MA 205 notes: Sheaves of $\mathcal{O}_X$-modules

Following Liu 5.1

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Today

Reminder

- A *scheme* is a (locally) ringed space $(X, \mathcal{O}_X)$ which has an open covering $X = \bigcup \operatorname{Spec} A_\alpha$. A morphism of schemes is a morphism of the corresponding locally ringed spaces.
- You learned a whole lot about schemes without considering sheaves other than $\mathcal{O}_X$.
- You can imagine that other sheaves might be of interest. For instance, the ideal $\mathcal{I}_Y$ of a closed subscheme $Y \subset X$ is naturally a sheaf which holds the key to understanding Y.
- Differential forms must say something about a scheme just like in the theory of manifolds.
- We have come to a point where we have to use them.

Definition

An $\mathcal{O}_X$-module is a sheaf of abelian groups $\mathcal{F}$ with a bilinear map $\mathcal{O}_X \times \mathcal{F} \to \mathcal{F}$ with the usual module axioms. *

Note: you know what the product of sheaves is.

Note: this is the same as an $\mathcal{O}_X$-module in the category of sheaves of abelian groups. So these form a category in the natural way!

Note: One can define direct products, more generally limits, of sheaves of $\mathcal{O}_X$-modules in the obvious manner. Direct sums also work.

Note: The tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$. Same care is needed with colimits.

Note: $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is the sheaf given by
$U \mapsto \underline{\mathrm{Hom}}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$.

Global generation

$\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module. We have an $\mathcal{O}_X(X)$-module homomorphism $\mathcal{F}(X) \to \mathcal{F}_x$.

Definition

$\mathcal{F}$ is globally generated* if the image of $\mathcal{F}(X) \to \mathcal{F}_x$ generates $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module. In other words, $\mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_{X,x} \to \mathcal{F}_x$ is surjective.

For instance $\mathcal{O}_X$ is globally generated, any $\oplus_I \mathcal{O}_X$ or its quotient is also.

Lemma (5.1.3)

$\mathcal{F}$ is globally generated if and only if there is an epimorphism $\oplus_I \mathcal{O}_X \to \mathcal{F}$.

Indeed if $\mathcal{F}$ is globally generated the homomorphism $\oplus_{\mathcal{F}(X)} \mathcal{O}_X \to \mathcal{F}$ sending the basis element corresponding to s to $s|_U$ is surjective. The other direction is the remark above.

Quasi-coherence

Definition

A sheaf of $\mathcal{O}_X$-modules is quasi-coherent if every $x \in X$ has an open neighborhood $x \in U \subset X$ and an exact sequence $\oplus_J \mathcal{O}_X|_U \to \oplus_I \mathcal{O}_X|_U \to \mathcal{F}|_U \to 0$.

This generality is important for instance in complex analysis. In algebraic geometry there is a wonderful coincidence.*
For a module M over a ring A we define a natural sheaf $\widetilde{M}$. The main result is

Theorem

A sheaf $\mathcal{F}$ is quasicoherent if and only if for every affine open $U = \operatorname{Spec} A \subset X$ there is an A-module M such that $\mathcal{F}|_U \simeq \widetilde{M}$.. We get an equivalence A-mod $\simeq QCoh(\operatorname{Spec} A)$.*

Sheaves associated to modules

- Let $X = \operatorname{Spec} A$ and M an A-module. We define a sheaf $\widetilde{M}$ by specifying it on principal opens: $\widetilde{M}(D(f)) := M[f^{-1}]$.
- One needs to check that this satisfies the $\mathcal{B}$-sheaf axiom. This turns out to require exactly the same proof as for $\mathcal{O}_X$, using the "partition of unity" $\sum a_i f_i^\ell = 1$ whenever $\cup D(f_i) = X$.*
- Consequently $\widetilde{M}_\mathfrak{p} = M_\mathfrak{p}$ and $\widetilde{M}$ is an $\mathcal{O}_X$-module.
- One has a functor $M \mapsto \widetilde{M}$, which respects direct sums since localization does: $\widetilde{\oplus M_i} = \oplus \widetilde{M}_i$.

Lemma

$M \to N \to L$ exact if and only if $\widetilde{M} \to \widetilde{N} \to \widetilde{L}$ exact.

Indeed $M \to N \to L$ exact if and only if $M_\mathfrak{p} \to N_\mathfrak{p} \to L_\mathfrak{p}$ exact $\forall \mathfrak{p}$, if and only if $\widetilde{M}_\mathfrak{p} \to \widetilde{N}_\mathfrak{p} \to \widetilde{L}_\mathfrak{p}$ exact $\forall \mathfrak{p}$, if and only if $\widetilde{M} \to \widetilde{N} \to \widetilde{L}$ exact.

