MA 252 notes: Commutative algebra
(Distilled from [Atiyah-MacDonald])

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Basics

- Recall: Modules, module homomorphisms, submodules, quotients, isomorphism theorems.
- $\underline{\mathrm{Hom}}_A(M, N)$ is an A-module.
- $\mathfrak{a}M \subset M$ the submodule generated by products.
- $(N : P)$ an ideal, $(0 : P) = Ann(P)$.
- In general $(N : P) = Ann(P/(N \cap P)) = Ann((N + P/N)$.
- M is a faithful module if $Ann(M) = 0$.
- Any A-module M is naturally a faithful $A/Ann(M)$-module.
- $x \in M \Rightarrow Ax \subset M$ submodule.
- $M = \sum Ax_i \Rightarrow \{x_i\}$ generates M.
- direct sums, direct products of modules.
- $A = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$ as a module $\Leftrightarrow A \simeq \prod A/\mathfrak{b}_i$ (where $\mathfrak{b}_i = \oplus_{i \neq j} \mathfrak{a}_j$).

Finite generation and ideals

M is finitely generated as module $\Leftrightarrow$ M quotient of A^r.

Proposition

M finitely generated, $\mathfrak{a}$ ideal, and $\phi \in End_A(M)$ with $\phi(M) \in \mathfrak{a}M$. Then there is $f \in A[x]$, monic with all other coefficients in $\mathfrak{a}$, such that $f(\phi) = 0$.

Let x_i generate M and $\phi(x_i) = \sum a_{ij}x_j$ (in essence lifting ϕ to A^n). The element $f(t) = \det(tI - B)$ has $f(B) = 0$ so $f(\phi) = 0$. See sleek argument in the book which includes Cayley-Hamilton.

Corollary

If M finitely generated and $\mathfrak{a}M = M$ then there is $x \equiv 1 \mod \mathfrak{a}$ such that $xM = 0$.

Taking $\phi = id$ get $f(id) = 1 + a_1 + \cdots$ with $a_i \in \mathfrak{a}$.

Nakayama

Proposition

If M finitely generated, $\mathfrak{a} \subset \mathfrak{R}(A)$, and $\mathfrak{a}M = M$, then $M = 0$.

We get $xM = 0$ where $x \equiv 1 \mod \mathfrak{R}(A)$. We have seen $x \in A^\times$. So $M = 0$.

Corollary

M finitely generated, $N \subset M$, $\mathfrak{a} \subset \mathfrak{R}(A)$, and $M = \mathfrak{a}M + N$. Then $M = N$.

Apply proposition to M/N: note that $\mathfrak{a}(M/N) = (\mathfrak{a}M + N)/N$.

Theorem (Nakayama)

M finitely generated, $(A, \mathfrak{m})$ local, $\bar{x}_i$ generate $M/\mathfrak{m}M$. Then x_i generate M.

If $N = \sum Ax_i \subset M$ then $M = \mathfrak{m}M + N$ so $N = M$.

Exact sequences

Recall:

- complexes, exact sequences $\to M_{i-1} \to M_i \to M_{i+1} \to$.
- injective and surjective as exact sequences
- Short exact sequences
- Splitting an exact sequence into short exact sequences using $N_i = Im(f_i) = Ker(f_{i+1})$.

Left exactness of $\underline{\mathrm{Hom}}$

Proposition

- *$M' \to M \to M'' \to 0$ is exact if and only if for all N the sequence $0 \to \underline{\mathrm{Hom}}(M'', N) \to \underline{\mathrm{Hom}}(M, N) \to \underline{\mathrm{Hom}}(M', N)$ exact.*
- *$0 \to N' \to N \to N''$ is exact if and only if for all M the sequence $0 \to \underline{\mathrm{Hom}}(M, N') \to \underline{\mathrm{Hom}}(M, N) \to \underline{\mathrm{Hom}}(M, N'')$ exact.*

For instance, if v surjective, $f : M'' \to N$ and $f \circ v = 0$ then $\forall m f(v(m)) = 0$ so $\forall m'' f(m'') = 0$ so $f = 0$ v^* injective. In the other direction take $f : M'' \to M''/v(M)$, so $f \circ v = 0$ so $v^* f = 0$ so $f = 0$ so $v(M) = M''$. Also if $g : M \to N$ such that $g \circ u = 0$ then $g = \bar{g} \circ v$ for well-defined $\bar{g}$ so $g \in Im(v^*)$. In the other direction take $g : M \to M/u(M')$, so $g \circ u = 0$ so $g \in Im(v^*)$ so $g = \bar{g} \circ v$ so $u(M') \supset \mathrm{Ker}(v)$ and equality follows.

Snake lemma

Given commutative diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & N' & \longrightarrow & N & \longrightarrow & N'' & \longrightarrow & 0 \end{array}$$

get long exact sequence

$$0 \to \mathrm{Ker}(f') \to \mathrm{Ker}(f) \to \mathrm{Ker}(f'') \stackrel{d}{\to}$$
$$\mathrm{Coker}(f') \to \mathrm{Coker}(f) \to \mathrm{Coker}(f'') \to 0.$$

see

`https://en.wikipedia.org/wiki/It's_My_Turn_(film)`

Homework: find a source on Ext and Tor and write a 4 page summary of construction and main properties, with examples.

Tensor products

- M, N, P modules, one looks at *A-blinear maps* $M \times N \to P$.
- The universal one is the tensor product $M \times N \to M \otimes_A N$.
- One constructs this as the quotient of the free module $\oplus_{M\times N} A(m,n)$ modulo the bilinear relations.
- It suffices to use just generators of M, N.
- The universal property gives associativity, commutativity, distributivity, unit $A \otimes_A M = M$.
- Tensor product is functorial.
- Can do multilinear tensor of many modules.

