MA 205/206 notes: Derived functors and cohomology

Following Hartshorne

Dan Abramovich

Brown University

February 2, 2019

Abelian categories

Definition

An abelian category is a category $\mathfrak{A}$ with each $Hom(A, B)$ provided the structure of an abelian group, such that

- Compositions laws $Hom(B, C) \times Hom(A, B) \to Hom(A, C)$ are bilinear (pre-additive category)
- Finite[a] products and coproducts exist and coincide (additive category), in particular an initial and final object 0 exists.
- Every morphism has a kernel and cokernel (pre-abelian).
- Every monomorphism is the kernel of its cokernel.
- Every epimorphism is the cokernel of its kernel.
- Every morphism factors into an epimorphism followed by a monomorphism (image and coimage in some order).

[a] 0 and 2 enough

- In an additive category, a kernel of $f : X \to Y$ is a fibered product $X \times_Y 0$.
- In an additive category, a cokernel of $f : X \to Y$ is a fibered coproduct $Y \oplus^X 0$.
- In any category, $f : X \to Y$ is a monomorphism if $g \mapsto f \circ g$ is injective.
- In any category, $f : X \to Y$ is an epimorphism if $g \mapsto g \circ f$ is injective.

Examples of abelian categories

- The prototypical example is $\mathfrak{Ab}$.
- The typical example is $\mathfrak{Mod}(A)$.
- We will use $\mathfrak{Ab}(X)$, $\mathfrak{Mod}(\mathcal{O}_X)$, $\mathfrak{Q}coh(\mathcal{O}_X)$,
- and, if X noetherian, $\mathfrak{C}oh(\mathcal{O}_X)$.

There is an embedding theorem saying any abelian $\mathcal{F}$ is a full subcategory of $\mathfrak{Ab}$.

Complexes

- A complex $A^\bullet$ in an abelian category $\mathfrak{A}$ is a sequence with maps $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$.
- Complexes in $\mathfrak{A}$ form an abelian category by requiring arrows to commute with d and doing things componentwise.
- $h^i(A^\bullet) = \mathrm{Ker}(d^i)/Im(d^{i-1})$. It is a functor $\mathfrak{C}omp(\mathfrak{A}) \to \mathfrak{A}$.
- If $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ exact, there is $\delta^i : h^i(C^\bullet) \to h^{i+1}(A^\bullet)$ with a long exact sequence.
- A homotopy between $f, g : A^\bullet \to B^\bullet$ is a collection $k^i : A^i \to B^{i-1}$ with $f - g = dk + kd$.
- If there is a homotopy we say f and g are homotopic, $f \sim g$.
- if $f \sim g$ then $h^i(f) = h^i(g)$.

Additive, left exact functors

- A covariant $F : \mathfrak{A} \to \mathfrak{B}$ is additive if $Hom(A, A') \to Hom(FA, FA')$ is a homomorphism.
- Such F is left exact if $0 \to A' \to A \to A'' \to 0$ exact implies $0 \to FA' \to FA \to FA''$ exact.
- For contravariant, require $0 \to FA'' \to FA \to FA'$ exact.

Prototypical example: for fixed A the covariant functor $Hom(A, \bullet) : \mathfrak{A} \to \mathfrak{A}b$

$$A' \mapsto Hom(A, A')$$

and the contravariant functor $Hom(\bullet, A) : \mathfrak{A} \to \mathfrak{A}b$

$$A'' \mapsto Hom(A'', A)$$

are left exact, by the abelian category axioms.

- $I \in Ob(\mathfrak{A})$ is injective if $Hom(\bullet, I)$ is exact.
- An injective resolution of A is an exact sequence

$$0 \to A \to I^0 \to I^1 \to \cdots$$

  with I^j injective.
- $\mathfrak{A}$ has enough injectives if every A is a subobject of an injective, so (inductively) has an injective resolution.

Fix an abelian category $\mathfrak{A}$ with enough injectives, and for each object A fix an injective resolution $I_A^\bullet$.

Definition

For an additive covariant left exact $F : \mathfrak{A} \to \mathfrak{B}$ define the right derived functors

$$R^i F(A) = h^i(F(I_A^\bullet)).$$

Basic Properties of $R^i F$

Theorem

- $R^i F : \mathfrak{A} \to \mathfrak{B}$ *is additive.*
- *Independence: Changing* I_A^* *results in a naturally isomorphic functor.*
- $R^0 F \simeq F$
- *For exact* $E_A : 0 \to A' \to A \to A'' \to 0$ *there is* $\delta^i : R^i F(A'') \to R^{i+1} F(A')$ *with long exact sequence*

$$R^i F(A') \to R^i F(A) \to R^i F(A'') \xrightarrow{\delta^i} R^{i+1} F(A') \cdots$$

- *This is compatible with morphisms of exact sequences* $E_A \to E_B$.
- *If* I *injective and* $I > 0$ *then* $R^i F(I) = 0$.