So both $M \mapsto \widetilde{M}$ and $\widetilde{M} \mapsto \widetilde{M}(X) = M$ are exact functors!

$\widetilde{M}$ is quasicoherent, and a converse

Proposition

$\widetilde{M}$ is quasicoherent

Take a presentation $\oplus_J A \to \oplus_I A \to M \to 0$. By compatibility with direct sums and exactness it induces a presentation $\oplus_J \mathcal{O}_X \to \oplus_I \mathcal{O}_X \to \widetilde{M} \to 0$, as needed.♠

Note: For any sheaf of $\mathcal{O}_X$-modules on an affine X there is a canonical homomorphism $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$.

Proposition

Suppose X affine and $\oplus_J \mathcal{O}_X \to \oplus_I \mathcal{O}_X \xrightarrow{\alpha} \mathcal{F} \to 0$ a presentation. Then $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$ is an isomorphism.

Write $M = Im(\alpha(X))$. So $\oplus_J A \to \oplus_I A \to M \to 0$ is exact, and so $\oplus_J \mathcal{O}_X \to \oplus_I \mathcal{O}_X \to \widetilde{M} \to 0$ exact, hence $\widetilde{M} \to \mathcal{F}$ an isomorphism, and $M = \mathcal{F}(X)$, as needed.♠

Proof of the theorem

The proposition implies that $\mathcal{F}$ is quasicoherent if and only if there is some covering by $U_i = \operatorname{Spec} A_i$ with $\mathcal{F}_{U_i} = \widetilde{M}_i$.

Proposition (1.6)**

Assume X is either Noetherian or separated and quasicompact, $\mathcal{F}$ quasi-coherent, and $f \in \mathcal{O}_X(X)$. Then $\mathcal{F}(X)[f^{-1}] \to \mathcal{F}(X_f)$ is an isomorphism.*

- Given the proposition, take any affine open $U \subset X$, for which the proposition applies;
- hence $\mathcal{F}(U)[f^{-1}] \to \mathcal{F}(D(f))$ for any $f \in \mathcal{O}_X(U)$.
- Since these form a basis $\widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$ is an isomorphism, and the theorem follows.♠

Proof of the proposition

- To prove the proposition cover $X = \cup_{\text{finite}} U_i$ where $\mathcal{F}_{U_i} = \widetilde{\mathcal{F}(U_i)}$.
- Write $V_i = X_f \cap U_i = D(f|_{U_i})$, which are also affine, so $\mathcal{F}(U_i)[f^{-1}] \to \mathcal{F}(V_i)$ is an isomorphism.
- This means that in the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{F}(X)[f^{-1}] & \longrightarrow & \oplus\mathcal{F}(U_i)[f^{-1}] & \longrightarrow & \oplus\mathcal{F}(U_i \cap U_j)[f^{-1}] \\ & \downarrow & & \downarrow \beta & & \downarrow \\ 0 \longrightarrow & \mathcal{F}(X_f) & \longrightarrow & \oplus\mathcal{F}(V_i) & \longrightarrow & \oplus\mathcal{F}(V_i \cap V_j) \end{array}$$

  the arrow β is an isomorphism.
- It follows that $\mathcal{F}(X)[f^{-1}] \to \mathcal{F}(X_f)$ is injective.
- The assumption propagates from X to opens such as $U_i \cap U_j$, so the right arrow is injective too;
- by the Snake Lemma the left arrow is surjective, hence an isomorphism ♠

A surprisingly useful result (and vanishing of $\check{H}^1$)

Proposition

Suppose X affine and $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ an exact sequence of $\mathcal{O}_X$-modules with $\mathcal{F}$ quasicoherent. Then $0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X) \to 0$ is exact.

- We need to take $s \in \mathcal{H}(X)$ and lift it to $\mathcal{G}(X)$.
- By definition of surjectivity we can lift to $t_i \in \mathcal{G}(U_i)$ with $U_i = D(f_i)$ principal opens.
- The difference $t_j - t_i \in \mathcal{G}(U_{ij})$ is $v_{ij} \in \mathcal{F}(U_{ij})$, and these satisfy the cocycle condition $v_{ij}|_{U_{ijk}} - v_{ik}|_{U_{ijk}} + v_{jk}|_{U_{ijk}} = 0$.

Lemma

There are $w_i \in \mathcal{F}(U_i)$ so that $v_{ij} = w_j - w_i$.

- With the lemma the elements $t_i' := t_i - w_i \in \mathcal{G}(U_i)$ have the property $t_i' - t_j' = 0$, hence define a section $t' \in \mathcal{G}(X)$ mapping to s, implying the proposition.

