MA 252 notes: Commutative algebra
(Distilled from [Atiyah-MacDonald])

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Primary ideals

Definition

$\mathfrak{q} \subsetneq A$ is primary if whenever $xy \in \mathfrak{q}$ then either $x \in \mathfrak{q}$ or there is n such that $y^n \in \mathfrak{q}$.

This is the same as saying: in the nonzero ring $A/\mathfrak{q}$ every zero-divisor is nilpotent.
The inverse image of a primary ideal is a primary ideal.

Radicals of primary ideals

Lemma

If 0 is primary then $\mathfrak{N}(A)$ is prime.

Note: since $\mathfrak{N}(A)$ is the intersection of all primes, it is the smallest prime in this ring.
If $xy \in \mathfrak{N}(A)$ then $x^m y^m = 0$, so either $x^m = 0$, in which case $x \in \mathfrak{N}(A)$, or $y^{mn} = 0$, in which case $y \in \mathfrak{N}(A)$.
The lemma implies:

Proposition

If $\mathfrak{q}$ is primary then $r(\mathfrak{q})$ is the smallest prime containing $\mathfrak{q}$.

If $\mathfrak{q}$ is primary and $\mathfrak{p} = r(\mathfrak{q})$ then $\mathfrak{q}$ is said to be $\mathfrak{p}$-primary.

Examples

- (p^n) is primary. In general $\mathfrak{p}^n$ are the primary ideals in a PID.
- (x^k, y^m) is (x, y)-primary in $k[x, y]$, even though not the power of a prime.
- $\mathfrak{p} = (x, z)$ is prime in $k[x, y, z]/(xy - z^2)$ but $\mathfrak{p}^2$ is not $\mathfrak{p}$-primary: $xy = z^2 \in \mathfrak{p}^2$ but no power of either y is in $\mathfrak{p}$.

$\mathfrak{m}$-primaries, Intersection

Proposition

If $r(\mathfrak{a}) = \mathfrak{m}$ maximal then $\mathfrak{a}$ is $\mathfrak{m}$-primary.

This implies that $\mathfrak{m}^n$ is $\mathfrak{m}$-primary.
Since $\mathfrak{m}/\mathfrak{a}$ is maximal and the minimal prime, it is the only one. Every element of $A/\mathfrak{a}$ is either a unit (outside $\mathfrak{m}/\mathfrak{a}$) or nilpotent (inside). A zero divisor is not a unit hence nilpotent.

Lemma

If $\mathfrak{q}_i$ are $\mathfrak{p}$-primary then $\mathfrak{q} = \cap_{i=1}^n \mathfrak{q}_i$ is $\mathfrak{p}$-primary.

Indeed $r(\cap_{i=1}^n \mathfrak{q}_i) = \cap_{i=1}^n r(\mathfrak{q}_i) = \mathfrak{p}$. If $xy \in \mathfrak{q}$ but $y \notin \mathfrak{q}$ then $xy \in \mathfrak{q}_i$ but $y \notin \mathfrak{q}_i$ for some i, and then $x \in \mathfrak{p}$ so $x^n \in \mathfrak{q}$.

Colon ideals

Lemma

Suppose $\mathfrak{q}$ is $\mathfrak{p}$-primary, $x \in A$.

(i) *If $x \in \mathfrak{q}$ then $(\mathfrak{q} : x) = 1$.*
(ii) *If $x \notin \mathfrak{q}$ then $(\mathfrak{q} : x)$ is $\mathfrak{p}$-primary, and $r(\mathfrak{q} : x) = \mathfrak{p}$.*
(iii) *If $x \notin \mathfrak{p}$ then $(\mathfrak{q} : x) = \mathfrak{q}$.*

(i) is the definition of an ideal.
(iii) follows since $xy \in \mathfrak{q}$ implies $y \in \mathfrak{q}$.
For (ii), assume $y \in (\mathfrak{q} : x)$ so $xy \in \mathfrak{q}$ but $x \notin \mathfrak{q}$, so $y \in \mathfrak{p}$. Namely $\mathfrak{q} \subset (\mathfrak{q} : x) \subset \mathfrak{p}$. So $r((\mathfrak{q} : x)) = \mathfrak{p}$. Now if $yz \in (\mathfrak{q} : x)$ but $y \notin \mathfrak{p}$ then $xyz \in \mathfrak{q}$ so $xz \in \mathfrak{q}$ so $z \in (\mathfrak{q} : x)$.

