MA 252 notes: Commutative algebra

(Distilled from [Atiyah-MacDonald])

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The Poincaré series of a graded module

Recall that an additive function $\lambda : A_0\text{-mod} \to \mathbb{Z}$ is a function such that $\lambda(M') + \lambda(M'') = \lambda(M)$ whenever $0 \to M' \to M \to M'' \to 0$ is exact.

The key case is $A_0 = k$ a field and $\lambda = \dim_k$, but the flexibility will be useful.

Say $A = \oplus A_n$ is a graded ring, $M = \oplus M_n$ a graded module. Then M_n is an A_0 module.

We assume A noetherian, generated by homogeneous $x_1, \ldots, x_s$ of degrees $k_1, \ldots, k_s$ over A_0. The key case is $k_i = 1$. Assuming M is ginitely generated, then each M_n is a finitely generated A_0-module.

Definition

The Poincaré series of M is $P_\lambda(M, t) := \sum \lambda(M_n) t^n$.

Rationality

Theorem

$P_\lambda(M,t) = f(t)/\prod(1-t^{k_i})$, with $f(t) \in \mathbb{Z}[t]$, in particular it is a rational function.

Apply induction on the number of generators s of A over A_0.
We have an exact sequence $0 \to K \to M \stackrel{\cdot x_s}{\to} M \to L \to 0$.
This breaks up as $0 \to K_n \to M_n \stackrel{\cdot x_s}{\to} M_{n+k_s} \to L_{n+k_s} \to 0$, giving
$\lambda(K_n) - \lambda(M_n) + \lambda(M_{n+k_s}) - \lambda(L_{n+k_s}) = 0$.
Each of K and L are $A/(x_s)$-modules, so induction applies.
To balance degrees we multiply by t^{k_s} and get
$t^{k_s}P(K,s) - t^{k_s}P(M,s) + P(M,s) - P(L,t) = g(t) \in \mathbb{Q}[t]$.

Dimension, act one.

Define $d(M)$ to be the order of pole of $P(M,t)$ at $t = 1$.

Corollary

If each $k_i = 1$ we have that $\lambda(M_n)$ is eventually a polynomial $\in \mathbb{Q}[n]$ of degree $d-1$.

(Read the proof in the book. Relies on Taylor expansion of $1/(1-t)^d$.)
This is called the Hilbert polynomial. The actual function $\lambda(M_n)$ is the Hilbert function.

Proposition

If $x \in A_k$ not a zero divisor in M then $d(M/xM) = d(M) - 1$.

Follow the proof of the theorem! $K = 0$ and $L = M/xM$.
If $A = k[x_1, \dots x_s]$ and $\lambda = \dim$ then $P(A,t) = 1/(1-t)^s$, so $d =$ "dimension".

Hilbert-Samuel functions

$(A, \mathfrak{m})$ noetherian local ring, $\mathfrak{q}$ an $\mathfrak{m}$-primary having s generators, M finitely generated, (M_n) stable $\mathfrak{q}$-filtration.

Proposition

(i) M/M_n of finite length.
(ii) $\ell(M/M_n)$ is eventually a polynomial $g(n)$, of degree $\leq s$.
(ii) $\deg(g(t))$ and the leading coefficient do not depend on the chosen filtration, only on $M, \mathfrak{q}$.

$Gr(A) = \oplus \mathfrak{q}^n/\mathfrak{q}^{n+1}$, $Gr_0(A) = A/\mathfrak{q}$ Artin local,
$Gr(M) = \oplus M_n/M_{n+1}$ finitely generated $Gr(A)$-module.
$Gr_n(M) = M_n/M_{n+1}$ noetherian $A/\mathfrak{q}$ module, so finite length.
So $\ell(M/M_n) = \sum_{r=0}^{n-1} \ell(M_r/M_{r+1})$ is finite, giving (i).