Proof that $\check{H}^1(\text{quasicoherent}) = 0$ on affine

Lemma

There are $w_i \in \mathcal{F}(U_i)$ so that $v_{ij} = w_j - w_i$. *

- Writing $\mathcal{F} = \widetilde{M}$ let $v_{ij} = m_{ij}/(f_i f_j)^r$, with $m_{ij} \in M$, for sufficiently large r good for all ij.
- The cocycle condition means that for all a, i, j we have $m_{ai} f_j^r - m_{aj} f_i^r + m_{ij} f_a^r = 0 \in M[(f_a f_i f_j)^{-1}]$, so for large ℓ we have $f_a^\ell(m_{ai} f_j^r - m_{aj} f_i^r + m_{ij} f_a^r) = 0 \in M[(f_i f_j)^{-1}]$
- Write $\sum h_a f_a^{\ell+r} = 1$.
- Define $w_i = \sum_a h_a f_a^\ell (m_{ai}/f_i^r) \in M[f_i^{-1}]$. Now*

$$(w_j - w_i)|_{U_{ij}} = \sum_a h_a f_a^\ell (m_{aj} f_i^r - m_{ai} f_j^r)/(f_i f_j)^r$$
$$= \sum_a h_a f_a^\ell (m_{ij} f_a^r)/(f_i f_j)^r = m_{ij}/(f_i f_j)^r = v_{ij}$$

as needed.♠

Finite generation and coherence

Definition

$\mathcal{F}$ is (locally) finitely generated if for all $x \in X$ there is a neighborhood $x \in U \subset X$ and epimorphism $\mathcal{O}_U^m \to \mathcal{F}|_U$. $\mathcal{F}$ is coherent if further for any* $\alpha : \mathcal{O}_U^n \to \mathcal{F}|_U$ also $\ker \alpha$ is finitely generated.

This is absolutely crucial for complex analytic spaces, but again in algebraic geometry it is well-behaved, at least for locally noetherian schemes.

Proposition

Let X be a locally noetherian scheme. Then a quasi-coherent $\mathcal{F}$ is coherent if and only if it is locally finitely generated, if and only if $\mathcal{F}(U)$ is finitely generated over $\mathcal{O}_X(U)$ for every affine open U.

Proof of proposition

- Coherent implies locally finitely generated by definition.
- If $\mathcal{F}$ is locally finitely generated and U affine, we can find a finite covering $U = \cup U_i$ by principal opens and epimorphisms $\mathcal{O}_{U_i}^{m_i} \to \mathcal{F}|_{U_i}$.
- By quasi-coherence we have $\mathcal{O}_{U_i}^{m_i}(U_i) \to \mathcal{F}(U_i)$ surjective, and $\mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(U_i) \to \mathcal{F}(U_i)$ an isomorphism.
- Find a finitely generated submodule $M \subset \mathcal{F}(U)$ such that $M \otimes_{\mathcal{O}(U)} \mathcal{O}(U_i) \to \mathcal{F}(U_i)$ surjective for all i.
- Now $\widetilde{M} \to \mathcal{F}|_U$ surjective so $\mathcal{F}(U)$ is finitely generated.
- We may take $M = \mathcal{O}(U)^m$.
- Now let $\mathcal{O}^n \to \mathcal{F}$ be a homomorphism on some affine. It is determined by $A^n \to M$. Since A noetherian the kernel is also finitely generated, as needed.

Basic properties

- Quasi-coherence is preserved by direct sums. Local finite generation preserved by finite direct sums.
- If $\mathcal{F}, \mathcal{G}$ are quasi-coherent then $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is quasi-coherent, with module $\mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{G}(U)$ on any affine open.
- The kernel and cokernel of a homomorphism of quasi-coherent sheaves is quasi-coherent. Same for coherent on a locally Noetherian scheme.
- An extension of quasi-coherent sheaves is quasi-coherent. Same for coherent on a locally Noetherian scheme.

Indeed the useful result gives an exact sequence $0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U) \to 0$, so a diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \widetilde{\mathcal{F}(U)} & \to & \widetilde{\mathcal{G}(U)} & \to & \widetilde{\mathcal{H}(U)} & \to & 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow \simeq & & \\ 0 & \to & \mathcal{F} & \to & \mathcal{G} & \to & \mathcal{H} & \to & 0 \end{array}$$

Pull back and push forward

- Given $f: X \to Y$ the structure arrow $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ is equivalent to $f^\#: f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.
- Define $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$.*
- Note $f^*\mathcal{O}_Y = \mathcal{O}_X$ and f^* commutes with direct sums.

Propoerties:

- $f^*\mathcal{G}_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$.
- If $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A, \mathcal{G} = \widetilde{M}$ affine then $f^*\mathcal{G} = \widetilde{M \otimes_A B}$.
- If $\mathcal{G}$ quasi-coherent then f^*G quasi-coherent.