Finiteness of vanishing

Usually one does not use a particular construction. The following is an example where the book does. I'd like to see a *nice* proof avoiding this.

Corollary

If $x_i \in M, y_i \in N$ and $\sum x_i \otimes y_i = 0 \in M \otimes_A N$ then there are finitely generated submodules $\{x_i\} \subset M_0 \subset M, \{y_j\} \subset N_0 \subset N$, such that $\sum x_i \otimes y_i = 0 \in M_0 \otimes_A N_0$

The element $\sum(x_i, y_i) \in \oplus_{M\times N} A(m,n)$ lies in the submodule generated by the various bilinear relations. In particular it is the sum of finitely many such bilinear relations, involving finitely many $m_j \in M$ and $n_j \in N$. Let M_0 be the submodule generated by $\{x_i, m_j\}$ and N_0 be the submodule generated by $\{y_i, n_j\}$. Then $\sum(x_i, y_i) \in \oplus_{M_0\times N_0} A(m,n) \subset \oplus_{M\times N} A(m,n)$ and it is a combination of bilinear relations inside $\oplus_{M_0\times N_0} A(m,n)$, as needed.

Restriction and extension of scalars

If $f : A \to B$ a ring homomorphism, and N a B-module, then it inherits the structure of an A-module by $(a, n) \mapsto f(a)n$. Sometime I denote this by ${}_AN$. In particular $B =_A B$ is an A-module. Evidently we have

Proposition

if B is a finitely generated A-module and N a finitely generated B module then ${}_AN$ is a finitely generated A-module.

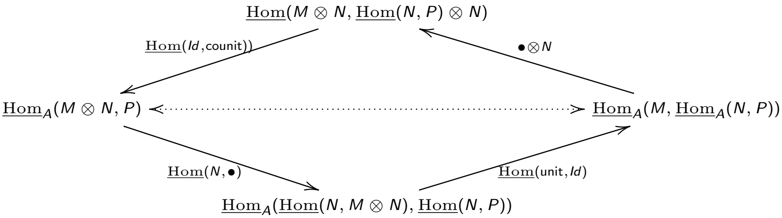
Let M be an A-module. Then $M_B := B \otimes_A M$ is an A-module, with a compatible B-module structure by $b(b' \otimes m) = bb' \otimes m$. (Think of it in terms of multiplication $B \otimes_A B \to B$). Evidently we have

Proposition

If M is a finitely generated A-module them M_B is a finitely generated B-module.

Fundamental isomorphism

- By the universal property, $\underline{\mathrm{Hom}}_A(M \otimes N, P) = Bil_A(M \times N, P)$ as A-modules.
- On the other hand by definition of bilinear maps $Bil_A(M \times N, P) = \underline{\mathrm{Hom}}_A(M, \underline{\mathrm{Hom}}_A(N, P))$ as A-modules.
- So the functors $\bullet \otimes N$ and $\underline{\mathrm{Hom}}(N, \bullet)$ are adjoint: $\underline{\mathrm{Hom}}_A(M \otimes N, P) = \underline{\mathrm{Hom}}_A(M, \underline{\mathrm{Hom}}_A(N, P))$.
- Counit: $\underline{\mathrm{Hom}}(N, P) \otimes N \to P$; Unit: $M \to \underline{\mathrm{Hom}}(N, M \otimes N)$.



Exactness of tensor products

Proposition

If $M' \to M \to M'' \to 0$ is exact and N an A-module then $M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact.

This can be proven by adjunction: since $M' \to M \to M'' \to 0$ is exact then for any P

$$0 \to \underline{\mathrm{Hom}}(M'', \underline{\mathrm{Hom}}(N, P)) \to \underline{\mathrm{Hom}}(M, \underline{\mathrm{Hom}}(N, P)) \to \underline{\mathrm{Hom}}(M', \underline{\mathrm{Hom}}(N, P))$$

is exact, so for any P

$$0 \to \underline{\mathrm{Hom}}(M'' \otimes N, P) \to \underline{\mathrm{Hom}}(M \otimes N, P) \to \underline{\mathrm{Hom}}(M' \otimes N, P)$$

is exact, so $M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact, as needed. (A left adjoint is right exact and a right adjoint is left exact)

Flatness

In general $\otimes N$ is not an exact functor. For instance $\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$.
A module for which $\otimes N$ is exact is called flat.
The notion was introduced as a technical tool in a magnificent paper by Serre (1956). It took over like a wildfire.

Proposition

The following are equivalent: (1) N flat, (2) $N \otimes \bullet$ preserves short exact sequences, (3) $N \otimes \bullet$ preserves injectivity, (4) $N \otimes \bullet$ preserves injectivity for $M' \hookrightarrow M$ with M', M finitely generated.

To prove (4) $\Rightarrow$ (3), if $f : M' \to M$ injective and $\sum m_i' \otimes n_i$ such that $\sum f(m_i') \otimes n_i = 0$. Write $M_0' = \sum Am_i' \subset M'$. There is a finitely generated submodule $M_0 \subset M$ containing $f(m_i')$ such that $\sum f(m_i') \otimes n_i = 0 \in M_0 \otimes N$. Now $f_0 : M_0' \to M_0$ still injective and $\sum f_0(m_i') \otimes n_i = 0$, so by assumption $\sum m_i' \otimes n_i = 0$, as needed