Primary decompositions

Definition

A primary decomposition of an ideal $\mathfrak{a}$ is an expression $\mathfrak{a} = \cap_{i=1}^n \mathfrak{q}_i$, where $\mathfrak{q}_i$ are primary.

The decomposition is minimal / irredundant if (i) the radicals $r(\mathfrak{q}_i)$ are distinct and (ii) $\forall i, \mathfrak{q}_i \not\supset \cap_{i \neq j} \mathfrak{q}_j$.

- By a previous lemma we can replace $\mathfrak{p}$-primary $\mathfrak{q}_i$ by their intersection, achieving (i). To get (ii) we can strike out violating $\mathfrak{q}_i$.
- Existence is not always assured.
- Regarding uniqueness, we have to be careful: $(xy, y^2) = (y) \cap (x, y)^2 = (y) \cap (x, y^2)$.
- We need a weaker statement which does hold.
- Geometrically in the example there is "something along the x-axis" and "something embedded at the origin".

Uniqueness of the primes

Theorem

The radicals $\mathfrak{p}_i = r(\mathfrak{q}_i)$ of a minimal decomposition $\mathfrak{a} = \cap_{i=1}^n \mathfrak{q}_i$ are the primes among $\{(\mathfrak{a} : a) | a \in A\}$.

- Indeed, $(\mathfrak{a} : x) = (\cap_{i=1}^n \mathfrak{q}_i : x) = \cap_{i=1}^n (\mathfrak{q}_i : x)$ so $r((\mathfrak{a} : x)) = \cap_{i=1}^n r(\mathfrak{q}_i : x) = \cap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i$..
- If $r(\mathfrak{a} : x)$ is prime, then by an early lemma it coincides with one of the $\mathfrak{p}_j$. On the other hand, minimality allows to choose $x \in \cap_{i \neq j} \mathfrak{q}_j \setminus \mathfrak{q}_i$, so $r(\mathfrak{a} : x) = \mathfrak{p}_i$. ♠
- We call $\mathfrak{p}_i$ the primes associated to $\mathfrak{a}$ (or the associated primes of $\mathfrak{a}$).

Isolated primes

Minimal associated primes are called "isolated", and the primary ideals are "isolated primary components". Others are "embedded", as in the picture!

Proposition

If $\mathfrak{a}$ decomposable, $\mathfrak{p}$ prime and $\mathfrak{p} \supset \mathfrak{a}$ then $\mathfrak{p} \supset \mathfrak{p}_i$ for some isolated $\mathfrak{p}_i$.
The isolated $\mathfrak{p}_i$ are the minimal elements among primes containing $\mathfrak{a}$.

If $\mathfrak{p} \subset \mathfrak{a} = \cap \mathfrak{q}_i$ then $\mathfrak{p} = r(\mathfrak{p}) \supset \cap r(\mathfrak{q}_i) = \cap \mathfrak{p}_i$, so it contais one of them by the ancient lemma. So it contains a minimal one among these.

The union of primes

Proposition

Given minimal decomposition $\mathfrak{a} = \cap \mathfrak{q}_i$, *with* $\mathfrak{p}_i = r(\mathfrak{q}_i)$, *we have* $\cup \mathfrak{p}_i = \{x | (\mathfrak{a} : x) \neq \mathfrak{a}\}$.