Hilbert-Samuel functions - continued

(i) M/M_n of finite length. ✓
(ii) $\ell(M/M_n)$ is eventually a polynomial $g(n)$, of degree $\leq s$.
(ii) $\deg(g(n))$ and the leading coefficient do not depend on the chosen filtration, only on $M, \mathfrak{q}$.

If $x_1, \ldots, x_s$ generate $\mathfrak{q}$ the ring $Gr(A)$ is generated by $\bar{x}_1, \ldots, \bar{x}_s$.
So $\ell(M_r/M_{r+1})$ is eventually a polynomial $f(n) \in \mathbb{Q}[n]$ of degree $\leq s-1$.
As $\ell_{n+1} - \ell_n = f(n)$ we have ℓ_n is eventually a polynomial $g(n)$, of degree $\leq s$, giving (ii).
Two stable filtrations $(M_n), (M_n^\dagger)$ are of bounded difference so $g(n+n_0) \geq g^\dagger(n)$, $g^\dagger(n+n_0) \geq g(n)$.
So $\lim(g(n)/g^\dagger(n)) = 1$, so they have the same degree and leading term, giving (iii).

Dimension, Act 2

Notation: $\chi_{\mathfrak{q}}^M(n) := \ell(M/\mathfrak{q}^n M)$ for large n. $\chi_{\mathfrak{q}}(n) := \chi_{\mathfrak{q}}^A(n)$.
Reformulate:

Corollary

$\ell(A/\mathfrak{q}^n)$ is a polynimial $\chi_{\mathfrak{q}}(n)$ for large n, of degree $\leq s$.

Proposition

$\deg \chi_{\mathfrak{q}}(n) = \deg \chi_{\mathfrak{m}}(n)$, *in particular independent on* $\mathfrak{q}$.

$\mathfrak{m} \supset \mathfrak{q} \supset \mathfrak{m}^r$ so $\mathfrak{m}^n \supset \mathfrak{q}^n \supset \mathfrak{m}^{rn}$ so $\chi_{\mathfrak{m}}(n) \leq \chi_{\mathfrak{q}}(n) \leq \chi_{\mathfrak{m}}(nr)$ for large n, so the degrees agree.
Notation $d(A) := \deg \chi_{\mathfrak{q}}(n)$.
Note: $d(A) = d(Gr(A))$.

Inequalities

For $(A, \mathfrak{m})$ noetherian local define $\delta(A) =$ minimum number of generators of a $\mathfrak{m}$-primary ideal. Since $s \geq d(A)$ we have

Proposition

$\delta(A) \geq d(A)$.

Proposition

If $x \in A$ is not a zero divisor for elements of M and $M' = M/xM$ then $\deg \chi_{\mathfrak{q}}^{M'} < \deg \chi_{\mathfrak{q}}^{M}$.
In particular if x is not a zero divisor then $d(A/(x)) \leq d(A) - 1$.

Writing $N = xM$ we have $N \simeq M$. Writing $N_n = N \cap \mathfrak{q}^n M$ we have exact sequence $0 \to N/N_n \to M/\mathfrak{q}^n M \to M'/\mathfrak{q}^n M' \to 0$. So $\ell(N/N_n) - \chi_{\mathfrak{q}}^{M}(n) + \chi_{\mathfrak{q}}^{M'}(n) = 0$ for large n. Now (N_n) is a stable $\mathfrak{q}$-filtration by Artin-Rees, so $\ell(N/N_n)$ is eventually a polynomial with the same leading term as $\chi_{\mathfrak{q}}^{M}(n)$, so $\chi_{\mathfrak{q}}^{M'}(n)$ has lower degree.

Inequalities - continued

Keeping $(A, \mathfrak{m})$ noetherian local, we have

Proposition

$d(A) \geq \dim A$.

We apply induction on $d = d(A)$. If $d = 0$ then $\ell(A/\mathfrak{m}^n)$ is eventually constant so $\mathfrak{m}^n = \mathfrak{m}^{n+1}$. By Nakayama $\mathfrak{m}^n = 0$ so A Artinian and $\dim A = 0$.