Acyclic objects and resolutions

We say J is F-acyclic if $R^iF(J) = 0$ for all $i > 0$. An F-acyclic resolution $J^\bullet$ of A is what you think.

Proposition

If $A \to J^\bullet$ is an F-acyclic resolution then $R^iF(A) \simeq h^i(F(J^\bullet))$.

Try it at home!

δ-functors

- One might think that the injective resolution definition is artificial. Read in Hartshorne (or Grothendieck) about δ-functors, universal δ-functors, effaceable δ-functors.
- It is shown that that an effaceable δ-functors is universal, and there can be at most one universal δ-functor up to isomorphism.
- Then it is shown that if there are enough injectives, the derived functor is effaceable so it is (the unique) universal δ-functor.

Enough injectives in $\mathfrak{Ab}$ and $\mathfrak{Mod}(A)$

- It is not too hard to see that an abelian group is injective if and only if it is divisible.
- Also one can embed any abelian group in a divisible group, so there are enough injectives in $\mathfrak{Ab}$.
- If M is an A-module and $M \to I_{\mathbb{Z}}$ be an embedding into a divisible group, then the natural embedding $M \to Hom_{\mathbb{Z}}(A, I_{\mathbb{Z}})$ provide an embedding into an injective A-module.
- You can find all this explained in a few pages e.g. in http://www.math.leidenuniv.nl/~edix/tag_2009/michiel_2.pdf.

Enough injectives in $\mathfrak{Mod}(\mathcal{O}_X)$

Proposition (Proposition III.2.2)

If $(X, \mathcal{O}_X)$ a ringed space[a] then $\mathfrak{Mod}(\mathcal{O}_X)$ has enough injectives.

[a] not necessarily lrs

Embedding:

- Embed the stalks $\mathcal{F}_x \hookrightarrow I_x$ in injective $\mathcal{O}_{X,x}$-modules.
- Define $\mathcal{J} = \prod_x j_{x*}(I_x)$.
- We have an embedding $\mathcal{F} \hookrightarrow \prod_x j_{x*}(\mathcal{F}_x) \hookrightarrow \prod_x j_{x*}(I_x) = \mathcal{J}$.

Injectivity:

- We have
  $Hom_{\mathcal{O}_X}(\mathcal{G}, \mathcal{J}) = \prod Hom_{\mathcal{O}_X}(\mathcal{G}, j_{x*}(I_x)) = \prod Hom_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$.
- $\mathcal{G} \mapsto \mathcal{G}_x$ and $\mathcal{G}_x \mapsto Hom_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$ are exact,
- so $\mathcal{G} \mapsto Hom_{\mathcal{O}_X}(\mathcal{G}, \mathcal{J})$ is exact, so $\mathcal{J}$ injective.

Since $(X, \mathbb{Z}_X)$ is a ringed space, $\mathfrak{Ab}(X)$ has enough injectives.

Definition

The sheaf cohomology $H^i(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F})$, the right derived functor of

$$\mathcal{F} \quad \mapsto \quad \Gamma(X, \mathcal{F}) = \mathcal{F}(X),$$

as a functor

$$\mathfrak{Ab}(X) \to \mathfrak{Ab}.$$

This is clean, but also disturbing, since it ignores any possible structure on X.

Flasque sheaves and cohomology

We say $\mathcal{F}$ is flasque[1] if for any opens $V \subset U$ the restriction $\mathcal{F}(U \to \mathcal{F}(V)$ is surjective.

Proposition

A flasque sheaf $\mathcal{F}$ of abelian groups is Γ-acyclic, i.e. $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Proposition

An injective $\mathcal{O}_X$-module is flasque.

Corollary

The derived functors of $\Gamma(A, \bullet) : \mathfrak{Mod}(\mathcal{O}_X) \to \mathfrak{Ab}$ are $H^i(X, \bullet)$.

Corollary

if $Y \overset{i}{\hookrightarrow} X$ closed and $\mathcal{F}$ on Y, then $H^k(Y, \mathcal{F}) = H^k(X, i_*\mathcal{F})$.

