MA 206 notes: introduction to resolution of singularities

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Resolution of singularities

Let k be a field and X a reduced variety over k. The set of regular points X^{reg} of X is Zariski-dense. (Read Liu 4.3, 6.1, 6.2)

Definition

A *resolution of singularities* of a reduced variety X is a *proper birational morphism* $f : X' \to X$, where X' is regular and irreducible, and f restricts to an isomorphism $f^{-1}(X^{reg}) \xrightarrow{\sim} X^{reg}$.

(We'll assume X geometrically reduced from now on.)

Theorem (Hironaka's Main Theorem 1)

Let X be a reduced variety over k with $\mathrm{Char}(k) = 0$. Then there is a projective resolution of singularities $X' \to X$.

We'll mostly assume $\mathrm{Char}(k) = 0$ from now on.

Method: blowing up and embedded resolution

The main tool: if $X \subset Y$ closed with Y regular, find $Z \subset X$ closed and regular with blowing up $\tilde{Y} \to Y$ such that the strict transform $\tilde{X} \subset \tilde{Y}$ gets better.

Theorem (Embedded resolution)

Suppose $X \subset Y$ is a closed subvariety of a regular variety Y. There is a sequence of blowings up $Y_n \to Y_{n-1} \to \cdots \to Y_0 = Y$, with regular centers $Z_i \subset Y_i$ and strict transforms $X_i \subset Y_i$, such that Z_i does not contain any irreducible component of X_i and such that X_n is regular.

From embedded resolution to resolution

How to get resolution in general?

- If X is embedded then you get resolution.
- In general can blow up so that X is embedded: Chow's[1] Lemma.
- Best option: X is always locally embedded. We'll make sure the procedure on X is independent of embedding.

[1] pronounced Zhou's

Normal crossings

To go further one needs to describe a desirable property of the exceptional divisor E_i and its interaction with the center Z_i.

Definition

We say that a closed subset $E \subset Y$ of a regular variety Y is a *simple normal crossings divisor* if in its decomposition $E = \cup E_j$ into irreducible components, each component E_j is regular, and these components intersect transversally: locally at a point $p \in E$ there are local parameters $x_1, \ldots, x_m$ such that E is the zero locus of a reduced monomial $x_1 \cdots x_k$.

We further say that E and a regular subvariety Z *have normal crossings* if such coordinates can be chosen so that $Z = V(x_{j_1}, \ldots, x_{j_l})$ is the zero set of a subset of these coordinates.

- When the set of coordinates $x_{j_1}, \dots, x_{j_l}$ is disjoint from $x_1, \dots, x_k$ the strata of E meet Z transversely, but the definition above allows quite a bit more flexibility.
- This definition works well with blowing up: If E is a simple normal crossings divisor, E and Z have normal crossings, $f : Y' \to Y$ is the blowing up of the regular center Z with exceptional divisor E_Z, and $E' = f^{-1}E \cup E_Z$ then E' is a simple normal crossings divisor.

Principalization

Theorem (Principalization)

Let Y be a regular variety and $\mathcal{I}$ an ideal sheaf. There is a sequence of blowings up $Y_n \to Y_{n-1} \to \cdots \to Y_0 = Y$, regular subvarieties $Z_i \subset Y_i, i = 0, \ldots, n-1$ and simple normal crossings divisors $E_i \subset Y_i, i = 1, \ldots, n$ such that

- $f_i : Y_{i+1} \to Y_i$ *is the blowing up of* Z_i *for* $i = 0, \ldots, n-1$,
- E_i *and* Z_i *have normal crossings for* $i = 1, \ldots, n-1$,
- $\mathcal{I}\mathcal{O}_{Y_i}$ *vanishes on* Z_i *for* $i = 0, \ldots, n-1$,
- E_{i+1} *is the union of* $f_i^{-1}E_i$ *with the exceptional locus of* f_i *for* $i = 0, \ldots, n-1$

and such that the resulting ideal sheaf $\mathcal{I}_n = \mathcal{I}\mathcal{O}_{Y_n}$ *is an invertible ideal with zero set* $V(\mathcal{I}_n)$ *supported in* E_n.