The first follows from $f^{-1}\mathcal{G}_x = \mathcal{G}_{f(x)}$.
Fix presentation $\oplus_I A \to \oplus_J A \to M \to 0$ giving presentation $\oplus_I \mathcal{O}_Y \to \oplus_J \mathcal{O}_Y \to \mathcal{G} \to 0$. Direct sums and right exactness give presentation $\oplus_I \mathcal{O}_X \to \oplus_J \mathcal{O}_X \to f^*\mathcal{G} \to 0$. On the other hand we also have presentation $\widetilde{\oplus_I B} \to \oplus_J B \to M \otimes_A B \to 0$ giving $\widetilde{\oplus_I \mathcal{O}_X} \to \oplus_J \mathcal{O}_X \to \widetilde{M \otimes_A B} \to 0$. This gives an isomorphism $M \otimes_A B \to f^*\mathcal{G}$.

Properties of pushforward

- Assume either X noetherian or f separated and quasi-compact. If $\mathcal{F}$ quasi-coherent then $f_*\mathcal{F}$ quasi-coherent.
- If f finite and $\mathcal{F}$ quasi-coherent and L.F.G then $f_*\mathcal{F}$ quasi-coherent and L.F.G.

We may assume Y affine, so X either noetherian or separated and quasi-compact.

- We want to show $f_*\mathcal{F} = \widetilde{\mathcal{F}(X)}$. Enough to evaluate $f_*\mathcal{F}(D(g))$. Write $g' = f^*g$. $f_*\mathcal{F}(D(g)) = \mathcal{F}(f^{-1}D(g)) = \mathcal{F}(X_{g'}) \simeq \mathcal{F}(X)[g'^{-1}] = \mathcal{F}(X) \otimes_A A[g^{-1}]$, as needed.
- f finite implies affine, so separated and quasi-compact, so $f_*\mathcal{F} = \widetilde{\mathcal{F}(X)}$. Now $\mathcal{F}(X)$ is finitely generated over $\mathcal{O}_X(X)$ so finitely generated over $\mathcal{O}_Y(Y)$.

Proposition

The correspondence $\{Z \subset X\} \leftrightarrow \{\mathrm{Ker}\, i^{\#}\}$ *is an order-reversing 1-1 correspondence between closed subschemes and quasi-coherent ideal sheaves.*

We may assume $X = \operatorname{Spec} A$ affine. There is already a correspondence between sheaves of ideals and closed ringed subspaces.

If $Z = V(I)$ is a closed subscheme then it was shown that $\operatorname{Ker} i^{\#}(D(g)) = I[g^{-1}]$ so $\operatorname{Ker} i^{\#} = \widetilde{I}$ is quasi-coherent. If $\widetilde{I} := \mathcal{I} \subset \mathcal{O}_X$ a quasi-coherent ideal sheaf then it was shown that the ringed subspace $V(\mathcal{I})$ is the closed subscheme $\operatorname{Spec} A/I$.

Sheaf associated to a graded module

- Let $B = \oplus_{n\geq 0} B_n$ be a graded ring, $M = \oplus_{n\in\mathbb{Z}} M_n$ a graded module.
- For a homogeneous $f \in B_+$ one defined an affine open $D_+(f) \subset X := \operatorname{Proj} B$ with ring $B_{(f)} = B[f^{-1}]_0$, the elements of degree 0 in the $\mathbb{Z}$-graded ring $B[f^{-1}]$.
- Define $M_{(f)} = M[f^{-1}]_0$,* the elements of degree 0 in the graded $B[f^{-1}]$-module $M[f^{-1}]$, namely $M_{(f)} = \{mf^{-d} | m \in M_{d\deg f}, d \in \mathbb{N}\}$.

Proposition

There exists a unique quasi-coherent sheaf $\widetilde{M}$ on X such that $\widetilde{M}|_{D_+(f)} = \widetilde{M_{(f)}}$. For $\mathfrak{p} \in \operatorname{Proj} B$ we have $\widetilde{M}_\mathfrak{p} = (M_\mathfrak{p})_0$.

For this note that $M_{(fg)} = M_{(f)}[(g^{\deg f}/f^{\deg g})^{-1}]$ or repeat the argument of $\mathcal{O}_X$.

This is not an equivalence.