[1] flabby

An injective $\mathcal{O}_X$-module is flasque

This uses the extension by 0 functor $j_!$ (II.1.19):

- For an open $U \stackrel{j_U}{\hookrightarrow} X$ consider the $\mathcal{O}_X$-module $\mathcal{O}_U := j_{U!}(\mathcal{O}_X|_U)$.
- $Hom(\mathcal{O}_U, \mathcal{F}) = \mathcal{F}(U)$.
- For $V \subset U$ get inclusion $\mathcal{O}_V \to \mathcal{O}_U$.
- Take $\mathcal{J}$ injective, and so $Hom(\mathcal{O}_U, \mathcal{J}) \to Hom(\mathcal{O}_V, \mathcal{J})$ surjective.
- So $\mathcal{J}(U) \to \mathcal{J}(V)$ surjective.

If $\mathcal{F}$ flasque then $H^i(X,\mathcal{F}) = 0$ for $i > 0$

This follows from general properties of flasque sheaves (exercise II.1.16), and requires an induction on i for all flasque seheaves:

- Let $\mathcal{F} \to I$ be an embedding in an injective sheaf in $\mathfrak{A}b(X)$, with s.e.s
$$0 \to \mathcal{F} \to I \to \mathcal{G} \to 0.$$
- Since $\mathcal{F}$ and I flasque it follows that $\mathcal{G}$ flasque.
- It also implies $0 \to \mathcal{F}(X) \to I(X) \to \mathcal{G}(X) \to 0$ exact.[2]
- Since $H^i(X,I) = 0$ for $i > 0$, the l.e.s gives $H^1(X,\mathcal{F}) = 0$ and $H^i(X,\mathcal{G}) = H^{i+1}(X,\mathcal{F})$ for $i > 0$.
- Induction gives the result.

[2]we did something similar

The corollaries

The derived functors of $\Gamma(A, \bullet) : \mathfrak{Mod}(\mathcal{O}_X) \to \mathfrak{Ab}$ coincide with $H^i(X, \bullet)$.

- We compute the derived functor by taking an $\mathcal{O}_X$-injective resolution $\mathcal{F} \to I^\bullet$.
- This is a flasque resolution, which is acyclic, hence computes cohomology.

if $Y \stackrel{i}{\hookrightarrow} X$ closed then $H^k(Y, \mathcal{F}) = H^k(X, i_*\mathcal{F})$.

If $I^\bullet$ is an injective, hence flasque, resolution of $\mathcal{F}$ then $i_* I^\bullet$ is a flasque resolution of $i_*\mathcal{F}$.

Grothendieck's dimension theorem

Theorem

Let X be a noetherian topological space of dimension n. Then for $i > n$ and for all sheaf $\mathcal{F}$ on X we have $H^i(X, \mathcal{F}) = 0$.

This is an involved but elegant sequence of reduction steps, with some general input about colimits.

- One uses i_* and $j_!$ and l.e.s to reduce to the case of irreducibles.
- One uses induction on dimension, the case of dimension 0 being trivial.
- One reduces using colimits to sheaves generated by finitely many sections.
- One uses the l.e.s to reduce to $j_!\mathbb{Z}_U$ and subsheaves of such.
- One uses the l.e.s to reduce to $j_!\mathbb{Z}_U$.
- One uses the l.e.s for $j_!\mathbb{Z}_U \subset \mathbb{Z}_X$ and flasqueness of $\mathbb{Z}_X$ to conclude.

Cohomology of Quasicohrent sheaves on affine noetherian schemes

Proposition

Let A be a noetherian ring, I an injective A-module. Let $\tilde{I}$ be the associated sheaf on $\operatorname{Spec} A$. Then $\tilde{I}$ is flasque.

The proof is non trivial and goes by way of showing that the sheaves of sections with support along an ideal $\mathfrak{a}$ is also injective.

Theorem

Let A be a noetherian ring, $X = \operatorname{Spec} A$, and $\mathcal{F}$ quasicoherent. Then for all $i > 0$ we have $H^i(X, \mathcal{F}) = 0$.