If $d > 0$ consider a chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ a chain of primes. Pick $x_1 \in \mathfrak{p}_1$ but $x_1 \notin \mathfrak{p}_0$, and set $A' = A/\mathfrak{p}_0$, an integral domain, with $x' \neq 0$ the image of x. We therefore have $d(A'/(x')) < d(A')$. Also we have $A/\mathfrak{m}^n \to A'/\mathfrak{m}'^n$ so $\ell(A/\mathfrak{m}^n) \geq \ell(A'/\mathfrak{m}'^n)$ so $d(A) \geq d(A')$.

So $d(A'/(x')) \leq d(A) - 1 = d - 1$. We have the primes $\mathfrak{p}_1/(\mathfrak{p}_0, x) \subsetneq \cdots \subsetneq \mathfrak{p}_r/(\mathfrak{p}_0, x)$, so by induction $r - 1 \leq \dim A'/(x') \leq d(A'/(x')) \leq d - 1$, hence $r \leq d$.

Height of a prime

Corollary

If A noetherian local then $\dim A$ *finite.*

Moreover, define the height $ht(\mathfrak{p})$ of $\mathfrak{p}$ to be the length of chain $\mathfrak{p} = \mathfrak{p}_r \supsetneq \cdots \supsetneq \mathfrak{p}_0$.

Corollary

Let A be noetherian. Then $ht(\mathfrak{p})$ is finite. In particular primes in A satisfy the descending chain condition.

Indeed we can pass to the noetherian local ring $A_{\mathfrak{p}}$.
(The depth of a prime ideal is not necessarily finite.)

Dimension, Act 3

Proposition

Let $(A, \mathfrak{m})$ be noetherian local. There is an $\mathfrak{m}$-primary $\mathfrak{q}$ generated by $\dim A$ elements.

This implies

Theorem (The dimension theorem)

If A noetherian local, then $\dim A = \delta(A) = d(A)$.

Existence of good primary ideal - proof

Let $(A, \mathfrak{m})$ be noetherian local. There is an $\mathfrak{m}$-primary $\mathfrak{q}$ generated by $d = \dim A$ elements.

One constructe $(x_1, \ldots, x_d)$ inductively so that if $\mathfrak{p} \supset (x_1, \ldots, x_i)$ then $ht(\mathfrak{p}) \geq i$.
Take $\mathfrak{p}_j$ the minimal primes containing $(x_1, \ldots, x_i)$, having height $i - 1$.
We have $i - 1 < \dim A = ht(\mathfrak{m})$. So $\mathfrak{m} \neq \mathfrak{p}_j$ and so $\mathfrak{m} \neq \cup \mathfrak{p}_j$.
Take $x_i \in \mathfrak{m} \smallsetminus \cup \mathfrak{p}_j$. If $\mathfrak{q}$ contains $(x_1, \ldots, x_i)$, it contains a minimal prime $\mathfrak{p}$ of $(x_1, \ldots, x_i)$.
If $\mathfrak{p} = \mathfrak{p}_j$ then $\mathfrak{q} \supsetneq \mathfrak{p}$ since $x_i \notin \mathfrak{p}$. So $ht(\mathfrak{q}) \geq i$.
If $\mathfrak{p}$ is another, then $ht(\mathfrak{p}) \geq i$ so $ht(\mathfrak{q}) \geq i$ anyway.
It follows that any prime containing $(x_1, \ldots, x_d)$ as height at least d, but since $ht(\mathfrak{m}) = d$ we have that $(x_1, \ldots, x_d)$ is $\mathfrak{m}$-primary.

Consequences

One gets immediately that $\dim(k[x_1,\ldots,x_m])_{(x_1,\ldots,x_m)} = m$.

Corollary

Let $(A,\mathfrak{m})$ be noetherian local. Then $\dim A \leq \dim_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

Corollary

Let A be noetherian. Every prime containing $(x_1,\ldots,x_r)$ has height $\leq r$.

