MA 205/206 notes: Crash course on cohomology

Following Liu 5.2-3

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Reminder: sheaves and sections

- We are working with schemes X.
- The structure is governed by sheaves of abelian groups, such as $\mathcal{O}_X$.
- Most important are Sheaves of $\mathcal{O}_X$-modules.
- Particularly useful are Quasi-coherent sheaves of $\mathcal{O}_X$-modules.
- We want to understand their sections.
- For instance: we classified morphisms $X \to \mathbb{P}^n$ through sections of an invertible sheaf.

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Reminder: failure of right-exactness

- Recall the sheaf axiom $0 \to \mathcal{F}(U) \to \prod \mathcal{F}(U_i) \to \prod \mathcal{F}(U_{ij})$.
- If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact then $0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X)$ exact...
- but right exactness fails in general:
- say $Y =$ two points in $X = \mathbb{P}^1$;
- then $0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$, but
- $0 \to 0 \to k \to k^2 \to 0$ is not.

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Comments on how this is resolved

- We'll follow Liu, following SERRE, *Faisceax algébriques cohérents*, to resolve using Čech cohomology. This works for sections of quasi-coherent sheaves.
- Hartshorne follows GROTHENDIECK, *Sur quelques points d'algèbre homologique*[1], to resolve this using derived finctors. This works in the context of left-exact additive functors on abelian categories with enough injective objects.
- An important modern approach uses derived categories (GELFAND–MANIN, WEIBEL), still in the additive realm.
- Homotopy theory has even loftier approaches (model categories, ...)

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Overview of Čech cohomology of sheaves

- Given a covering $\mathcal{U} := \{U_i\}$ of X one defines a complex
  $0 \to \mathcal{F}(X) \to C^0(\mathcal{U}, \mathcal{F}) \stackrel{d_0}{\to} C^1(\mathcal{U}, \mathcal{F}) \stackrel{d_1}{\to} \cdots,$
- where $C^p(\mathcal{U}, \mathcal{F}) := \prod \mathcal{F}(U_{i_0,\ldots,i_p})$.
- For $f \in C^p(\mathcal{U}, \mathcal{F})$ one defines

$$df = \sum_0^{p+1} (-1)^k f_{i_0,\ldots\hat{i}_k,\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}.$$

- Exercise: $d^2 = 0$.
- Define $\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) = \mathrm{Ker}(d_p)/\Im(d_{p-1})$.

Proposition (5.2.6)

$\check{\mathrm{H}}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$.

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Shortcuts

Instead of $C(\mathcal{U}, \mathcal{F})$ one can work instead with alternating chains $C'(\mathcal{U}, \mathcal{F})$ or with the direct summand $C''(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0, \ldots, i_p})$.

Proposition

We have $\check{\mathrm{H}}(\mathcal{U}, \mathcal{F}) = \check{\mathrm{H}}'(\mathcal{U}, \mathcal{F}) = \check{\mathrm{H}}''(\mathcal{U}, \mathcal{F})$.

This is proved by Serre using a homotopy of chain complexes.

Corollary

If $\mathcal{U}$ *contains* n *opens then* $\check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}) = 0$ *for all* $p \geq n$.

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- Example: Consider $X = \mathbb{P}^1_A$ with the open sets $U_i = D_+(T_i)$.
- The Čech complex $C''(\mathcal{U}, \mathcal{F})$ of $\mathcal{O}_X$ is

$$0 \to A \to A[t] \oplus A[t^{-1}] \xrightarrow{d_0} A[t, t^{-1}] \to 0 \cdots .$$

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Refinements

- A refinement $\mathcal{V} = \{V_j\}_{j\in J}$ of $\mathcal{U} = \{U_i\}_{i\in I}$ is a covering $\mathcal{V}$ with a map $\sigma : J \to I$ such that $U_{\sigma(j)} \subset V_j$.
- Get a map $\sigma^* : C(\mathcal{U}, \mathcal{F}) \to C(\mathcal{V}, \mathcal{F})$ compatible with grading and differentials,
- giving $\sigma^* : \check{\mathrm{H}}(\mathcal{U}, \mathcal{F}) \to \check{\mathrm{H}}(\mathcal{V}, \mathcal{F})$.
- Serre shows this homomorphism is independent of σ.
- Two coverings are equivalent if each is a refinement of the other.
- Define
$$\check{\mathrm{H}}^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} \check{\mathrm{H}}^p(\mathcal{U}, \mathcal{F}),$$
the Čech cohomology of $\mathcal{F}$.
- For quasicompact spaces finite covers suffice. For schemes affine covers suffice.
- Read Theorem 5.2.12 on a criterion for $\check{\mathrm{H}}(U, \mathcal{F}) \to \check{\mathrm{H}}(X, \mathcal{F})$ to be an isomorphism (Leray's theorem)