Set $M = \Gamma(X, \mathcal{F})$, and let $0 \to M \to I^\bullet$ injective resolution.
We have learned that $0 \to \mathcal{F} \to \tilde{I}^\bullet$ exact, flasque resolution!
and taking sections $0 \to M \to \Gamma(X, \tilde{I}^\bullet)$ also exact.
So $H^i(X, \mathcal{F}) = h^i(\Gamma(X, \tilde{I}^\bullet)) = 0$ ♠

QC Flasque resolutions on noetherian schemes

Corollary

X noetherian, $\mathcal{F}$ quasicoherent, then there is and embedding $\mathcal{F} \hookrightarrow \mathcal{G}$ with $\mathcal{G}$ quasicoherent and flasque

Let $X = \bigcup_{\text{finite}} U_i$
with $U_i = \operatorname{Spec} A_i \stackrel{j_i}{\hookrightarrow} X$,
write $\mathcal{F}(U_i) = M_i$ and choose $M_i \subset I_i$, with I_i injective A_i modules.
Then $\mathcal{F} \hookrightarrow \oplus j_{i*}\widetilde{I_i}$ injective,
and $\widetilde{I_i}$ flasque hence $\oplus j_{i*}\widetilde{I_i}$ flasque.

Serre's criterion

Theorem

Suppose X either notherian or separated and quasicompact. Then the following are equivalent:

(i) *X affine.*

(ii) *$H^p(X, \mathcal{F})$ for every quasicoherent $\mathcal{F}$ and $p > 0$.*

(iii) *$H^1(X, \mathcal{F})$ for every quasicoherent $\mathcal{F}$.*

(iv) *$H^1(X, \mathcal{I})$ for every quasicoherent ideal $\mathcal{I}$.*

- Let $A = \mathcal{O}(X)$. Need $\phi : X \to \operatorname{Spec} A$ an isomorphism.
- For $f \in A$ we have $X_f = \phi^{-1} D(f)$ and by an old result $\mathcal{O}_X(X_f) = A[f^{-1}]$.
- If X_f affine then $\phi_{X_f} : X_f \to D(f)$ an isomorphism,
- so it suffices to show (1) each $x \in X$ lies in an affine X_f, and (2) ϕ surjective.

Serre's criterion, (1) each $x \in X$ lies in an affine X_f

- The closure $\overline{\{x\}}$ is quasicompact, hence has a closed point;
- might as well assume x closed.
- Let $\mathcal{M} = \mathcal{I}_{\{x\}}$. Let $U \ni x$ be an affine neighborhood. Let $J = \mathcal{I}_{X \setminus U}$.
- $0 \to \mathcal{M}\mathcal{J} \to \mathcal{J} \to \mathcal{J}/\mathcal{M}\mathcal{J} \to 0$ is exact.
- The latter is a skyscraper with fiber $k(x)$ at x.
- By assumption $H^1(X, \mathcal{M}\mathcal{J}) = 0$,
- and by the long exact sequence there is $f \in \mathcal{J}$ such that $f(x) \neq 0$.
- Note that $X_f = D_U(f)$ is an affine neighborhood of x.♠

Serre's criterion, (2) ϕ surjective.

- Take finitely many f_i so that $X = \cup X_{f_i}$.
- Need to show $A = \cup X_{f_i}$, namely $(f_1, \ldots, f_m) = (1)$.
- Consider $\psi : \mathcal{O}_X^n \to \mathcal{O}_X$, where $\psi(a_1, \ldots, a_n) = \sum a_i f_i$.
- $0 \to \text{Ker}\psi \to \mathcal{O}^n \to \mathcal{O} \to 0$ is an exact sequence of quasicoherent sheaves.
- Enough to show $H^1(X, \text{Ker}\psi) = 0$, since then we have $A^n \to A$ surjective, as needed.
- Write $\mathcal{K}_i = \text{Ker}\psi \cap \mathcal{O}_X^i$ and $\mathcal{Q}_i = \mathcal{K}_i/\mathcal{K}_{i-1}$.
- Note: $\mathcal{Q}_i \subset \mathcal{O}_X$ a sheaf of ideals.
- So $H^1(X, \mathcal{Q}_i) = 0$, and by l.e.s and induction $H^1(X, \text{Ker}\psi) = 0$, as needed ♠

Čech cohomology of sheaves

- Given a covering $\mathfrak{U} := \{U_i\}$ of X one defines a complex $0 \to \mathcal{F}(X) \to C^0(\mathfrak{U}, \mathcal{F}) \stackrel{d_0}{\to} C^1(\mathfrak{U}, \mathcal{F}) \stackrel{d_1}{\to} \cdots,$
- where $C^p(\mathfrak{U}, \mathcal{F}) := \prod_{i_0 < \ldots < i_p} \mathcal{F}(U_{i_0, \ldots, i_p})$.
- For $f \in C^p(\mathfrak{U}, \mathcal{F})$ one defines

$$df = \sum_0^{p+1} (-1)^k f_{i_0, \ldots \hat{i}_k, \ldots, i_{p+1}} |_{U_{i_0, \ldots, i_{p+1}}}.$$

- Exercise: $d^2 = 0$.
- Define $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = h^i(C^\bullet(\mathfrak{U}, \mathcal{F}))$.