We will later require a possibly nonempty $E_0 \subset Y_0$.

Principalization implies embedded resolution

- In local coordinates $x_1, \ldots, x_m$ on Y_n as above, this means that $\mathcal{I}_n = (x_1^{a_1} \cdots x_m^{a_k})$ is locally principal and monomial, hence the name "principalization".
- The condition that Z_i have normal crossings with E_i guarantees that E_{i+1} is a simple normal crossings divisor.
- Quoting Kollár (p. 137), principalization implies embedded resolution seemingly "by accident":
- suppose for simplicity that X is irreducible, and let the ideal of $X \subset Y$ be $\mathcal{I}$. Since $\mathcal{I}_n$ is the ideal of a divisor supported in the exceptional locus, at some point in the sequence the center Z_i must contain the strict transform X_i of X. Since $\mathcal{I}$ vanishes on Z_i, it follows that Z_i coincides with X_i at least near X_i. In particular X_i is regular!

Are we working too hard?

- Principalization seems to require "too much" for resolution: why should we care about exceptional divisors which lie outside X? Are we trying too hard?
- In the example of the cuspidal curve $X \subset Y = \mathbb{A}^2$ above, the single blowing up $Y_1 \to Y$ at the origin does *not* suffice for principalization: the resulting equation $x^2(z^2 - x) = 0$ with exceptional $\{x = 0\}$ is *not* monomial. One needs no less that *three* more blowings up!

The cusp example again

I'll describe just one key affine patch of each:

- Blowing up $x = z = 0$ one gets, in one affine patch where $x = zw$, the equation $z^3w^2(z - w) = 0$, strict transform $X_2 = \{z = w\}$ and exceptional $\{wz = 0\}$.
- Blowing up $z = w = 0$ one gets, in one affine patch where $z = wv$, the equation $v^3w^6(v - 1) = 0$. The exceptional in this patch is $\{wv = 0\}$, with one component (the old $\{z = 0\}$) appearing only in the other patch. The strict transform is $X_3 = \{v = 1\}$.
- In the open set $\{v \neq 0\}$ we blow up $\{v = 1\}$. This actually does nothing, except turning the function $u = v - 1$ into a monomial along $v = 1$, so the equation $v^3w^6(v - 1) = 0$ at these points can be written as $(u + 1)^3w^6u = 0$, which in this patch is equivalent to $w^6u = 0$, a monomial in the exceptional parameters u, w.

The cusp example: discussion

- The fact that we could blow up $\{v = 1\}$ means that X_3 is regular, giving rather late evidence that we obtained resolution of singularities for X.
- These "redundant" steps add to the sense that this method works "by accident".
- It turns out that principalization itself is quite useful in the study of singularities.
- Also the fact that it provides the prize of resolution is seen as sufficient justification.

Accident or not, we will continue to pursue principalization.

Order

Finally, principalization of an ideal is proven by way of *order reduction*.

- The *order* $\mathrm{ord}_p(\mathcal{I})$ of an ideal $\mathcal{I}$ at a point p of a regular variety Y is the maximum integer d such that $\mathfrak{m}_p^d \supseteq \mathcal{I}$; here $\mathfrak{m}_p$ is the maximal ideal of p. It tells us "how many times every element of $\mathcal{I}_p$ vanishes at p."
- In particular we have $\mathrm{ord}_p(\mathcal{I}) \geq 1$ precisely if $\mathcal{I}$ vanishes at p.
- We write $\mathrm{maxord}(\mathcal{I}) = \max\{\mathrm{ord}_p(\mathcal{I}) | p \in X\}$.
- For instance we have $\mathrm{maxord}(\mathcal{I}) = 0$ if and only if $\mathcal{I}$ is the unit ideal, which vanishes nowhere.
- Another exceptional case is $\mathrm{maxord}(\mathcal{I}) = \infty$ which happens if $\mathcal{I}$ vanishes on a whole component of Y.