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The long exact sequence

- The construction of $C(\mathcal{U}, \mathcal{F})$ and $\check{\mathrm{H}}(\mathcal{U}, \mathcal{F})$ is functorial in $\mathcal{F}$.
- Hence $\check{\mathrm{H}}(\mathcal{U}, \mathcal{F})$ is functorial in $\mathcal{F}$.
- Suppose now $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact,
- and suppose further
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$$\xrightarrow{\partial} \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}') \to \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}) \to \check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}'') \xrightarrow{\partial}$$

  exat.
- If further this holds for a cofinal family of coverings, then

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The shorter exact sequence

In general we only have

Proposition (2.15)

Suppose $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *exact. Then there is a functorial* $\partial : \mathcal{F}''(X) \to \check{\mathrm{H}}^1(X, \mathcal{F}')$ *with exact sequence*

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In fact in general one uses other cohomology constructions.

Quasicoherent on affines

Lemma

Let X be affine, $\mathcal{F}$ quasicoherent, $\mathcal{U}$ a finite covering by principal opens. Then $\check{\mathrm{H}}^p(\mathcal{U},\mathcal{F}) = 0$ for $p \geq 1$.

- When we proved Proposition 5.1.8, I showed this for you as a Lemma in case $p = 1$. The proof is "the same", with slightly more horrendous indices.
- This boils down to constructing a homotopy using a "partition of unity" $\sum h_i g_i^m = 1$, where $U_i = D(g_i)$.

Theorem (2.18)

Let X be affine, $\mathcal{F}$ quasicoherent. Then $\check{\mathrm{H}}^p(X,\mathcal{F}) = 0$ for $p \geq 1$.

Indeed the family of finite coverings by principal opens is cofinal.

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Quasicoherent cohomology of separated schemes

Theorem (2.18)

Let X be separated, $\mathcal{F}$ quasicoherent, $\mathcal{U}$ affine covering. Then $\check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(X, \mathcal{F})$ is an isomorphism.

- This implies that $\check{H}^p(\mathbb{P}^1_A, \mathcal{O}) = 0$ for all $p > 0$.
- This is proven in the book as a consequence of Leray's acyclicity, which is not proven there.
- One can prove directly using the total complex of a double complex
- One deduces that $\check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{W}, \mathcal{F})$ is an isomorphism.

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The long exact sequence

Corollary

Let X be separated, $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact, then we have a long exact sequence

$$\begin{aligned} 0 &\to \mathcal{F}'(x) \to \mathcal{F}(x) \to \mathcal{F}''(x) \\ &\xrightarrow{\partial} \check{\mathrm{H}}^1(X, \mathcal{F}') \to \check{\mathrm{H}}^1(X, \mathcal{F}) \to \check{\mathrm{H}}^1(X, \mathcal{F}'') \\ &\xrightarrow{\partial} \check{\mathrm{H}}^2(X, \mathcal{F}') \to \check{\mathrm{H}}^2(X, \mathcal{F}) \to \check{\mathrm{H}}^2(X, \mathcal{F}'') \\ &\xrightarrow{\partial} \dots \end{aligned}$$

Serre's criterion

Theorem

Suppose X either notherian or separated and quasicompact. Then the following are equivalent:

(i) *X affine.*

(ii) *$\check{\mathrm{H}}^p(X, \mathcal{F})$ for every quasicoherent $\mathcal{F}$ and $p > 0$.*

(iii) *$\check{\mathrm{H}}^1(X, \mathcal{F})$ for every quasicoherent $\mathcal{F}$.*

- Let $A = \mathcal{O}(X)$. Need to show $\phi : X \to \operatorname{Spec} A$ an isomorphism.
- For $f \in A$ we have $X_f = \phi^{-1} D(f)$ and by an old result $\mathcal{O}_X(X_f) = A[f^{-1}]$.
- If X_f affine then $\phi_{X_f} : X_f \to D(f)$ an isomorphism,
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Serre's criterion, (1) each $x \in X$ lies in an affine X_f

- The closure $\overline{\{x\}}$ is quasicompact, hence has a closed point;
- might as well assume x closed.
- Let $\mathcal{M} = \mathcal{I}_{\{x\}}$. Let $U \ni x$ be an affine neighborhood. Let $J = \mathcal{I}_{X \smallsetminus U}$.
- $0 \to \mathcal{M}\mathcal{J} \to \mathcal{J} \to \mathcal{J}/\mathcal{M}\mathcal{J} \to 0$ is exact.
- The latter is a skyscraper with fiber $k(x)$ at x.
- By assumption $H^1(X, \mathcal{M}\mathcal{J}) = 0$,
- and by the general exact sequence there is $f \in \mathcal{J}$ such that $f(x) \neq 0$.
- Note that $X_f = D_U(f)$ is an affine neighborhood of x.♠