In particular, if 0 is decomposable, then the union of associated primes forms the set D of zero-divisors. On the other hand going to $A/\mathfrak{a}$ this is the main case.
We have seen that $D = \cup_{x \neq 0} r(0 : x) = \cup_x \cap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i$, and $\cap_{x \notin \mathfrak{q}_i} \mathfrak{p}_i \subset \mathfrak{p}_i$, so $D \subset \cup \mathfrak{p}_i$. But we have just seen that $\mathfrak{p}_i = r(0, x)$ for some x, so equality holds.

Primary ideals in localization

Proposition

S multiplicative, $\mathfrak{q}$ a $\mathfrak{p}$-primary.

- *If $S \cap \mathfrak{p} \neq \emptyset$ then $\mathfrak{q}[S^{-1}] = A[S^{-1}]$.*
- *If $S \cap \mathfrak{p} = \emptyset$ then $\mathfrak{q}[S^{-1}]$ is $\mathfrak{p}[S^{-1}]$-primary and contracts to $\mathfrak{q}$.*

If $s \in S \cap \mathfrak{p}$ then $s^n \in \mathfrak{q}$ so the unit $s^n/1 \in \mathfrak{q}[S^{-1}]$.
If $S \cap \mathfrak{p} = \emptyset$ and $sa \in \mathfrak{q}$ then $a \in \mathfrak{q}$ by definition of primary. So no element of S is a zero divisor in $A/\mathfrak{q}$, so $\mathfrak{q}$ is a contracted ideal.
Also $r(\mathfrak{q}[S^{-1}]) = r(\mathfrak{q})[S^{-1}] = \mathfrak{p}[S^{-1}]$.
If $(a/s)(b/t) \in \mathfrak{q}[S^{-1}]$ then $abu \in \mathfrak{q}$, and since $u \notin \mathfrak{p}$ we get $ab \in \mathfrak{q}$ so either $a/s \in \mathfrak{q}[S^{-1}]$ or $(b/t)^n \in \mathfrak{q}[S^{-1}]$ as needed.
Finally we have seen that the contraction of a primary is primary.

Primary decomposition in localization

Denote by $S(\mathfrak{a})$ the ideal $\mathfrak{a}^{ec}$.

Proposition

Suppose $\mathfrak{a} = \cap \mathfrak{q}_i$ *minimal, and* S *meets only* $\mathfrak{q}_{m+1}, \ldots, \mathfrak{q}_n$. *Then we have a primary decomposition* $\mathfrak{a}[S^{-1}] = \cap_{i=1}^{m} \mathfrak{q}_i[S^{-1}]$ *and also* $S(\mathfrak{a}) = \cap_{i=1}^{m} \mathfrak{q}_i$

First, localization commutes with finite intersections. Then we have described $\mathfrak{q}_i[S^{-1}]$, which is prime if $i \leq m$ and A otherwise. This is minimal since the $\mathfrak{p}_i[S^{-1}]$ remain distinct in localization. Contracting gives the second statement.

Isolated sets of associated primes and uniqueness

- A set $\Sigma \subset \{\mathfrak{p}_i\}$ is isolated if whenever $\mathfrak{p}' \in \{\mathfrak{p}_i\}$ and $\mathfrak{p}' \subset \mathfrak{p} \in \Sigma$ then $\mathfrak{p}' \in \Sigma$.
- If Σ isolated and $S = A \smallsetminus \cup_\Sigma \mathfrak{p}_i$ then S is multiplicative (it is the intersection of submonoids). Clearly $\mathfrak{p} \in \Sigma \Rightarrow \mathfrak{p} \cap S = \emptyset$. But if $\mathfrak{p} \notin \Sigma$ then $\mathfrak{p} \cap S \neq \emptyset$. This implies

Theorem

$\mathfrak{a} = \cap \mathfrak{q}_i$, $\Sigma \subset \{\mathfrak{p}_i\}$ isolated, then $\cap_{\mathfrak{p}_i \in \Sigma} \mathfrak{q}_i$ is independent of the decomposition.

Corollary

The isolated primary components are independent of the decomposition.