MA 252 notes: Commutative algebra
(Distilled from [Atiyah-MacDonald])

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Topological groups

Recall: a topological group G is a group in the category *Top*: product, inverse and unit $\{1\} \to G$ are continuous maps. They form a category.

Since translation by a is continuous the topology is determined by neighborhoods of 1.

Lemma

The intersection H of neighborhoods of 1 is a closed subgroup, the closure of 1. The quotient G/H is Haussdorff, the maximal (initial) Hausdorff quotient.

Completion and inverse limit

We work with abelian groups, additively. We suppose 0 has a countable fundamental system of neighborhoods. We can therefore form the completion $G \to \hat{G}$ using Cauchy sequences, a continuous hmomorphism with kernel H as above. Its formation is a functor $TopGroups \to TopGroups$.
Assume now 0 has a countable descending system of (normal) open subgroups, say $G = G_0 \supset G_1 \supset \cdots$. We have seen they are automatically closed. Then

Theorem

$G \to \varprojlim G/G_n$ extends to continuous $\hat{G} \to \varprojlim G/G_n$. If the system of subgroups is fundamental then $\hat{G} \simeq \varprojlim G/G_n$.

Exactness

An inverse system is a projective system parametrized by $\mathbb{N}$ with reverse ordering. There is a category of these, and we can talk about exact sequences.

Proposition

If $0 \to (A_n) \to (B_n) \to (C_n) \to 0$ is an exact sequence of inverse systems then $0 \to \varprojlim(A_n) \to \varprojlim(B_n) \to \varprojlim(C_n)$ is exact, and if the structure maps $A_i \to A_{i-1}$ are surjective then $0 \to \varprojlim(A_n) \to \varprojlim(B_n) \to \varprojlim(C_n) \to 0$ is exact.

In essence, one shows that $\varprojlim^1(A_n) = 0$, represented as a quotient of the product. A more general condition than surjectivity is the Mittag-Leffler condition (find and read).

Exactness of completions

Corollary

Let $0 \to G' \to G \to G'' \to 0$ *be an exact sequence of groups,* $G = G_0 \supset G_1 \supset \cdots$ *sequence of subgroups,* $G'_n = i^{-1}G_n$ *and* $G''_n = j(G_n)$*. Then* $0 \to \hat{G}' \to \hat{G} \to \hat{G}'' \to 0$ *is exact*

Corollary

$\hat{G}_n \to \hat{G}$ *injective,* $\hat{G}/\hat{G}_n \simeq G/G_n$ *discrete.*

Proposition

$\hat{\hat{G}} = \hat{G}$.

We say that $\hat{G}$ is complete.

Topological rings and $\mathfrak{a}$-adic completions

We apply the considerations to a ring A additively abelian group, and ideals $\mathfrak{a}^n$ a descending sequence of subgroups, giving rise to a topological ring $\hat{A} = \varprojlim A/\mathfrak{a}^n$, with $\ker(A \to \hat{A}) = \mathfrak{a}^\infty := \cap \mathfrak{a}^n$.
Same for modules $M \supset \mathfrak{a}^n M$, so $\hat{M} = \varprojlim M/\mathfrak{a}^n M$.
These are functors *Rings* $\to$ *TopRings*, and A-mod $\to$ $\hat{A}$-mod.
Distant goal: A noetherian implies $\hat{A}$ noetherian.

Filtration

Other sequences of ideals or submodules may give the same topology.

Definition

A filtration on M is a chain $M = M_0 \supset M_1 \ldots$.
A filtration is an $\mathfrak{a}$-filtration if $\mathfrak{a} M_n \subset M_{n+1}$.
A filtration is a stable $\mathfrak{a}$-filtration if for some n_0 and all $n > n_0$ we have $\mathfrak{a} M_n = M_{n+1}$.

For instance $M_n = \mathfrak{a}^n M$ is stable.

Filtrations and completions

Lemma

Given $(M_n), (M'_n)$ stable $\mathfrak{a}$-filtrations, they define the same topology and isomorphic completions. Moreover they have bounded difference: there is n_0 such that $M_{m+n_0} \subset M'_m$ and $M'_{m+n_0} \subset M_m$ for all m.

We might as well assume $M'_n = \mathfrak{a}^n M$.
The $\mathfrak{a}$-filtration assumption gives indictively $\mathfrak{a}^n M \subset M_n$.
Stability gives $M_{n_0+n} = \mathfrak{a}^n M_{n_0} \subset \mathfrak{a}^n M$.