The twisting sheaf and twists of sheaves

- Define a graded module $B(n)$ by $B(n)_d = B_{n+d}$. Denote $\mathcal{O}_X(n) := \widetilde{B(n)}$.
- If $f \in B_1$ then $B(n)_{(f)} = f^n B_{(f)}$. So $\mathcal{O}_X(n)|_{D_+(f)} = (\mathcal{O}_X|_{D_+(f)})f^n$.
- We have $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(n+m)$. The sheaf $\mathcal{O}_X(1)$ is called the twisting sheaf.
- For a scheme Y one has a canonical morphism $f : \mathbb{P}^d_Y \to \mathbb{P}^d_{\mathbb{Z}}$. One defines $\mathcal{O}_{\mathbb{P}^d_Y}(n) = f^*\mathcal{O}_{\mathbb{P}^d_{\mathbb{Z}}}(n)$.
- It is an exercise (5.1.20) to show that when $Y = \operatorname{Spec} A$ affine then $\mathcal{O}_{\mathbb{P}^d_Y}(n)$ coincides with $\widetilde{A[T_0, \ldots, T_d](1)}$.

Definition

Let $X = \mathbb{P}^n_A$, let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The n-th twist of $\mathcal{F}$ is $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.
For an immersion* $Y \overset{\iota}{\hookrightarrow} X$ write $\mathcal{O}_Y(1) = \iota^*\mathcal{O}_X(1)$ and for a quasi-coherent $\mathcal{F}$ write $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n)$.

Proposition (1.22)

Let $B = A[T_0, \ldots, T_d], d \geq 1*$ *and* $X = \operatorname{Proj} B$. *Then* $(\mathcal{O}_X(n))(X) = B_n$, *so* $\oplus_{\mathbb{Z}}(\mathcal{O}_X(n))(X) = B$.

- A section of $(\mathcal{O}_X(n))(X)$ restricts to sections of $(\mathcal{O}_X(n))(D_+(T_i))$ which agree on intersections.
- These are submodules of homogeneous elements of $A[T_0, \ldots, T_d, T_0^{-1}, \ldots, T_n^{-1}]$ where only T_i can be in the denominator.
- Since $d > 0$ this means that no T_i can be in the denominator, giving a homogeneous polynomial of degree n.

Note that if $d = 0$ one gets $\oplus_{\mathbb{Z}}(\mathcal{O}_X(n))(X) = B[T_0^{-1}]$

Invertible sheaves

- We say that an $\mathcal{O}_X$-module $\mathcal{L}$ is invertible if every $x \in X$ has a neighborhood $x \in U \subset X$ and an isomorphism $\mathcal{L}|_U \simeq \mathcal{O}_X|_U$.
- Since $\mathbb{P}^n_Y$ is covered by $D_+(x_0)$ which have degree 1, we see that $\mathcal{O}_{\mathbb{P}^d_Y}(n)$ is invertible.
- If $\mathcal{L}$ invertible and $s \in \mathcal{L}(X)$ one defines an open $X_s = \{x \in X : \mathcal{L}_x = \mathcal{O}_{X,x} \cdot s_x\} \subset X$.
- It generalizes X_f when $f \in \mathcal{O}_X(X)$.

Proposition (* 1.25)

Let X be either noetherian or separated and quasicompact, $\mathcal{L}$ invertible, $\mathcal{F}$ quasi-coherent. Fix $s \in \mathcal{L}(X)$.

(1) *Let $f \in \mathcal{F}(X)$ with $f|_{X_s} = 0$. Then there is $n \geq 0$ such that $fs^n = 0 \in \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$.*

(2) *Let $g \in \mathcal{F}(X_s)$, Then there is $n \geq 0$ and $f \in (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)(X)$ such that $f|_{X_s} = gs^n$.*

Proof of the proposition, (1)

Let X_i be finitely many opens covering X with isomorphisms $\mathcal{L}_i \stackrel{\phi_i}{\simeq} \mathcal{O}_{X_i}$, and $t_i = \phi_i(s)$.

Let $f \in \mathcal{F}(X)$ with $f|_{X_s} = 0$. Then there is $n \geq 0$ such that $fs^n = 0 \in \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n$.

- As $X_i \cap X_s = (X_i)_{t_i}$ we can use Proposition 1.6.
- So there is n such that $fs^n|_{X_i} = ft_i^n = 0$, for all i.
- By the sheaf axiom $fs^n = 0$ ♠

Proof of the proposition, (2)

Let $g \in \mathcal{F}(X_s)$, Then there is $n \geq 0$ and $f \in (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)(X)$ such that $f|_{X_s} = gs^n$.

- By 1.6 there is k and $h_i \in (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^n)(X_i)$ such that $gs^k|_{(X_i)_s} = gt_i^k = h_i|_{(X_i)_s}$, for all i.
- Note that $(h_i - h_j)|_{(X_{ij})_s} = 0$, and the assumption propagates to the opens X_{ij}.
- So by (1) there is m so that $(h_i s^m - h_j s^m)|_{X_{ij}} = 0$ for all i, j.
- By the sheaf axiom there is f so that $f|_{X_i} = h_i s^m$ as needed.♠

Serre's eventual global generation

For an immersion $Y \stackrel{\iota}{\hookrightarrow} \mathbb{P}^d_A$ we wrote $\mathcal{O}_Y(1) = \iota^*\mathcal{O}_{\mathbb{P}^d_A}(1)$ and for a quasi-coherent $\mathcal{F}$ we wrote $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(n)$.