Indeed, the ideal $(x_1,\ldots,x_r)$ becomes $\mathfrak{m}$-primary in $A_\mathfrak{p}$.

Krull's theorem

Theorem (Krulls Hauptidealsatz)

A noetherian, $x \in A$ not a unit and not a zero divisor. Every minimal prime of (x) has height 1.

The height is ≤ 1 by a previous result.
Every element of a prime of height 0 is a zero divisor by a result on primary decompositions.
So the height is 1.

Corollary

If A noetherian local and $x \in \mathfrak{m}$ not a zero divisor, then $\dim A/(x) = \dim A - 1$.

We have seen that $d' := \dim A/(x) \leq \dim A - 1$. If $(\bar{x}_1, \ldots, \bar{x}_{d'})$ is an $\mathfrak{m}'$ primary ideal then $(x_1, \ldots, x_{d'}, x)$ is $\mathfrak{m}$ primary, giving $d' + 1 \geq \dim A$.

More consequences

Corollary

Let $\hat{A}$ be the completion of $(A, \mathfrak{m})$. Then $\dim A = \dim \hat{A}$.

Indeed $A/\mathfrak{m}^n \simeq \hat{A}/\hat{\mathfrak{m}}^n$ so $\chi_{\mathfrak{m}}(A) = \chi_{\hat{\mathfrak{m}}}(\hat{A})$.
We say $(x_1, \ldots x_d)$ a system of parameters if they generate a primary ideal with $d = \dim A$.

Proposition

Assume $(x_1, \ldots x_d)$ a system of parameters with primary $\mathfrak{q}$.
Assume $f(t_1, \ldots, t_d) \in A[t_1, \ldots, t_d]_s$, and $f(x_1, \ldots, x_d) \in \mathfrak{q}^{s+1}$.
Then $f \in \mathfrak{m}[t_1, \ldots, t_d]$.

First consider $\alpha : A/\mathfrak{q}[t_1, \ldots, t_d] \to Gr_{\mathfrak{q}}(A)$ sending $t_i \mapsto x_i + \mathfrak{q}$.
This is surjective since t_i generate $\mathfrak{q}$.
It follows that $f \mod \mathfrak{q}$ is in $\mathrm{Ker}\alpha$. (...continued)

Parameters continued

Assume $(x_1, \dots x_d)$ a system of parameters with primary $\mathfrak{q}$.
Assume $f(t_1, \dots, t_d) \in A[t_1, \dots, t_d]_s$, and $f(x_1, \dots, x_d) \in \mathfrak{q}^{s+1}$.
Then $f \in \mathfrak{m}[t_1, \dots, t_d]$.

$\bar{f}$ is in $\mathrm{Ker}(\alpha : A/\mathfrak{q}[t_1, \dots, t_d] \to Gr_{\mathfrak{q}}(A))$.
Assume by contradiction a coefficient of f is a unit. Then $\bar{f}$ can't be a zero divisor by an old statement you did in homework.

$$\begin{aligned}\dim(Gr_{\mathfrak{q}}(A)) &\leq \dim(A/\mathfrak{q}[t_1, \dots, t_d]/(f)) \\ &= \dim(A/\mathfrak{q}[t_1, \dots, t_d] - 1 \\ &= d - 1\end{aligned}$$

But by the dimension theorem $\dim(Gr_{\mathfrak{q}}(A)) = d$.

Parameters with coefficient field

A coefficient field for $(A, \mathfrak{m})$ is a field $k \subset A$ such that $k \to A/\mathfrak{m}$ an isomorphism.
So $k[[x]]$ has a coefficient field by $\mathbb{Z}_p$ does not.

Corollary

Assume A has a coefficient field k and $(x_1, \ldots x_d)$ parameters. Then $x_1, \ldots x_d$ are algebraically independent.

Assume $f(x_1, \ldots, x_d) = 0$ where f has coefficients in k. Write $f = f_s + h.o.t.$ with $f_s \neq 0$. The proposition applies of f_s, so it has its coefficients on $\mathfrak{m}$, and since the coefficients are in k they are 0, contradiction!