Order subsets and admissibility

- Given an integer a, we write $V(\mathcal{I}, a)$ for the locus of points p where $\text{ord}_p(\mathcal{I}) \geq a$.
- A regular closed subvariety $Z \subset Y$ is said to be $(\mathcal{I}, a)$-*admissible* if and only if $Z \subset V(\mathcal{I}, a)$, in other words, the order of $\mathcal{I}$ at every point of Z is at least a.
- Admissibility is related to blowings up:
- if $\text{maxord}(\mathcal{I}) = a$, and if $Y' \to Y$ is the blowing up of an $(\mathcal{I}, a)$-admissible $Z \subset Y$, with exceptional divisor E having ideal $\mathcal{I}_E$, then $\mathcal{I}\mathcal{O}_{Y'} = (\mathcal{I}_E)^a \mathcal{I}'$, with $\text{maxord}(\mathcal{I}') \leq a$.
- In other words, orders do not grow in admissible blowings up

Order reduction

Order reduction is the following statement:

Theorem (Order reduction)

Let Y be a regular variety, $E_0 \subset Y$ a simple normal crossings divisor, and $\mathcal{I}$ an ideal sheaf, with

$$\operatorname{maxord}(\mathcal{I}) = a.$$

There is a sequence of $(\mathcal{I}, a)$-admissible blowings up $Y_n \to Y_{n-1} \to \cdots \to Y_0 = Y$, with regular centers $Z_i \subset Y_i$ having normal crossings with E_i such that $\mathcal{I}\mathcal{O}_{X_n} = \mathcal{I}_n \mathcal{I}'_n$with $\mathcal{I}_n$ an invertible ideal supported on E_n and such that

$$\operatorname{maxord}(\mathcal{I}'_n) < a.$$

Order reduction implies principalization

- Order reduction implies principalization simply by induction on the maximal order $\operatorname{maxord}(\mathcal{I}) = a$: once $\operatorname{maxord}(\mathcal{I}'_n) = 0$ we have $\mathcal{I}\mathcal{O}_{X_n} = \mathcal{I}_n$ so only the exceptional part remains, which is supported on a simple normal crossings divisor by induction.
- Hironaka himself used the *Hilbert–Samuel function*, an invariant much more refined than the order.
- It is a surprising phenomenon that resolution becomes easier to explain when one uses just the order, thus less information, see [Encinas-Villamayor].

It remains to prove order reduction.

Tangent and cotangent sheaves

Consider a separated morphism $Y \to S$, the scheme $Y^2_Y := Y \times_S Y$, and the diagonal $\Delta_Y \subset Y^2_S$. It is a closed subscheme isomorphic to Y with ideal $\mathcal{I}_\Delta$.

Definition

- The relative cotangent sheaf $\Omega^1_{Y/S} = \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$, naturally an $\mathcal{O}_{\Delta_Y}$-module. Also known as sheaf of relative Kähler differentials.
- The sheaf $T_{Y/S} = \mathcal{H}om_Y(\Omega^1_{Y/S}, \mathcal{O}_Y)$ also denoted $\Theta_{Y/S}$ is the tangent sheaf.
- The universal $\mathcal{O}_Y$-derivation $d : \mathcal{O}_Y \to \Omega^1_{Y/S}$ relative to S is defined by sending a function f to $\pi_1^* f - \pi_2^* f \mod \mathcal{I}_\Delta^2$.

- If $Y \to S$ is a smooth morphism then these are locally free sheaves, just as you learned in Manifolds.
- We'll mostly work with $S = \operatorname{Spec} k$ and drop it from notation, but not always.

