}x_n = b_k.$$

(this is the k-th row)

and only free variables have nonzero coefficients other than x_i, which is a basic variable.

1. Give arbitrary values to the free variables.

2. Set the basic variables

$$x_i = b_k - (a_{k\,i+1}x_{i+1} + \dots a_{kn}x_n).$$

Consequences:

(Uniqueness) for a consistent system, the solution is unique if and only if there are no free parameters.

(Dimension) we will formalize a deeper notion of dimension. The number of free parameters is the dimension of the set of solutions.

(Uniqueness of reduced echelon form - part 1) say x_j is a free variable. Set $x_j = 1$ and all other free variables to be 0. This determines all the basic variables. We get the equation

$$x_i + a_{kj}x_j = b_k,$$

which, since $x_j = 1$, means

$$a_{kj} = b_k - x_i.$$

This determines U, if we know which variables are free and which basic.