Theorem

Say X projective over A and $\mathcal{F}$ locally finitely generated quasi-coherent. There exists n_0 so that for all $n \geq n_0$ the sheaf $\mathcal{F}(n)$ is globally generated.

- By assumption there is $\iota : X \hookrightarrow \mathbb{P}^d_A$.
- There is a nice exercise (5.1.6) showing $(\iota_*\mathcal{F})(n) = \iota_*(\mathcal{F}(n))$.
- Since $\iota_*(\mathcal{F}(n))(\mathbb{P}^d_A) = (\mathcal{F}(n))(X)$, and since $\iota_*(\mathcal{F}(n))_{f(x)} = (\mathcal{F}(n))_x$ we may and do replace X by $\mathbb{P}^d_A$.
- For each i we have $\mathcal{F}(D_+(T_i))$, and thus $\mathcal{F}|_{D_+(T_i)}$, generated by finitely many s_{ij}. Thus $\mathcal{F}(n)|_{D_+(T_i)}$ is generated by $s_{ij}T_i^n$.
- By the proposition there is n such that $s_{ij}T_i^n = u_{ij}|_{D_+(T_i)}$, with $u_{ij} \in (\mathcal{F}(n))(X)$, for all i, j, as needed. ♠

Corollary

Again X projective over A and $\mathcal{F}$ locally finitely generated quasi-coherent. There is m and an epimorphism $\mathcal{O}_X^r(m) \to \mathcal{F}$.

- Indeed for any n in the theorem take an epimorphism $\mathcal{O}_X^r \to \mathcal{F}(n)$.
- Taking $\otimes_{\mathcal{O}_X} \mathcal{O}_X(-n)$ get an epimorphism $\mathcal{O}_X^r(-n) \to \mathcal{F}$,
- so $m = -n$ works.

- A quasi-coherent sheaf $\mathcal{F}$ on $\operatorname{Spec} A$ satisfies $\mathcal{F} = \mathcal{F}\widetilde{(\operatorname{Spec} A)}$.
- Suppose instead $X = \operatorname{Proj} B$ with $B = A[T_0, \ldots, T_d]$. The A-module $\Gamma_\bullet(\mathcal{F}) := \oplus_{n\geq 0}\Gamma(X, \mathcal{F}(n))$ is a graded B-module via $\Gamma(X, \mathcal{O}_X(n)) \otimes_Z \Gamma(X, \mathcal{F}(m)) \to \Gamma(X, \mathcal{F}(n+m))$.

Lemma

we have an isomorphism $\widetilde{\Gamma_\bullet(\mathcal{F})} \to \mathcal{F}$.

- Note that in general $M \to \Gamma_\bullet(\widetilde{M})$ is not an isomorphism.*
- Let $T = T_0$ and $U = D_+(T_0)$. By affine case it suffices to show that $\Gamma_\bullet(\mathcal{F})_{(T)} \to \mathcal{F}(U)$ is an isomorphism.**
- Let $T^{-n}t \in \Gamma_\bullet(\mathcal{F})_{(T)}$, with $t \in \Gamma(\mathcal{F}(n))$.
- So $t|_U \in \Gamma(U, \mathcal{F}(n)) = T^n\Gamma(U, \mathcal{F})$, and $t|_U = T^n s$ for unique $s \in \mathcal{F}(U)$.
- This is surjective by Proposition 1.25 (2) and injective by Proposition 1.25 (1).

Closed subschemes of projective space

Proposition

*With A, B, X as above, let $Z \subset X$ be closed. Then $Z = \operatorname{Proj} B/I$ for some homogeneous ideal $I \subset B$. In particular any projective A-scheme is of the form $\operatorname{Proj} C$.**

- Let $\mathcal{I}$ be the quasi-coherent sheaf of ideals defining Z and $I = \Gamma_\bullet(\mathcal{I})$.
- $\mathcal{I}(n) \subset \mathcal{O}_X(n)$ because $\mathcal{O}(n)$ is locally free hence flat.
- So by Proposition 1.22, $I \subset \Gamma_\bullet(\mathcal{O}_X) = B$.
- By the Lemma $\widetilde{I} = \mathcal{I}$. So $\mathcal{I}_{V(I)} = \widetilde{I} = \mathcal{I}$ and $V(I) = Z$.*

Maps to projective space

For an invertible $\mathcal{L}$ and section $s \in \mathcal{L}(X)$ one has an isomorphism $\mathcal{O}_{X_s} \simeq \mathcal{L}_{X_s}$ via $1 \mapsto s$. The inverse maps $t \mapsto t/s$.