Differential operators: characteristic 0 naïvely

Sections of $T_{Y/S}$ are by definition derivations $\mathcal{O}_Y \to \mathcal{O}_Y$ which are zero on $\mathcal{O}_S$. Let us start by a naïve definition.

Definition

The sheaf of rings generated over $\mathcal{O}_Y$ by the operators in T_Y is the *sheaf of differential operators* $\mathcal{D}_Y$.

- As a sheaf of $\mathcal{O}_Y$ modules it looks locally like the symmetric algebra $\mathrm{Sym}^\bullet(T_Y) = \oplus_{n\geq 0}\mathrm{Sym}^n(T_Y)$, but its ring structure is very different, as $\mathcal{D}_Y$ is non-commutative.
- Still for any integer a there is a subsheaf $\mathcal{D}_Y^{\leq a} \subset \mathcal{D}_Y$ of differential operators of order $\leq a$, those sections which can be written in terms of monomials of order at most a in sections of T_Y.
- As a special case, one always has a splitting $\mathcal{D}_Y^{\leq 1} = \mathcal{O}_Y \oplus T_Y$, the projection $\mathcal{D}_Y^{\leq 1} \to \mathcal{O}_Y$ given by applying $\nabla \mapsto \nabla(1)$.

Differential operators in general

Here is a better way to define things, which detects p-th powers in characteristic p:

Definition

The sheaf of principal parts of order a of Y is defined as $\mathcal{PP}_Y^a = \mathcal{O}_{Y\times Y}/\mathcal{I}_\Delta^{a+1}$

- This is a sheaf of $\mathcal{O}_Y$-modules via either projection; its fiber at $p \in Y$ describes functions on Y up to order a at p.
- Its dual sheaf is $\mathcal{D}^{\leq a} := (\mathcal{PP}_Y^a)^\vee$, which *in characteristic 0* admits the concrete description given earlier.
- The natural projection $\mathcal{PP}_Y^a \to \mathcal{PP}_Y^{a-1}$ gives rise to an inclusion $\mathcal{D}_Y^{\leq a-1} \subset \mathcal{D}_Y^{\leq a}$, and one defines in general $\mathcal{D}_Y = \cup_a \mathcal{D}_Y^{\leq a}$.
- This is nice enough, but the fact that in positive characteristics sections of $\mathcal{D}_Y$ are not written as polynomials in sections of T_Y is the source of much trouble.

Derivatives and order

Definition

Let $\mathcal{I}$ be an ideal sheaf on Y and $y \in Y$ a point. Write $\mathcal{D}_Y^{\leq a}\mathcal{I}$ for the ideal generated by elements $\nabla(f)$, where ∇ an operator in $\mathcal{D}_Y^{\leq a}$ and f a section of $\mathcal{I}$.

We have the following characterization:

$$\operatorname{ord}_y(\mathcal{I}) = \min\{a : (\mathcal{D}^{\leq a}\mathcal{I})_p = \mathcal{O}_{Y,p}\}.$$

In other words, the order of $\mathcal{I}$ at y is the minimum order of a differential operator ∇ such that for some $f \in \mathcal{I}_y$ the element $\nabla(f)$ does not vanish at y.

The order-a scheme

- We define $MC(\mathcal{I}, a) := \mathcal{D}^{\leq a-1}\mathcal{I}$
- In these terms, the set $V(\mathcal{I}, a)$ can be promoted to a *scheme*, the zero locus of an ideal: $V(\mathcal{I}, a) := V(MC(\mathcal{I}, a))$.[2]
- This is not too surprising in characteristic 0 since we all learned calculus, but it may seem strange in characteristic $p > 0$.
- For instance, the order of (x^p) is p, since there is always an operator ∇ of order p such that $\nabla(x^p) = 1$.
- In characteristic 0 we can write

$$\nabla = \frac{1}{p!}\left(\frac{\partial}{\partial x}\right)^p,$$

but in characteristic p we have no such expression!