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- Take finitely many f_i so that $X = \cup X_{f_i}$.
- Need to show $A = \cup X_{f_i}$, namely $(f_1, \ldots, f_m) = (1)$.
- Consider $\psi : \mathcal{O}_X^n \to \mathcal{O}_X$, where $\psi(a_1, \ldots, a_n) = \sum a_i f_i$.
- $0 \to \mathrm{Ker}\psi \to \mathcal{O}^n \to \mathcal{O} \to 0$ is an exact sequence of quasicoherent sheaves.
- Since $H^1(X, \mathrm{Ker}\psi) = 0$ we have $A^n \to A$ surjective, as needed!♠

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Vanishing above dimension

Recall: if Y projective over noetherian A and Z closed, then there is homogeneous $f \in \mathcal{I}_Z$ not vanishing at any generic point of $Y \setminus Z$.

Write d for the maximal dimension of a fiber of $X \to \operatorname{Spec} A$.

Proposition

If X quasiprojective over Noetherian A there is a covering of X by $d+1$ affines. In particular $H^p(X, \mathcal{F}) = 0$ for $\mathcal{F}$ quasicoherent and $p > d$.

- Write $Y = \bar{X}$ and $Z = Y \setminus X$.
- Write $Y_1 = V(f), X_1 = X \cap Y_1, Z_1 = Z \cap Z_1$.
- We have by induction d affines covering X_1,
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Relative cohomology: affine case

We say $f : X \to Y$ is quasicompact if preimage of affine open is quasicompact.

Lemma

Let $f : X \to \operatorname{Spec} A$ be separated and quasicompact, $\mathcal{F}$ quasicoherent on X, and M an A-module. Denote $\mathcal{F} \otimes_A M = \mathcal{F} \otimes f^*\tilde{M}$. Then there is a canonical morphism $H^p(X, \mathcal{F}) \otimes_A M \to H^p(X, \mathcal{F} \otimes_A M)$, which is an isomorohism when M is flat.

- Taking a finite affine covering all the intersections are affine.
- One verifies term by term that $C^p(\mathcal{U}, \mathcal{F}) \otimes M = C^p(\mathcal{U}, \mathcal{F} \otimes M)$.
- If $K^\bullet$ is a complex of A-modules there is a canonical map $h^p(K^\bullet) \otimes M \to h^p(K^\bullet \otimes M)$, which is isomorphic if M is flat, as needed.

Flat base change

Corollary

Assume further B is a flat A-algebra, and $\rho : X_B \to X$ the base change. Then $H^p(X, \mathcal{F}) \otimes_A B \simeq H^p(X_B, \rho^*\mathcal{F})$.

One notes that $C(\mathcal{U}_B, \rho^*\mathcal{F}) = C(\mathcal{U}, \mathcal{F} \otimes_A B)$.

Higher direct image

Say $f : X \to Y$ separated and quasicompact, $\mathcal{F}$ quasicoherent on X. For $V \subset Y$ affine open and $p \geq 0$ define $R^p f_* \mathcal{F}(V) := H^p(f^{-1}V, \mathcal{F})$.

Proposition

This is a quasicoherent sheaf on Y.

We call it the p-th higher direct image sheaf.
If $W \subset V$ principal open we have a homomorphism

$$H^p(f^{-1}V, \mathcal{F}) \otimes_{\mathcal{O}(V)} \mathcal{O}(W) \to H^p(f^{-1}W, \mathcal{F}),$$

which is an isomorphism since $\mathcal{O}(W)$ is a flat $\mathcal{O}(V)$ algebra.

Flat sheaves

We say $\mathcal{F}$ is flat at x if $\mathcal{F}_x$ is a flat $\mathcal{O}_{X,x}$-module.
If $f : X \to Y$ we say that $\mathcal{F}$ is flat over Y at x if $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,f(x)}$-module.
We say $\mathcal{F}$ is flat over Y if t is flat over Y at all $x \in X$.

Lemma

Assume $\mathcal{F}$ quasicoherent. Then $\mathcal{F}$ is flat over X if and only if for all affine opens $\mathcal{F}(U)$ is a flat $\mathcal{O}(U)$-module.
If furthermore X locally noetherian and $\mathcal{F}$ coherent, then $\mathcal{F}$ is flat over X if and only if $\mathcal{F}$ is locally free.