Basic facts

Proposition

$\check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{F}) = \mathcal{F}(X).$

Indeed for a collecton of sections $s_i \in \mathcal{F}(U_i)$ do have $d_0((s_i)) = 0$ we need precisely $s_i|_{U_{ij}} - s_j|_{U_{ij}} = 0$, and the sheaf axiom gives $s \in \mathcal{F}(X)$.

Observation

If $\mathfrak{U}$ contains n opens then $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = 0$ for all $p \geq n$.

Instead of $C(\mathfrak{U}, \mathcal{F})$ one can work with alternating chains $C'(\mathfrak{U}, \mathcal{F})$. We have $\check{\mathrm{H}}(\mathfrak{U}, \mathcal{F}) = h^i(C'(\mathfrak{U}, \mathcal{F}))$.

- Example: Consider $X = \mathbb{P}^1_A$ with the open sets $U_i = D_+(T_i)$.
- The Čech complex $C(\mathfrak{U}, \mathcal{O}_X)$ is

$$0 \to A[t] \oplus A[t^{-1}] \stackrel{d_0}{\to} A[t, t^{-1}] \to 0 \cdots .$$

- $\check{\mathrm{H}}(\mathfrak{U}, \mathcal{O}_X) = \mathrm{Ker}(d_0) = A$,
- $\check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{O}_X) = \mathrm{Coker}(d_0) = 0$,
- and the rest is 0.
- The Čech complex $C(\mathfrak{U}, \Omega_X)$ is

$$0 \to A[t]dt \oplus A[s]ds \stackrel{d_0}{\to} A[t, t^{-1}]dt \to 0 \cdots .$$

- Here ds maps to $-dt/t^2$.
- $\check{\mathrm{H}}(\mathfrak{U}, \Omega_X) = \mathrm{Ker}(d_0) = 0$,
- $\check{\mathrm{H}}^1(\mathfrak{U}, \Omega_X) = \mathrm{Coker}(d_0) = A\, dt/t$,
- (and the rest is 0).

Čech complex of sheaves

Theorem

If X noetherian separated, $\mathfrak{U}$ affine covering, $\mathcal{F}$ quasicoherent, there is a functorial isom $\check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F}) \tilde{\to} H^p(X,\mathcal{F})$.

- Set $\mathfrak{C}^p(\mathfrak{U},\mathcal{F}) = \prod_{i_0<\cdots<j_p} (j_{i_0,\ldots,i_p})_*(j_{i_0,\ldots,i_p})^*\mathcal{F}$.
- Set $d : \mathfrak{C}^p(\mathfrak{U},\mathcal{F}) \to \mathfrak{C}^{p+1}(\mathfrak{U},\mathcal{F})$ as before.
- Note: $\Gamma(X,\mathfrak{C}^p(\mathfrak{U},\mathcal{F})) = C^p(\mathfrak{U},\mathcal{F})$.

Lemma

For any $\mathcal{F}$ we have a resolution $0 \to \mathcal{F} \to \mathfrak{C}^\bullet(\mathfrak{U},\mathcal{F})$.

Proposition

If $\mathcal{F}$ is flasque then $\check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F}) = 0$ for all $p > 0$.

Lemma

There are functorial maps $\check{\mathrm{H}}^p(\mathfrak{U},\mathcal{F}) \to H^p(X,\mathcal{F})$.

Lemma: we have a resolution $0 \to \mathcal{F} \to \mathfrak{C}^\bullet(\mathfrak{U}, \mathcal{F})$

- Check on stalks, say at $x \in U_j$.
- Construct homotopy $k : \mathfrak{C}^p(\mathfrak{U}, \mathcal{F})_x \to \mathfrak{C}^{p-1}(\mathfrak{U}, \mathcal{F})_x$.
- A section $\alpha_x \in \mathfrak{C}^p(\mathfrak{U}, \mathcal{F})_x$ lifts to $\alpha \in \mathfrak{C}^p(\mathfrak{U}, \mathcal{F})(V)$ for some neighborhood $x \in V \subset U_j$.
- Viewing α as an alternating cochain define

$$k(\alpha)_{i_0,\ldots,i_{p-1}} = \alpha_{j,i_0,\ldots,i_{p-1}}.$$

- This is well defined on V, hence at x, since $V \subset U_j$.
- Check $(dk + kd)(\alpha) = \alpha$, so $id \sim 0$ and $h^p(\mathfrak{C}^\bullet_x) = 0$.