Regular local rings

Theorem

Let $(A, \mathfrak{m})$ *noetherial local,* $\dim A = d, k = A/\mathfrak{m}$*. The following are equivalent:*

(i) $Gr_{\mathfrak{m}}(A) \simeq k[t_1, \ldots, t_d]$,

(ii) $\dim_k \mathfrak{m}/\mathfrak{m}^2 = d$

(iii) $\mathfrak{m}$ *can be generated by* d *elements.*

(i) implies (ii) by definition of grading. Elements x_i generating $\mathfrak{m}/\mathfrak{m}^2$ generate $\mathfrak{m}$ by Nakayama so (ii) implies (iii), and (iii) implies (i) since as in the proof of the proposition $k[t_1, \ldots, t_d] \to Gr_{\mathfrak{m}}(A)$ an isomorphism.

Such rings are called regular local rings.

For instance $k[x_1, \ldots, x_d]_{\mathfrak{m}_0}$ regular, since its gradation is $k[x_1, \ldots, x_d]$.

Domains, regular local rings and completions

Lemma

Let $\mathfrak{a} \subset A$ such that $\cap \mathfrak{a}^n = 0$, and assume $Gr_{\mathfrak{a}} A$ a domain. Then A is a domain.

This evidently implies that a regualr local ring is a domain.
Assume $x, y \in A$ nonzero. Let r maximal so $x \in \mathfrak{a}^r$ and s maxial for $y \in \mathfrak{a}^s$. So $\bar{x} \in Gr_r(A)$ and $\bar{y} \in Gr_s(A)$ are nonzero, so $\bar{x}\bar{y} = \bar{(}xy) \in Gr_{r=s}(A)$ nonzero, so $xy \neq 0$.

Proposition

For A noetherian, $(A, \mathfrak{m})$ regular if and only if $(\hat{A}, \hat{\mathfrak{m}})$ regular.

Indeed $\hat{A}$ noetherian of the same dimension and with isomorphic graded ring!
So $k[\![x_1, \ldots, x_d]\!]$ is regular.

Dimensions of varieties

Assume k algebraically closed. The dimension $\dim V$ of an affine variety V with coordinate ring $A(V) = k[x_1, \ldots, x_n]/\mathfrak{p}$ is the transcendence degree of the fraction field $k(V)$ of $A(V)$.

Theorem

For any V and $\mathfrak{m} \subset A(V)$ we have $\dim V = \dim(A(V)_\mathfrak{m})$.

Lemma

Assume $B \subset A$ domains, B integrally closed, A integral over B. Let $\mathfrak{m} \subset A$ maximal and $\mathfrak{n} = B \cap \mathfrak{m}$. Then $\mathfrak{n}$ maximal and $\dim A_\mathfrak{m} = \dim B_\mathfrak{n}$.

We have seen by going up that $\mathfrak{n}$ is maximal, and any chain of primes $\mathfrak{m} \supsetneq \mathfrak{q}_1 \cdots \supsetneq \mathfrak{q}_d$ in A restricts to distinct primes in B, giving $\dim A_\mathfrak{m} \leq \dim B_\mathfrak{n}$. The opposite follows by going down.

Dimensions of varieties - conclusion

For any V and $\mathfrak{m} \subset A(V)$ we have $\dim V = \dim(A(V)_{\mathfrak{m}})$.

By a previous corollary we have that $\dim V \geq \dim(A(V)_{\mathfrak{m}})$, since a system of parameters is algebraically independent.
By Noether normalization there is a polynomial ring and integral extension $B = k[x_1, \ldots, x_d] \subset A(V)$, with $d = \dim V$. Since B integrally closed the lemma applies so we need to show $d = \dim B_{\mathfrak{n}}$. By weak nullstellensatz we may as well assume $\mathfrak{n}$ is the ideal of the origin, and the local ring has dimension d.