Proposition (5.1.31)

Let Z/A be a scheme and $X = \operatorname{Proj} A[T_0, \ldots, T_d]$.

1) *If $f : Z \to X$ an A-morphism then $f^*\mathcal{O}_X(1)$ is generated by the $d+1$ sections f^*T_i.*
2) *If $\mathcal{L}$ an invertible sheaf on Z generated by sections $s_0, \ldots, s_d$ there is a unique morphism $f : Z \to X$ with $s_i = f^*T_i$.*

1) Note T_i generate $\mathcal{O}_X(1)$, with epimorphism $\mathcal{O}_X^{d=1} \to \mathcal{O}_X(1)$ giving epimorphism $\mathcal{O}_Z^{d=1} \to f^*\mathcal{O}_Z(1)$ by right-exactness of $\otimes$.
2) Define $Z_{s_i} \to D_+(T_i)$ via the A-algebra homomorphism $T_j/T_i \mapsto s_j/s_i \in \mathcal{O}_Z(Z_{s_i})$.
   These glue, and give an isomorphism $\mathcal{L} = f^*\mathcal{O}_X(1)$.

Maps to projective space: the fundamental question

Examples: power maps, Veronese, Segre.

Note: the morphism f is not changed by rescaling s_i simultaneously.

Note: An invertible linear transformation on s_i results in composing with the corresponding projective linear transformation on $\mathbb{P}^d$.

The fundamental question of projective geometry is: what are the possible ways to map Z to projective space?

The discussion says that this is equivalent to: what are the invertible sheaves with finitely many generating sections (up to chosen equivalence)?

Definition

Pic(X) = set of isomorphism classes of invertible sheaves, with group structure given by $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ and $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$.

Sections up to rescaling will be characterized in terms of linear systems of Cartier divisors.

Very ample and ample sheaves

Definition

A sheaf of the form $\iota^*\mathcal{O}_{\mathbb{P}^d_A}(1)$, with $\iota : X \hookrightarrow \mathbb{P}^d_A$ an immersion, is very ample over A.

An invertible sheaf $\mathcal{L}$ is ample if for every finitely generated quasi-coherent $\mathcal{F}$ there is n_0 such that $\mathcal{F} \otimes \mathcal{L}^n$ is globally generated for all $n \geq n_0$.

Theorem

Say $f : X \to \operatorname{Spec} A$ is of finite type, and either X noetherian or f separated. If $\mathcal{L}$ is ample on X there is $m \geq 1$ such that $\mathcal{L}^m$ is very ample.

Proof of the theorem: good affine neighborhoods

Lemma

Say either X noetherian or f separated, and $\mathcal{L}$ is ample on X. Each $x \in X$ has an affine neighborhood of the form $X_s, s \in \mathcal{L}^n(X)$.

- Let U be an affine neighborhood of x on which $\mathcal{L}|_U \simeq \mathcal{O}_X|_U$, and $\mathcal{J} = \mathcal{I}(X \setminus U)$.
- The inclusion $\mathcal{J} \subset \mathcal{O}_U$ gives $\mathcal{J} \otimes \mathcal{L}^n = \mathcal{J}\mathcal{L}^n \subset \mathcal{L}^n$, since $\mathcal{L}^n$ locally free.
- By ampleness there is a section $s \in (\mathcal{J}\mathcal{L}^n)(X)$ generating $(\mathcal{J}\mathcal{L}^n)_x = \mathcal{L}^n_x$, so $x \in X_s \subset U$.
- The isomorphism $\mathcal{L}|_U \simeq \mathcal{O}_X|_U$ carries s to $f \in \mathcal{O}_X(U)$, so $X_s = D_H(f)$ is affine.

Proof of the theorem: lifting coordinates to sections and embedding

- Say $\mathcal{O}_X(X_s) = A[f_1, \ldots, f_k]$. For some r we have that $s^r f_j$ lifts to $s_j \in \mathcal{L}^{nr}(X)$.
- X is covered by finitely many, X_s, renumbered X_{s_i}. May choose n, r good for all i, j.
- Consider $Y = \operatorname{Proj} A[\{T_i, T_{ij}\}]$. By Proposition 5.1.31 there is $\phi : X \to Y$ with $\phi^* \mathcal{O}_Y(1) = \mathcal{L}$.
- $X_{s_i} = \phi^{-1} D_+(T_i)$ and $\mathcal{O}_Y(D_+(T_i)) \to \mathcal{O}_X(X_{s_i})$ surjective.
- So ϕ a closed embedding, as needed.