Graded rings

Definition

A graded ring A is a direct sum $A = \oplus_{n=0}^{\infty} A_n$ of abelian subgroups A_n such that $A_n A_m \subset A_{n+m}$.

These form a category. Note that $\bullet A_n$ is a subring of A, $\bullet A$ is an A_0-algebra, and $\bullet A_n$ is an A_0-module.
Key example: $k[x_1, \ldots, x_n]$, with A_n the spaces of homogeneous polynomials of degree n.

Graded modules

Definition

If A a graded ring, then a graded A-module is a module which is a direct sum $M = \oplus_{n=0}^{\infty} M_n$ such that $A_n M_m \subset M_{n+m}$.

In particular M_n are A_0-modules.
These form a category gr$A$-mod.
An element $x \in M_n$ is said to be homogeneous. Every element $y \in M$ has a unique decomposition $y = \sum y_n$ as finite sum of its homogeneous components $y_n \in M_n$.

Definition

$A_+ := \oplus_{n=1}^{\infty} A_n$, an ideal.

Note that $A_+ = \ker(A \to A_0)$.

Noetherian graded rings

Proposition

A graded ring A is noetherian if and only if A_0 is noetherian and A a finitely generated A_0-algebra.

- If A_0 is noetherian and A a finitely generated A_0-algebra then A noetherian by Hilbert's basis theorem.
- Assume A is noetherian. Then $A_0 \simeq A/A_+$ so A_0 noetherian.
- Let $(x_1, \ldots, x_n)$ be generators of A_+.
- Replacing them by all their homogeneous components, we may assume $x_i \in A_{k_i}$ homogeneous.
- Let A' be the subalgebra generated by A_0 and $x_1, \ldots, x_n$.
- We claim $A' \supset A_m$ so $A' = A$.
- We use induction, $m = 0$ holds by assumption.
- Indeed if $y \in A_m, m > 0$ then $y = \sum a_i x_i$ with $a_i \in A_{m-k_i} \subset A'$, as needed.

Rees algebras

Definition

Let A be a ring, $\mathfrak{a}$ and ideal. The Rees algebra of $\mathfrak{a}$ is the graded algebra $A^* = \oplus_{n=0}^{\infty} \mathfrak{a}^n$, where we agree $\mathfrak{a}^0 := A$.

Definition

Let M be a module, abd M_n an $\mathfrak{a}$-filtration on M. Write $M^* = \oplus_n M_n$, an A^*-module.

Note: if A is Noetherian then $I = (x_1, \ldots, x_n)$ is finitely generated and then $A[x_1, \ldots, x_n] \to A^*$ surjective, so A^* noetherian.

Filtrations and noetherian modules

Lemma

Assume A noetherian, M finitely generated module, (M_n) an $\mathfrak{a}$-filtration.
Then M^* is a finitely generated A^* module if and only if (M_n) is $\mathfrak{a}$-stable.

- We have M_n finitely generated so $Q_n = \oplus_{r=0}^n M_r$ finitely generated.
- Write $M_n^* = M_0 \oplus \ldots \oplus M_n \oplus \mathfrak{a}M_n \oplus \mathfrak{a}^2 M_n \oplus \cdots$, an A^*-module generated by Q_n, so finitely generated.
- M_n^* an ascending chain with union M^*, so M^* is finitely generated
  if and only if the chain stabilizes at some $M^* = M_{n_0}^*$
  if and only if $M_{n_0+r}^* = \mathfrak{a}^r M_{n_0}^*$ for all r
  if and only if the filtration is stable.

Artin-Rees: filtrations and submodules

Assumptions: A noetherian, $\mathfrak{a}$ ideal, M finitely generated module, (M_n) stable $\mathfrak{a}$-filtration, $M' \subset M$ submodule.

Proposition (Artin-Rees 1)

The filtration $(M' \cap M_n) \subset M'$ is a stable $\mathfrak{a}$-filtration.

- Note that $\mathfrak{a}(M' \cap M_n) \subset \mathfrak{a}M' \cap \mathfrak{a}M_n \subset M' \cap M_{n+1}$ giving at $\mathfrak{a}$-filtration.
- So $(M')^* \subset M^*$ sum A^*-module.
- Finitely generated as A^* noetherian.
- By Lemma $(M' \cap M_n)$ stable $\mathfrak{a}$-filtration.

Corollary (Artin-Rees 2)

There is k such that $\mathfrak{a}^{n+k}M \cap M' = \mathfrak{a}^n(\mathfrak{a}^k M \cap M')$.

This follows from the proposition applied to the stable filtration $\mathfrak{a}^n M$.