A module is flat if and only if all its localizations are. A finite module over a noetherian local ring is flat if and only if it is free.

Flat base change

Proposition

Let $f : X \to Y$ be separated and quasicompact, $\mathcal{F}$ quasicoherent on X and $\mathcal{G}$ quasicoherent on Y. Then there is a canonical homomorphism $(R^p f_*\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G} \to R^p f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G})$ which is an isomorphism whenever $\mathcal{G}$ is flat over Y.

- If $\mathcal{G}$ is flat, this homomorphism is called "flat base change".
- For a morphism $g : Y' \to Y$ with pullback $g' : X' \to X$ and $f' : X' \to Y'$ this gives $g^*(R^p f_*\mathcal{F}) \to R^p f'_*(g'^*\mathcal{F})$, an isomorphism if g is flat.
- To prove let $V \subset Y$ be affine.
- $LHS = H^p(f^{-1}V, \mathcal{F}) \otimes_{\mathcal{O}(V)} \mathcal{G}(V)$,
- $RHS = H^p(f^{-1}V, \mathcal{F} \otimes_{\mathcal{O}(V)} \mathcal{G}(V))$.
- This was done under "affine case".

Vanishing above fiber dimension

Say $f : X \to Y$ a quasiprojective morphism, Y locally noetherian. Write $y = \max_{y \in Y} \dim X_y$.

Proposition

If $\mathcal{F}$ quasicoherent on X then $R^p f_* \mathcal{F} = 0$ whenever $p > r$.

Proof: pass to affines, where it was done.

Cohomology of $\mathcal{O}(n)$

Proposition

Say $B = A[X_0, \ldots, X_d]$ *and* $X = \operatorname{Proj} B$. *Then*

(a) $H^0(X, \mathcal{O}(n)) = B_n$,

(b) $H^i(X, \mathcal{O}(n)) = 0$ *for* $0 < i < d$

(c) $H^d(X, \mathcal{O}(n)) \simeq H^0(X, \mathcal{O}(-n-d-1))^\vee$.

(a) has been proven. (c) is an exercise assigned. (b) can be found in Hartshorne or FAC.

Serre vanishing

Theorem

If A noetherian, X/A projective, $\mathcal{L}$ ample, $\mathcal{F}$ coherent then

- *$H^p(X, \mathcal{F})$ is a finitely generated A-module for all p.*
- *For large n and any $p > 0$ we have $H^p(X, \mathcal{F} \otimes \mathcal{L}(n)) = 0$.*

- Say $\mathcal{L}^k$ is very ample giving an embedding $f : X \to \mathbb{P}^d_A$.
- $H^p(X, \mathcal{F}) = H^p(\mathbb{P}^d_A, f_*\mathcal{F})$ by an exercise you are doing for Friday. So may assume $X = \mathbb{P}^d_A$.
- Proving the result for $\mathcal{F} \otimes \mathcal{L}^j, 0 \leq j < r$ shows that it is enough to take $\mathcal{L} = \mathcal{O}(1)$.

Serre vanishing - completed

- $H^p(X, \mathcal{F})$ is a finitely generated A-module for all p.
- For large n and any $p > 0$ we have $H^p(X, \mathcal{F} \otimes \mathcal{L}(n)) = 0$.

- We know that $H^p(X, \mathcal{F}) = 0$ for $p > d$. Apply descending induction.
- Choose an exact sequence $0 \to \mathcal{G} \to \mathcal{O}(m)^r \to \mathcal{F} \to 0$. We get $H^p(\mathcal{O}(m)^r) \to H^p(\mathcal{F}) \to H^{p+1}(\mathcal{G})$ exact.
- We know the result for $\mathcal{O}(m)^r$ and for H^{p+1}, and it follows for $H^p(\mathcal{F})$

Corollary

Let $X \to Y$ be a projective morphism, Y noetherian, $\mathcal{F}$ coherent. Then $R^p f_* \mathcal{F}$ is coherent.

Serre's criterion for ampleness

Theorem

For a proper morphism $X \to \operatorname{Spec} A$ and invertible $\mathcal{L}$ the following are equivalent:

- $\mathcal{L}$ *is ample on* X
- *for any coherent* $\mathcal{F}$, *for all* p, *for large enough* n *we have* $H^p(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$.
- *For any ideal sheaf* $\mathcal{J}$, *for large enough* n *we have* $H^p(X, \mathcal{J} \otimes \mathcal{L}^n) = 0$.

We proved (i) $\Rightarrow$ (ii) $\to$ (iii).