Proposition: If $\mathcal{F}$ flasque then $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = 0$ for $p > 0$.

- Products and direct images of flasque sheaves are flasque.
- The sheaves $\mathfrak{C}^p(\mathfrak{U}, \mathcal{F})$ are thus flasque.
- We have a flasque resolution $0 \to \mathcal{F} \to \mathfrak{C}^\bullet(\mathfrak{U}, \mathcal{F})$,
- hence
  $H^p(X, \mathcal{F}) = h^p(\Gamma(\mathfrak{C}^\bullet(\mathfrak{U}, \mathcal{F}))) = h^p(C^\bullet(\mathfrak{U}, \mathcal{F})) = \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F})$.
- But since $\mathcal{F}$ is itself flasque we have

$$\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) = H^p(X, \mathcal{F}) = 0$$

for $p > 0$.

Lemma: There are functorial maps $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$.

- Take an injective resolution $0 \to \mathcal{F} \to \mathcal{J}^\bullet$.
- Since $\mathcal{F} \to \mathfrak{C}^0(\mathcal{F})$ injective there is an extension $\mathfrak{C}^0(\mathcal{F}) \to \mathcal{J}^0$.
- The composite $\mathfrak{C}^0(\mathcal{F}) \to \mathcal{J}^1$ factors through $\mathfrak{C}^0(\mathcal{F})/\mathcal{F}$.
- SInce $\mathfrak{C}^0(\mathcal{F})/\mathcal{F} \to \mathfrak{C}^1(\mathcal{F})$ injective, there is an extension $\mathfrak{C}^1(\mathcal{F}) \to \mathcal{J}^1$
- Induction provides an arrow $\mathfrak{C}^\bullet(\mathfrak{U}, \mathcal{F}) \to \mathcal{J}^\bullet$.
- Then apply $h^p(\Gamma(X, \bullet))$.
- Functoriality is trickier

X noetherian separated, $\mathfrak{U}$ affine , $\mathcal{F}$ quasicoherent $\Rightarrow \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{F}) \tilde{\to} H^p(X, \mathcal{F})$.

- Induction on p for all $\mathcal{F}$ etc.
- Take $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{Q} \to 0$ s.e.s with $\mathcal{G}$ quasicoherent flasque (so $\mathcal{Q}$ quasicoherent).
- X separated $\Rightarrow U_{i_0,\ldots,i_p}$ affine.
- $\Rightarrow 0 \to \mathcal{F}(U_{i_0,\ldots,i_p}) \to \mathcal{G}(U_{i_0,\ldots,i_p}) \to \mathcal{Q}(U_{i_0,\ldots,i_p}) \to 0$ exact.
- $\Rightarrow 0 \to C^\bullet(\mathfrak{U}, \mathcal{F}) \to C^\bullet(\mathfrak{U}, \mathcal{G}) \to C^\bullet(\mathfrak{U}, \mathcal{Q}) \to 0$ exact.
- Since $\mathcal{G}$ flasque $\check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{G}) = H^p(X, \mathcal{G}) = 0$ for $p > 0$.
- The long exact sequences and functoriality give

$$\begin{array}{ccccccccc} 0 \to & \check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{F}) & \to & \check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{G}) & \to & \check{\mathrm{H}}^0(\mathfrak{U}, \mathcal{Q}) & \to & \check{\mathrm{H}}^1(\mathfrak{U}, \mathcal{F}) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^0(X, \mathcal{F}) & \to & H^0(X, \mathcal{G}) & \to & H^0(X, \mathcal{Q}) & \to & H^1(X, \mathcal{F}) & \to 0 \end{array}$$

and

$$\begin{array}{ccc} \check{\mathrm{H}}^p(\mathfrak{U}, \mathcal{Q}) & = & \check{\mathrm{H}}^{p+1}(\mathfrak{U}, \mathcal{F}) \\ \downarrow & & \downarrow \\ H^p(X, \mathcal{Q}) & = & H^{p+1}(X, \mathcal{F}) \end{array}$$

Induction gives the required isomorphisms. ♠