Artin Rees: topologies

Theorem

A noetherian, $\mathfrak{a}$ ideal, M finitely generated module, $M' \subset M$ submodule.
Then $\mathfrak{a}^n M'$ and $\mathfrak{a}^n M \cap M'$ have bounded difference.
In particular the $\mathfrak{a}$-adic topology on M' is induced from the $\mathfrak{a}$-adic topology on M.

By the proposition we have that $\mathfrak{a}^n M \cap M'$ is a stable $\mathfrak{a}$-filtration, so be a previous lemma the two topologies have bounded difference.

Completions and exactness

Proposition

Let A be noetherian, $\mathfrak{a} \subset A$, and $0 \to M' \to M \to M'' \to 0$ exact sequence of finitely generated modules.
Then the sequence of $\mathfrak{a}$-adic completions $0 \to \hat{M}' \to \hat{M} \to \hat{M}'' \to 0$ is exact.

First note that $\mathfrak{a}^m M''$ is the image of $\mathfrak{a}^m M$.
By the theorem, $\hat{M}' = \varprojlim M'/\mathfrak{a}^n M' = \varprojlim M'/(\mathfrak{a}^m M \cap M')$.
An early exactness lemma implies the result.

Completion and tensor products

Note: we have an $\hat{A}$- homomorphism $\hat{A} \otimes_A M \to \hat{A} \otimes_A \hat{M} \to \hat{A} \otimes_{\hat{A}} \hat{M} = \hat{M}$.

Proposition

If M finitely generated then $\hat{A} \otimes_A M \to \hat{M}$ surjective.
If M finitely generated and A noetherian then $\hat{A} \otimes_A M \to \hat{M}$ an isomorphism.

First, completion commutes with direct sums, in particular $\hat{A} \otimes_A A^n = \hat{A}^n$.
If M finitely generated take a presentation $0 \to N \to F \to M \to 0$ with $F \simeq A^n$.

Completion and tensor products - continued

We get a commutative diagram of complexes

$$\begin{array}{ccccccccc} & & \hat{A}\otimes_A N & \longrightarrow & \hat{A}\otimes_A F & \longrightarrow & \hat{A}\otimes_A M & \longrightarrow & 0 \\ & & \downarrow \nu & & \downarrow \phi & & \downarrow \mu & & \\ 0 & \longrightarrow & \hat{N} & \longrightarrow & \hat{F} & \longrightarrow & \hat{M} & \longrightarrow & 0. \end{array}$$

The top row is exact by right exactness of $\otimes$.
The map $\hat{F} \to \hat{M}$ is surjective since the topology of M is induced from F (and an old lemma on completions of groups).
Since ϕ an isomorphism we have μ surjective as well.
Now assuming A noetherian, we have that N finitely generated and the bottom row is also exact.
From the above ν is surjective. By the snake lemma $\hat{A}\otimes_A M \to \hat{M}$ an isomorphism.

Flatness of completion

Assume A noetherian. The proposition implies that the functor $M \mapsto \hat{A} \otimes_A M$ is exact on finitely generated modules. But we have seen that this suffices for flatness:

Theorem

Assume A noetherian, $\hat{A}$ its $\mathfrak{a}$-adic completion. Then $\hat{A}$ is a flat A-algebra.

In general the functor $M \mapsto \hat{M}$ is not exact.

Completion of noetherian rings - first properties

Let A be noetherian, $\mathfrak{a}$ and ideal and $\hat{A}$ the $\mathfrak{a}$-adic completion.

Proposition

(i) $\hat{\mathfrak{a}} = \mathfrak{a}\hat{A} \simeq \mathfrak{a} \otimes_A \hat{A}$. *(ii)* $\hat{\mathfrak{a}}^n = \widehat{\mathfrak{a}^n}$. *(iii)* $\mathfrak{a}^n/\mathfrak{a}^{n+1} = \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1}$. *(iv)* $\hat{a} \subset \mathfrak{R}\hat{A}$, *the Jacobson radical.*

Since A noetherian the homomorphism $\mathfrak{a} \otimes_A \hat{A} \to \hat{\mathfrak{a}}$ is an isomorphism. Its image is $\mathfrak{a}\hat{A}$, giving (i), in fact for any ideal.
In particular $\widehat{\mathfrak{a}^n} = \mathfrak{a}^n\hat{A} = (\mathfrak{a}\hat{A})^n = \hat{\mathfrak{a}}^n$, giving (ii).
The topology on $\mathfrak{a}$ is induced from A. By an old result on groups we have $A/\mathfrak{a}^{n+1} \simeq \hat{A}/\hat{\mathfrak{a}}^{n+1}$. Taking the kernel of the map to $A/\mathfrak{a}^n \simeq \hat{A}/\hat{\mathfrak{a}}^n$ one gets (iii).
The above implies that $\hat{A}$ is complete for teh $\mathfrak{a}$-adic topology, so for $x \in \hat{\mathfrak{a}}$ the series $(1-x)^{-1} = 1 + x + x^2 + \cdots$ is convergent in $\hat{A}$, so $\in \mathfrak{R}\hat{A}$.