Ampleness and quasi-projective schemes

Lemma

Assume X either Noetherian or quasicompact and separated; $\mathcal{L}$ invertible.

a) *If $X = \cup X_{s_i}$ affine open cover for $s_i \in \mathcal{L}(X)$, then $\mathcal{L}$ ample.*
b) *If $\mathcal{L}$ ample and U subset$X$ open and quasicompact then $\mathcal{L}|_U$ ample.*

Corollary

If $X \to \operatorname{Spec} A$ as in theorem, then it is quasi-projective if and only if there is an ample sheaf.

The theorem gives ample $\Rightarrow$ quasi-projective. If quasi-projective then X open in a projective Y, which has an ample. By the lemma the restriction is ample.

Proof of Lemma

a)
- Suppose $\mathcal{F}$ finitely generated and quasicoherent. Say f_{ij} generate $\mathcal{F}(X_{s_i})$.
- Then $f_{ij}s_i^n$ lift to $t_{ij} \in (\mathcal{F} \otimes \mathcal{L}^n)(X)$ generating the sheaf on X_{s_i}, for some n and for all i, as needed.

b) By the Lemma there are sections of $\mathcal{L}^r$ satisfying (a). Also $U \cap X_{s_j}$ is covered by finitely many principal opens $D_{X_{s_j}}(h_{ij}) \subset X_{s_j}$. So $s_j^n h_{ij}$ lifts to $t_{ij} \in \mathcal{L}(X)$. Now $u_{ij} = t_{ij}|_U$ have $U_{u_{ij}} = D_{X_{s_j}}(h_{ij})$ are affine so satisfy (a) on U, and $\mathcal{L}|_U$ ample by (a).

Ampleness is local

Proposition

Let $f : X \to Y$ be a proper morphism of locally noetherian schemes, $\mathcal{L}$ invertible on X. Fix $y \in Y$ and $\phi : X_{Y,y} := X \times_Y \operatorname{Spec} \mathcal{O}_{Y,y} \to X$ the canonical base change morphism.

a) *If $\phi^*\mathcal{L}$ is globally generated then there is an open $y \in V \subset Y$ such that $\mathcal{L}_{X_V}$ is globally generated.*
b) *If $\phi^*\mathcal{L}$ is ample then there is an open $y \in V \subset Y$ such that $\mathcal{L}_{X_V}$ is ample.*

Lemma (Flat base change)

Assume $Y = \operatorname{Spec} A$ affine. Then $\mathcal{F}(X) \otimes_A \mathcal{O}_{Y,y} \to \mathcal{F}(X_{Y,y})$ an isomorphism.

Proof of Lemma

One only needs X noetherian or separated and quasi-compact. Pick a finite affine open covering $X = \cup U_i$. The sequence $0 \to \mathcal{F}(X) \to \oplus \mathcal{F}(U_i) \to \oplus \mathcal{F}(U_i \cap U_j)$ induces a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathcal{F}(X) \otimes_A \mathcal{O}_{Y,y} & \longrightarrow & \oplus \mathcal{F}(U_i) \otimes_A \mathcal{O}_{Y,y} & \longrightarrow & \oplus \mathcal{F}(U_i \cap U_j) \otimes_A \mathcal{O}_{Y,y} \\
& \downarrow & & \downarrow \beta & & \downarrow \\
0 \longrightarrow & \mathcal{F}(X_{Y,y}) & \longrightarrow & \oplus \mathcal{F}((U_i)_{Y,y}) & \longrightarrow & \oplus \mathcal{F}((U_i \cap U_j)_{Y,y})
\end{array}$$

the arrow β is an isomorphism. So the left arrow is injective. Same holds for X replaced by $U_i \cap U_j$. So it is an isomorphism.

We may replace Y by affine open.

a) Then $\phi^*\mathcal{L}(X_{Y,y}) = \mathcal{L}(X) \otimes_A \mathcal{O}_{Y,y}$ and $\phi^*\mathcal{L}_x = \mathcal{L}_x$. So $\mathcal{L}(X) \otimes_A \mathcal{O}_X \to \mathcal{L}$ is surjective along $f^{-1}y$. There is an open set of X containing $f^{-1}y$ where this is surjective. Since $X \to Y$ is proper tehre is a neighborhood of y where this is surjective.

b) By taking a high power we may assume $\phi^*\mathcal{L}$ is very ample and globally generated. Shrinking Y and using (a) we may assume $\mathcal{L}$ is globally generated. Since X is quasicompact a finite number of sections suffices, giving a morphism $X \to \mathbb{P}^d_A$. Let Z is the closure of the image, with $f : X \to Z$ the morphism. This is an isomorphism to the image along $X_{Y,y}$, so it is an isomorphism on an open set (this is exercise 3.2.5).