[2] This is the *right* scheme structure, as it satisfies an appropriate universal property.

Maximal contact

Definition

- Let $\mathcal{I}$ be an ideal of maximal order a.
- *A maximal contact hypersurface for* $(\mathcal{I}, a)$ *at* p is a hypersurface H *regular at* p,
- such that, in some neighborhood Y^0 of p we have $H \supseteq V(\mathcal{I}, a) = V(MC(\mathcal{I}, a))$,
- namely H contains the *scheme* of points where $\mathcal{I}$ has order a.

Proposition

In characteristic 0, a maximal contact hypersurface for $(\mathcal{I}, a)$ *at* p *exists.*

Existence

Proposition

In characteristic 0, a maximal contact hypersurface for $(\mathcal{I}, a)$ at p exists.

- Since $\mathcal{I}$ has maximal order a, we have $\mathcal{D}^{\leq a}(\mathcal{I}) = (1)$.
- Consider the ideal $MC(\mathcal{I}, a) = \mathcal{D}^{\leq a-1}(\mathcal{I})$.
- Since we are in characteristic 0, it must contain an antiderivative of 1, so $\text{ord}_p(MC(\mathcal{I}, a)) \leq 1$.
- Any local section x of $MC(\mathcal{I}, a)$ with order ≤ 1 gives a maximal contact hypersurface $\{x = 0\}$ at p.

Classic example

- Suppose $\mathcal{I} = (f)$ where

$$f(x, y) = y^a + g_1(x)y^{a-1} + \cdots + g_{a-1}(x)y + g_a(x).$$

- Then $\mathrm{ord}_{(0,0)}(f) = a$ exactly when $\mathrm{ord}_0(g_i) \geq i$ for all i.
- In characteristic 0 we may replace y by $y + g_1(x)/a$, so we may assume $g_1(x) = 0$.
- In this case $\partial^{a-1} f / \partial y^{a-1} = a! \cdot y$, so $\{y = 0\}$ is a maximal contact hypersurface at $(0, 0)$.

Maximal contact is local

- The definition I gave is pointwise.
- It is easy to see that if H is a maximal contact hypersurface for $(\mathcal{I}, a)$ at p then the same holds at any nearby p',
- so the concept is local.
- Unfortunately it is also not hard to cook up examples where there is no *global* maximal contact hypersurface which works everywhere.
- We will have to tackle this problem.

Positive characteristics: Alas, there are fairly simple examples in charactersitic $p > 0$ where maximal contact hypersurfaces do not exist [Narasimhan].
The whole discussion from here on simply does not work in characteristic > 0.

What to restrict to H?

- Consider $\mathcal{I} = (f)$ with $f(x,y) = y^a + g_2(x)y^{a-2} + \cdots + g_{a-1}(x)y + g_a(x)$, and $\operatorname{ord} g_i \geq i$ so $\operatorname{maxord}(\mathcal{I}) = a$.
- In order to reduce the order of $\mathcal{I}$ we need to reduce the order of at least one g_i so that $\operatorname{ord} g_i < i$.
- So the information necessary needs to involve all these g_i.
- The right generalization is given by the collection of ideals

$$\mathcal{D}^{\leq i}(\mathcal{I}), \quad i \leq a-1,$$

with maximal order $a - i$, and their restriction to H.