Completion at a maximal ideal

Proposition

Let A be noetherian, $\mathfrak{m}$ maximal. Then the $\mathfrak{m}$-adic completion $\hat{A}$ is a local ring with maximal ideal $\hat{\mathfrak{m}}$.

We have seen that $\hat{A}/\hat{\mathfrak{m}} = A/\mathfrak{m}$, which is a field, so $\hat{\mathfrak{m}}$ maximal. Since $\hat{\mathfrak{m}} \subset \mathfrak{R}\hat{A}$ and the latter is contained in all maximal ideals we have $\hat{\mathfrak{m}}$ unique, so $\hat{A}$ local.

Krull's intersection theorem

Note that the kernel of $M \to \hat{M}$ is $E = \cap \mathfrak{a}^n M$.

Theorem (Krull)

A noetherian, $\mathfrak{a}$ ideal, M finitely generated, $\hat{M}$ the $\mathfrak{a}$-adic completion. Then $x \in E = \cap \mathfrak{a}^n M$ if and only if there is $a \in \mathfrak{a}$ such that $(1 + a)x = 0$.

Note that this is the kernel of $M \to M[(1 + \mathfrak{a})^{-1}]$. Taking $M = A$, we note $1 + \mathfrak{a}$ maps to $\hat{A}^\times$. In particular we have an injective homomorphism $A[(1 + \mathfrak{a})^{-1}] \to \hat{A}$.

Proof.

The topology on E is trivial, but it coincides with the $\mathfrak{a}$-adic topology, so $\mathfrak{a}E = E$. Since E finitely generated, Nakayama gives an $a \in \mathfrak{a}$ with $(1 - a)E = 0$ giving one inclusion.
Conversely if $x = ax = a^2x = \cdots$ then $x \in \cap \mathfrak{a}^n M$. ♠

Implications of Krull on ideals and modules

Corollary

If A noetherian domain, $\mathfrak{a} \neq 1$, then $\cap \mathfrak{a}^n = 0$.

Indeed $1 + \mathfrak{a}$ contains no zero divisors.

Corollary

A noetherian $\mathfrak{a} \subset \mathfrak{R}A$ and M finitely generated, then $\cap \mathfrak{a}^n M = 0$.

Indeed $1 - \mathfrak{a} \subset A^\times$. This implies in particular:

Corollary

If $(A, \mathfrak{m})$ local noetherian, M finitely generated then the $\mathfrak{m}$-adic topology on M is Hausdorff.

The correspondence of $\mathfrak{p}$ primaries in A and $\mathfrak{m}$-primaries in $A_\mathfrak{p}$ gives

Corollary

If A noetherian, the intersection of $\mathfrak{p}$-primaries is $\operatorname{Ker}(A \to A_\mathfrak{p})$.

Associated graded

- If A a ring at $\mathfrak{a}$ an ideal define $Gr_{\mathfrak{a}}(A) = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}$, naturally a graded ring.
- If (M_n) an $\mathfrak{a}$-filtration then $Gr_{(M_n)}(M) = \bigoplus_{n=0}^{\infty} M_n/M_{n+1}$, naturally $Gr_{\mathfrak{a}}(A)$-module.

Proposition

A noetherian, then (1) $Gr_{\mathfrak{a}}(A)$ noetherian. (2) $Gr_{\mathfrak{a}}(A) \simeq Gr_{\hat{\mathfrak{a}}}(\hat{A})$. (3) if (M_n) stable then $Gr(M)$ is a finitely generated $Gr(A)$-module.

(1) Let x_i generate $\mathfrak{a}$ and $\bar{x}_i$ the images in $\mathfrak{a}/\mathfrak{a}^2$ then $A/\mathfrak{a}[\bar{x}_1, \ldots, \bar{x}_n] \to Gr(A)$ surjective.
Part (2) follows since $\mathfrak{a}^n/\mathfrak{a}^{n+1} \simeq \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1}$.
(3) Fix r so that $M_{r+n} = \mathfrak{a}^n M_r$. Then $Gr(M)$ is generated by $\oplus_{k\leq r}(M_k/M_{k+1})$. Each term is noetherian over $A/\mathfrak{a}$, so finitely generated, and thus the same generators generate $Gr(M)$.

Completions and gradings

Say (A_n) a filtration of abelian group A, (B_n) a filtration of B, and $\phi : A \to B$ a homomorphism respecting filtrations: $\phi(A_n) \subset B_n$. Write $Gr(\phi) : Gr(A) \to Gr(B)$ and $\hat{\phi} : \hat{A} \to \hat{B}$.

Lemma

(i) If $Gr(\phi)$ is injective then $\hat{\phi}$ is injective.
(ii) If $Gr(\phi)$ is surjective then $\hat{\phi}$ is surjective.

$$\begin{array}{ccccccccc} 0 & \to & A_n/A_{n+1} & \to & A/A_{n+1} & \to & A/A_n & \to & 0 \\ & & \downarrow{\scriptstyle Gr_n(\phi)} & & \downarrow{\scriptstyle \phi_{n+1}} & & \downarrow{\scriptstyle \phi_n} & & \\ 0 & \to & B_n/B_{n+1} & \to & B/B_{n+1} & \to & B/B_n & \to & 0 \end{array}$$ commutative.

By the snake lemma, $Gr_n(\phi)$ $\star$jective and ϕ_n $\star$jective then ϕ_{n+1} $\star$jective. Moreover $Gr_n(\phi)$ surjective implies $(\mathrm{Ker}\phi_n)$ is also a surjective inverse system.

Note $\phi_0 : 0 \to 0$ is bijective, so $Gr(\phi)$ $\star$jective implies $\hat{\phi}$ $\star$jective by the exactness properties of $\varprojlim$.

Finite generation of modules over complete rings

Assume $(\star)$ A is complete for $\mathfrak{a}$-adic topology, M a module with $\mathfrak{a}$-filtration (M_n). We assume the topology is Hausdorff, namely $\cap M_n = 0$.

Proposition

If $Gr(M)$ is a finitely generated $Gr(A)$-module then M is a finitely generated A-module.

Let $\xi_i \in Gr_{n(i)}(M)$ be finitely many homogeneous generators of $Gr(M)$, the images of $\xi_i \in M_{n(i)}$.
We make the map $x_i : A \to M$ a filtered map by giving $A = F(-n(i))$ the filtration $F_k(-n(i)) = \mathfrak{a}^{k-n(i)}$.
Write $F = \oplus F(-n(i))$ and $\phi : F \to M$ the filtered map given by x_i. By design $Gr(\phi)$ surjective, so $\hat{\phi}$ surjective.

Finite generation of modules - continued

If $Gr(M)$ is a finitely generated $Gr(A)$-module then M is a finitely generated A-module.

We obtained $\hat{\phi}$ surjective.
$$\begin{array}{ccc} F & \xrightarrow{\phi} & M \\ \alpha\downarrow & & \downarrow\beta \\ \hat{F} & \xrightarrow{\hat{\phi}} & \hat{M}. \end{array}$$

Now F being free, and $\hat{A} = A$, we have α an isomorphism. So $F \to \hat{M}$ surjective.
M being Hausdorff we have β injective. It follows that ϕ surjective (and β an isomorphism). In particular M finitely generated by the x_i.

Noetherian modules over complete rings

Keep assumption $(\star)$.

Corollary

If $Gr(M)$ is a noetherian $Gr(A)$-module then M is a noetherian A-module.

Let $M' \subset M$ and $M'_n = M' \cap M_n$. The embedding $i : M' \to M$ is a filtered map, with $Gr_n(i) : M'_n/M'_{n+1} \to M_n/M_{n+1}$ injective by design, namely $Gr(i)$ injective.

Since $Gr(M)$ noetherian $Gr(M')$ finitely generated. It is Hausdorff since $\cap M'_n \subset \cap M_n = 0$. So the proposition applies and M' finitely generated.

Completion of noetherian rings

Theorem

If A noetherian, then its $\mathfrak{a}$-adic completion $\hat{A}$ is noetherian.

We have seen that $Gr(A)$ is noetherian and $Gr(A) = Gr(\hat{A})$. Since $\hat{A} \to \hat{\hat{A}}$ an isomorphism $\hat{A}$ Hausdorff. We may apply the result to $M = \hat{A}$.

Corollary

If A noetherian then $A[\![x_1, \ldots, x_n]\!]$ is noetherian.