This is central and requires work:

Proposition

- *Any sequence of* $(\mathcal{I}, a)$*-admissible blowings up has centers lying in* H *and its successive strict transforms. The resulting sequence of blowings up on* H *is* $((\mathcal{D}_Y^{\leq i}\mathcal{I})_{|H}, a-i)$*-admissible for every* $i < a$*.*
- *Conversely, every sequence of blowings up on* H *which is* $((\mathcal{D}_Y^{\leq i}\mathcal{I})_{|H}, a-i)$*-admissible for every* $i < a$ *gives rise, by blowing up the same centers on* Y*, to a sequence of* $(\mathcal{I}, a)$*-admissible blowings up.*

The Włodarczyk ideal

Definition (Kollár)

Giving $\mathcal{D}_Y^{\leq i}\mathcal{I}$ with $i \leq a-1$ weight $a-i$, define

$$W(\mathcal{I}, a) = \sum_{\sum(a-i_j)=a!} \prod_j \mathcal{D}_Y^{\leq i_j}\mathcal{I}$$

to be the ideal generated by terms of total weight $a!$.

Order reduction of the collection of ideals is equivalent to that of the Włodarczyk ideal, hence

Corollary ((13) Kollár, Corollary 3.85)

A sequence of blowings up is an order reduction for $(\mathcal{I}, a)$ *if and only if it is an order reduction for* $(W(\mathcal{I}, a)_{|H}, a!)$.

Traditionally one uses a coefficient ideal instead.

Separation of exceptional loci

- This does not quite work with E_0, which is necessary.
- In the example of a cuspidal curve above, the ideal $\mathcal{I} = x^2(z^2 - x)$ is of the form $\mathcal{I}_E^2 \mathcal{I}'$. The unique maximal contact hypersurface for $(\mathcal{I}', 1)$ is precisely X', the vanishing locus of $\mathcal{I}'$, but since it is tangent to E it does not have normal crossings with E.
- One treats this is via a trick: one separates the relevant part of the ideal $\mathcal{I}'$ from the monomial part $\mathcal{I}_E$ by applying a suitable principalization for an ideal of the form $\mathcal{I}_E^\alpha + \mathcal{I}'^\beta$ describing the intersection of their loci.

Patching

Hypersurfaces of maximal contact are not unique and do not patch. However we have

Proposition ((4), (Włodarczyk))

Let $H_1, H_2 \subset Y$ be two local maximal contact hypersurfaces at $p \in V(\mathcal{I}, a)$. Then, after replacing Y by an étale neighborhood of p, there is an automorphism ϕ of Y fixing $p, V(\mathcal{I}, a)$, stabilizing $W(\mathcal{I}, a)$, and sending H_1 to H_2.

A sketch of the algorithm

Let us summarize how one *functorially* reduces the order of a nonzero ideal $\mathcal{I}$ of maximal order $a > 0$ on a regular variety Y.

- If $\dim(Y) = 0$ there is nothing to prove, since $\mathcal{I}$ is trivial hence of order 0. We assume proven order reduction in dimension $< \dim(Y)$.
- We cover $V(\mathcal{I}, a)$ with open patches U possessing maximal contact hypersurfaces H_U. The Włodarczyk ideal $W(\mathcal{I}, a)_{|H_U}$ has order $\geq a!$.
- If this order is infinite, it means that $\mathcal{I}_{|U} = \mathcal{I}^a_{H_U}$, we simply blow up H_U and automatically the order of $\mathcal{I}$ is reduced on U.
- Otherwise we can inductively reduce the order of this ideal by a functorial sequence of transformations $H_k \to \cdots \to H$ until the order drops below $a!$. By Corollary 13 these provide a local order reduction for $(\mathcal{I}, a)$ which patches together to a functorial order reduction by Proposition 4.

The re-embedding principle

Proposition (The re-embedding principle)

Suppose $\mathcal{I}$ is an ideal on a regular variety Y. Consider the embedding $Y \subset Y_1 := Y \times \mathbb{A}^1$ sending $y \mapsto (y, 0)$. Let $\mathcal{I}_1 = \mathcal{I}\mathcal{O}_{Y_1} + (z)$, where z is the coordinate on $\mathbb{A}^1$. Then the principalization described above of $\mathcal{I}_1$ on Y_1 is obtained by taking the principalization of $\mathcal{I}$ on Y and blowing up the same centers, embedded in Y_1 and is transforms.